Finer analysis of characteristic curves and its application to shock profile, exact and optimal controllability of a scalar conservation law with strict convex flux

Adimurthi[*], Shyam Sundar Ghoshal[†]
and G.D.Veerappa Gowda[‡]

Centre for Applicable Mathematics,
Tata Institute of Fundamental Research,
Post Bag No 6503, Sharadanagar,
Bangalore - 560065, India.

Abstract

Here we consider scalar conservation law in one space dimension with strictly convex flux. Goal of this paper is to study two problems. First problem is to know the profile of the entropy solution. In spite of the fact that, this was studied extensively in last several decades, the complete profile of the entropy solution is not well understood. Second problem is the exact controllability. This was studied for Burgers equation and some partial results are obtained for large time. It was a challenging problem to know the controllability for all time and also for general convex flux. In a seminal paper [8], Dafermos introduces the characteristic curves and obtain some qualitative properties of a solution of a convex conservation law. In this paper, we further study the finer properties of these characteristic curves. As a bi-product we solve these two problems in complete generality. In view of the explicit formulas of Lax - Oleinik [10], Joseph - Gowda [16], target functions must satisfy some necessary conditions. In this paper we prove that it is also sufficient. Method of the proof depends highly on the characteristic methods and explicit formula given

[*]aditi@math.tifrbng.res.in
[†]ssghoshal@math.tifrbng.res.in
[‡]gowda@math.tifrbng.res.in

by Lax - Oleinik and the proof is constructive. This method allows to solve the optimal controllability problem in a trackable way.

Key words: Hamilton-Jacobi Equation, Scalar Conservation Laws, Control and Optimal Control, Explicit Formula.

1 Introduction:

In this paper we consider the following scalar conservation law in one space dimension. Let $f : I\!R \to I\!R$ be a strictly convex C^2 (Regularity condition on f can be relaxed, see remark 3.21) function satisfying the super linear growth,

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = \infty. \tag{1.1}$$

Let $T > 0,\ I = (A, B),\ \Omega = I \times (0, T),\ u_0 \in L^\infty(I),\ b_0, b_1 \in L^\infty((0, T))$ and consider the problem

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad (x, t) \in \Omega, &(1.2)\\ u(x, 0) &= u_0(x) \quad x \in I, &(1.3)\\ u(A, t) &= b_0(t) \quad t \in (0, T), &(1.4)\\ u(B, t) &= b_1(t) \quad t \in (0, T). &(1.5) \end{aligned}$$

This problem was well studied from last several decades starting from the pioneering works of Lax-Oleinik [10], Kruzkov [14], Bardaux-Leraux-Nedeleck [5]. They have studied the existence and uniqueness of weak solutions to (1.2)-(1.5) satisfying the entropy condition. In spite of being well studied, still there are problems which are open. Notably among them are

1. Profile of a solution, for example how many shocks can a solution exhibit and the nature of the shocks.
2. Exact controllability of initial and initial-boundary value problem.
3. Optimal controllability for initial and initial-boundary value problem.

In this paper we study these problems for the entropy solution of (1.2) and we say a solution means a weak solution satisfying the entropy condition. The basic ingredient in studying all these problems comes from the analysis of characteristic curves $R_\pm$. Originally this was introduced by Hopf [12] and later by Dafermos [8], who studied them quite extensively to obtain information on the nature of solutions.

Independently this was used in [2] to obtain the explicit formula for solution of discontinuous flux.

The plan of the paper is as follows:
In section (2) we study the finer properties of the characteristics for the initial value problem, namely

(i). Comparison properties with respect to the initial data.

(ii). Failure of the continuity with respect to the initial data.

(iii). Behavior of the characteristics when one side of the initial data is large.

Main tool to study all these properties is the Lax-Oleinik formula [10] and we recall them without proof. Then proceed to obtain all the required properties. Next we use this information to study (1),(2) and (3).

1. Profile of a solution: In section (3) we study the profile of a solution. In order to understand the shock profile of a solution, we consider the following two basic examples which form the general pattern. Consider the Burgers equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad (x,t) \in I\!R \times I\!R_+.$$

Let $A_1 < A_2,\ u_-, u_+ \in I\!R,\ \bar{u}_0 \in L^\infty((A_1, A_2))$ and define the initial data u_0 by

$$u_0(x) = \begin{cases} u_- & \text{if } \ x < A_1, \\ \bar{u}_0(x) & \text{if } \ A_1 < x < A_2, \\ u_+ & \text{if } \ x > A_2. \end{cases} \tag{1.6}$$

Example 1.1 (Single shock case): (See Figure 1) Let $u_- > \alpha > u_+$ and $\bar{u}_0(x) = \alpha$. Define

$$\begin{aligned} \sigma_0 &= \frac{f(u_-) - f(u_+)}{u_- - u_+},\ \sigma_1 = \frac{f(u_-) - f(\alpha)}{u_- - \alpha},\ \sigma_2 = \frac{f(u_+) - f(\alpha)}{u_+ - \alpha},\ T_0 = \frac{A_2 - A_1}{\sigma_1 - \sigma_2}, \\ x_0 &= A_1 + \sigma_1 T_0,\ s_1(t) = A_1 + \sigma_1 t,\ s_2(t) = A_2 + \sigma_2 t,\ s_0(t) = x_0 + (t - T_0)\sigma_0. \end{aligned}$$

Then the solution u is given by,

(i). Let $0 < t < T_0$, then

$$u(x,t) = \begin{cases} u_- & \text{if } \ x < s_1(t), \\ \alpha & \text{if } \ s_1(t) < x < s_2(t), \\ u_+ & \text{if } \ x > s_2(t). \end{cases} \tag{1.7}$$

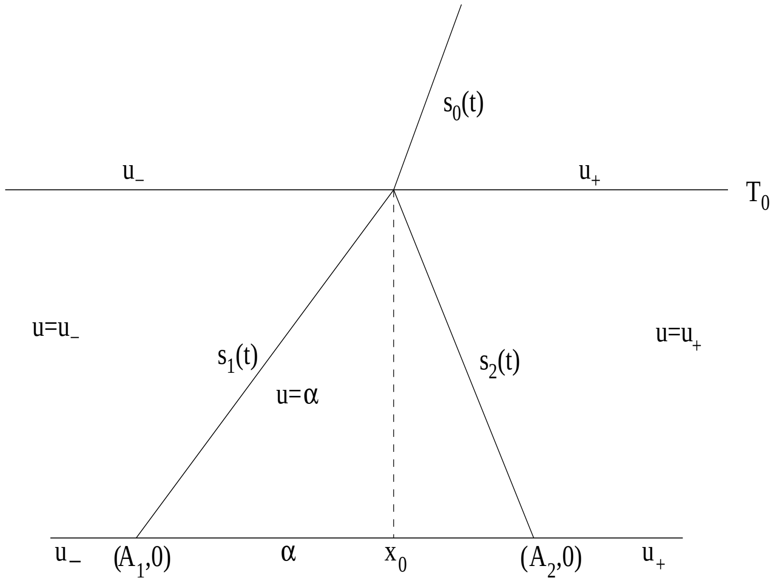

Figure 1:

(ii). Let $t > T_0$, then

$$u(x,t) = \begin{cases} u_- & \text{if} \quad x < s_0(t), \\ u_+ & \text{if} \quad x > s_0(t). \end{cases} \tag{1.8}$$

Example 1.2.(Infinitely many Shocks): (See Figure 2 and Figure 3) Let $I = (A_1, A_2)$ and define the ASSP (asymtotically single shock packet;see definition 3.9) $D(I)$ and single shock solution $u(x,t,I)$ as follows:

$$D(I) = I \times (0, \infty). \tag{1.9}$$

$$T(I) = \frac{A_2 - A_1}{2}. \tag{1.10}$$

$$\bar{u}_0(x,I) = \begin{cases} 1 & \text{if} \quad A_1 < x < \frac{A_1+A_2}{2}, \\ -1 & \text{if} \quad \frac{A_1+A_2}{2} < x < A_2. \end{cases} \tag{1.11}$$

Let $u(x,t,I)$ be a solution of (1.2) in $D(I)$ with initial condition $\bar{u}_0(x,I)$ and satisfying the boundary condition

$$u(A_1+, t, I) = u(A_2-, t, I) = 0, \tag{1.12}$$

and it is given by(see Figure 2).

(i). Let $0 < t < T(I)$, then

$$u(x,t,I) = \begin{cases} \frac{x-A_1}{t} & \text{if} \quad A_1 < x < A_1 + t, \\ 1 & \text{if} \quad A_1 + t < x < \frac{A_1+A_2}{2}, \\ -1 & \text{if} \quad \frac{A_1+A_2}{2} < x < A_2 - t, \\ \frac{x-A_2}{t} & \text{if} \quad A_2 - t < x < A_2. \end{cases} \tag{1.13}$$

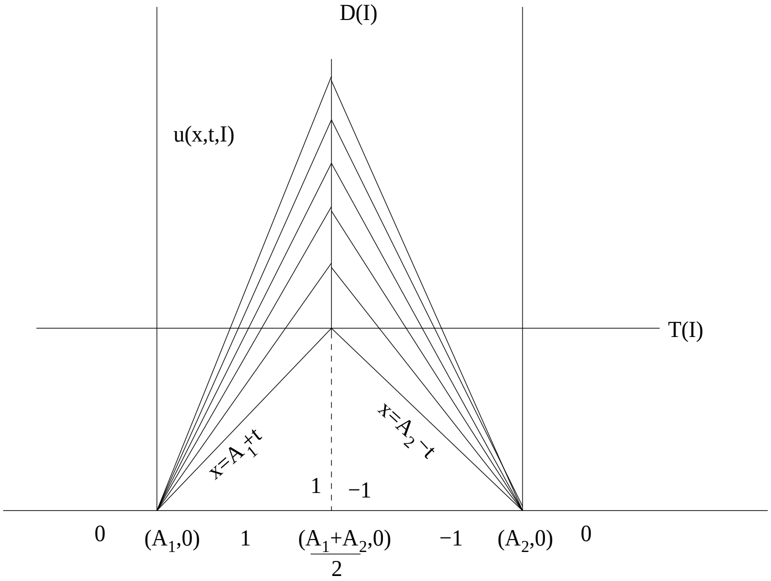

Figure 2:

(ii). Let $t > T(I)$, then

$$u(x,t,I) = \begin{cases} \frac{x-A_1}{t} & \text{if } A_1 < x < \frac{A_1+A_2}{2}, \\ \frac{x-A_2}{t} & \text{if } \frac{A_1+A_2}{2} < x < A_2. \end{cases} \quad (1.14)$$

Next we glue such solutions to produce a single solution having infinitely many shocks for each $t > 0$. Let $n \geq 1$ and define

$$\begin{aligned} I_+ &= (1,\infty),\ I_- = (-\infty,0),\ I_n = (\frac{1}{2n},\frac{1}{2n-1}),\ J_n = (\frac{1}{2n+1},\frac{1}{2n}),\ n \geq 1. \\ D_n &= I_n \times (0,\infty),\ x_n = \frac{1}{2}(\frac{1}{2n}+\frac{1}{2n-1}). \end{aligned}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \in (1,\infty) \cup_{n=1}^{\infty} (\frac{1}{2n}, x_n), \\ 0 & \text{if } x \in J_n \cup I_-, \\ -1 & \text{if } x \in \cup_{n=1}^{\infty} (x_n, \frac{1}{2n-1}), \end{cases} \quad (1.15)$$

and define the solution by (see Figure 3)

$$u(x,t) = \begin{cases} 1 & \text{if} \quad 1+t < x, \\ \frac{x-1}{t} & \text{if} \quad 1 < x < 1+t, \\ u(x,t,I_n) & \text{if} \quad (x,t) \in D_n, \\ 0 & \text{otherwise.} \end{cases} \quad (1.16)$$

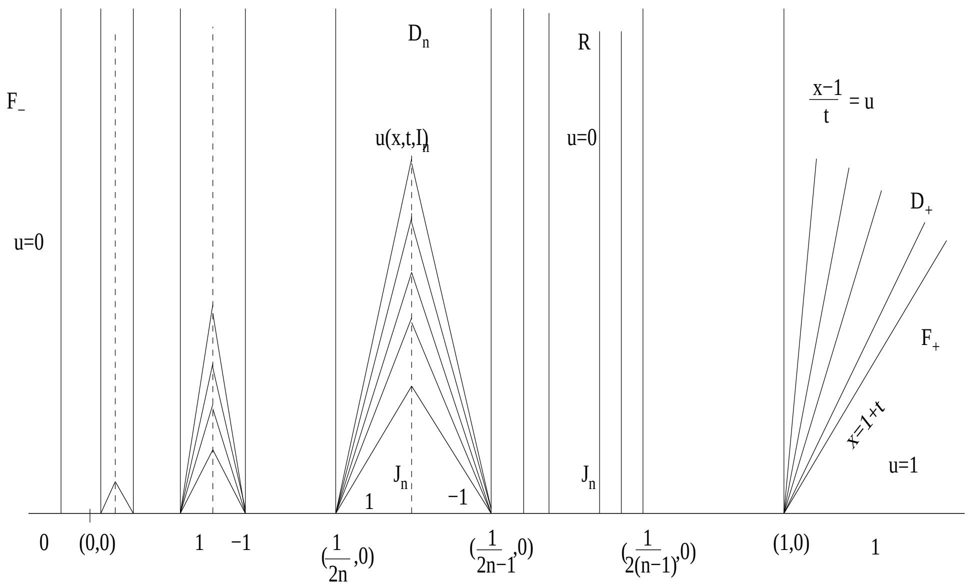

Figure 3:

In view of (1.12), u satisfies Rankine-Hugoniot condition across ∂D_n and hence u is a solution of (1.2) with u_0 as its initial condition.

Now the basic question is, under what conditions, u admits a single shock (as in example (1.1)) for t sufficiently large and what happens if it does not admits a single shock (as in example 1.2) for t large.

Analysis of shocks: By looking at the example (1.1), the data satisfies

$$u_- > u_+. \tag{1.17}$$

It was proved by Liu [19] that for $\bar{u}_0$ arbitrary and u is a piecewise continuous function, then u admits a single shock for t large provided (1.17) holds. Recently Shearer-Dafermos [9] have relaxed the condition of piecewise continuity of u but assuming (1.17) and for $x \in (A_1, A_2)$,

$$u_- \geq \bar{u}_0(x) \geq u_+, \tag{1.18}$$

they proved that for t sufficiently large, u admits a single shock.

Then the question is **the condition (1.18) is necessary ?**.
In section (3) of this paper (Theorem 3.12) we show that infact (1.17) is both necessary and sufficient conditions for u to admit a single shock for t sufficiently large.
Next we consider the case

$$u_- \leq u_+,$$

and want to know how the solution behaves. In this case we can show that (Theorem 3.12) any solutions behaves like as in example (1.2). That is there exists a countable number of disjoint regions $\{D_j\}, F_\pm, D_\pm$ (see Figure 3) such that

(i). $\Omega = I\!R \times I\!R_+ = F_+ \cup F_- \cup D_- \cup D_+ \cup_{i\in I} D_i \cup R$.

(ii). $F_\pm$ are closed and $u(x,t) = u_\pm$ in the interior of $F_\pm$. In example (1.2)

$$F_+ = \{(x,t) : x \le 0\},\ F_- = \{(x,t) : x \ge 1 + t\}.$$

(iii). u behaves like rarefaction in $D_- \cup D_+$, and in example (1.2), $D_- = \phi,\ D_+ = \{(x,t) : 1 < x < 1 + t\}$.

(iv). R is a closed set consists of characteristic lines and u is continuous in R. In example (1.2) R is given by

$$\begin{aligned} R &= \cup_{n=1}^{\infty}\{(x,t) : x \in \bar{J}_n\}, \\ u(x,t) &= 0 \qquad \text{for } (x,t) \in R. \end{aligned}$$

(v). ∂D_i are parallel characteristic lines and any two characteristics curves within D_i intersects after finite time. Asyptotically it represents a single shock packet. In example (1.2), $D_i = I_i \times (0,\infty)$.

(vi). There exists a continuous non-decreasing N curve in $\Omega \setminus F_- \cup F_+$ such that (see Figure 4)

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)| = O(\frac{1}{t^{1/2}})$$

provided $f'' \ge \beta > 0$. In example (1.2), N is given by

$$N(x,t) = \begin{cases} 0 & \text{if } \ x \le 1, \\ \frac{x-1}{t} & \text{if } \ 1 \le x \le 1+t, \\ 1 & \text{if } \ x > 1+t. \end{cases}$$

Main result of this section is Theorem 3.12, where (i) to (iv) has been obtained for a

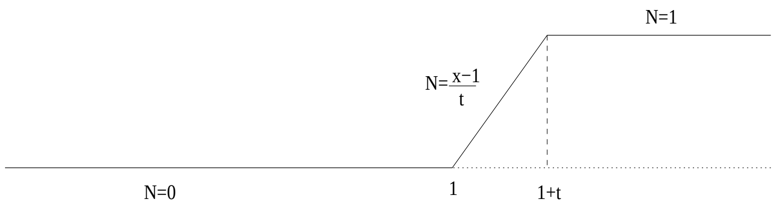

Figure 4: N Wave

general convex flux f. In case of (vi), we have relaxed the condition $f^{''} \ge \beta > 0$. We allow $f^{''}$ can have zeros finite order and then obtained the decay estimates. Earlier this problem was studied by Liu-Pierre [20] (for the power law). Kim [17] (for the algebric growth rate at zero) was relaxed the condition $f'' \geq \beta > 0$ and obtained the decay estimates. Here we have generalized these results and obtain the rate of decay of solutions with respect to the N-wave (see Remark 3.21).

It has been shown by Schaeffer [21] that for large class of smooth initial data u_0, the solution can develop atmost finitely many shocks for t large. If we remove the smoothness in the initial data, example (1.2) shows the existence of infinitely many shock pockets D_i and asymptotically, each one represents a single shock. In fact within D_i, one can have infinitely many shocks and all of them merge at infinity (see example 3.17). On the other hand if the data is smooth, Schaeffer [21] showed that for a large class of smooth initial data for which the solution can exhibit a finite number of shocks. In example (see example 3.14), we construct a $u_0 \in C_c^\infty(I\!R)$ for which the solution admits infinitely many shocks and hence Schaeffer's result cannot be improved.

Main results of this section are Theorems 3.12, (where we prove the above decomposition and obtain optimal decay estimates for N curves) 3.13, 3.16 and example (3.17).

2. Exact Controllability: Normally for the non linear evolution equations, technique of linearization is adopted to study controllability problems. Unfortunately this method does not work (see Horsin [15]) and very few results are available on this subject. Here we consider the following three problems of controllability. Let $u_0 \in L^\infty(I\!R)$,

(I) **Controllability for pure initial value problem:** Assume that $I = I\!R, \Omega = I\!R \times (0,T)$. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2),\ g \in L^\infty(J_1)$, a target be given. The question is, does there exists a $\bar{u}_0 \in L^\infty(J_2)$ and u in $L^\infty(\Omega)$ such that u is a solution of (1.2) satisfying

$$u(x,T) = g(x) \quad x \in J_1, \tag{1.19}$$

$$u(x,0) = \begin{cases} u_0(x) & \text{if} \quad x \notin J_2, \\ \bar{u}_0(x) & \text{if} \quad x \in J_2. \end{cases} \tag{1.20}$$

(II) **Controllability for one sided initial boundary value problem:** Assume that $I = (0,\infty)$, $\Omega = I\!R \times (0,T)$, $J = (0,C)$ and a target function $g \in L^\infty(J)$ be given. The question is, does there exists a $u \in L^\infty(\Omega)$ and a $b \in L^\infty((0,T))$ such that u is a solution of (1.2) satisfying

$$u(x,T) = g(x) \quad \text{if } x \in J, \tag{1.21}$$

$$u(x,0) = u_0(x) \quad \text{if } x \in (0,\infty), \tag{1.22}$$

$$u(0,t) = b(t) \quad \text{if } t \in (0,T). \tag{1.23}$$

(III) **Controllability from two sided initial boundary value problem:**

(a). Let $\Omega = I\!R \times (0,T)$, $I_1 = (B_1, B_2)$, $B_1 \leq C \leq B_2$. Given the target functions $g_1 \in L^\infty(B_1, C)$, $g_2 \in L^\infty(C, B_2)$, does there exists a $\bar{u}_0 \in L^\infty(I\!R \setminus I_1)$ and $u \in L^\infty(\Omega)$ such that u is a solution of (1.2) satisfying

$$u(x,0) = \begin{cases} g_1(x) & \text{if} \quad B_1 < x < C, \\ g_2(x) & \text{if} \quad C < x < B_2. \end{cases} \tag{1.24}$$

and

$$u(x,0) = \begin{cases} u_0(x) & \text{if} \quad B_1 < x < B_2, \\ \bar{u}_0(x) & \text{if} \quad x < B_1 \text{ or } x > B_2. \end{cases} \tag{1.25}$$

(b). Here we consider controllability in a strip. Let $I = (B_1, B_2)$, $\Omega = I \times (0,T), B_1 < C < B_2$. Let $g_1 \in L^\infty((B_1, C)), g_2 \in L^\infty((C, B_2))$ be given. The question is, does there exist $b_0, b_1 \in L^\infty((0,T))$ and a $u \in L^\infty(\Omega)$ such that u is a solution of (1.2) and satisfying

$$u(x,0) = u_0(x), \tag{1.26}$$

$$u(x,T) = \begin{cases} g_1(x) & \text{if} \quad B_1 < x < C, \\ g_2(x) & \text{if} \quad C < x < B_2. \end{cases} \tag{1.27}$$

$$u(B_1,t) = b_0(t), \tag{1.28}$$

$$u(B_2,t) = b_1(t). \tag{1.29}$$

In view of the Lax-Oleinik (Chapter (3) of [10]) explicit formula for solutions of pure initial value problem and by Joseph-Gowda [16] for initial boundary value problem, the targets g or g_1, g_2 cannot be arbitrary. They must satisfy the compatibility condition, for example in the case of problem (I), there exists a non-decreasing function ρ in (B_1, B_2) such that for a.e $x \in (B_1, B_2)$

$$f'(g(x)) = \frac{x - \rho(x)}{T}. \tag{1.30}$$

In the case of problem (II), there exists a non-decreasing function ρ in $(0, C)$ such that

$$f'(g(x)) = \frac{x}{T - \rho(x)}. \tag{1.31}$$

Assuming that the target functions satisfies the compatibility conditions, then the question is **whether the problems (I),(II) and (III) admit a solution?**. In fact, **it is true** and we have the following results. First we describe the class of functions satisfying compatibility conditions.

Definition (Admissible functions): Let $J = (M, N)$ and $T > 0$,

$$S(J) = \{\rho : J \to I\!R : \rho \text{ is monotone and left or right condinuous function}\}.$$

Then define admissible class of target functions by

(i) Target space for initial value problem (IA):

$$IA(J) = \{g;\ f^{'}(g(x)) = \frac{x-\rho(x)}{T},\ \rho \in S(J),\ \rho \text{ is a non-decreasing funtion}\}. \tag{1.32}$$

(ii) Target space for left boundary problem (LA):

$$LA(J) = \{g;\ \ f^{'}(g(x)) = \ \tfrac{x-M}{T-\rho(x)},\ \rho \in S(J),\ \rho \text{ is a non-increasing right continuous function}\}. \tag{1.33}$$

(iii) Target space for right boundary problem (RA):

$$RA(J) = \{g;\ \ f^{'}(g(x)) = \ \tfrac{x-N}{T-\rho(x)},\ \rho \in S(J),\ \rho \text{ is a non-decreasing left continuous function}\}. \tag{1.34}$$

THEOREM 1.1 *Let* $J_1 = (C_1, C_2), J_2 = (B_1, B_2)$. *Let* $g(x) = (f')^{-1}\left(\frac{x-\rho(x)}{T}\right)$ *be in* $IA(J_1)$ *and* $B_1 < A_1 < A_2 < B_2$, *satisfying*

$$A_1 \le \rho(x) \le A_2 \quad \text{if } x \in J_1, \tag{1.35}$$

then there exists a $\bar{u}_0 \in L^\infty(J_2),\ u \in L_\infty(\Omega)$ *such that* $(u, \bar{u}_0)$ *is a solution to problem (I) (see Figure 5).*

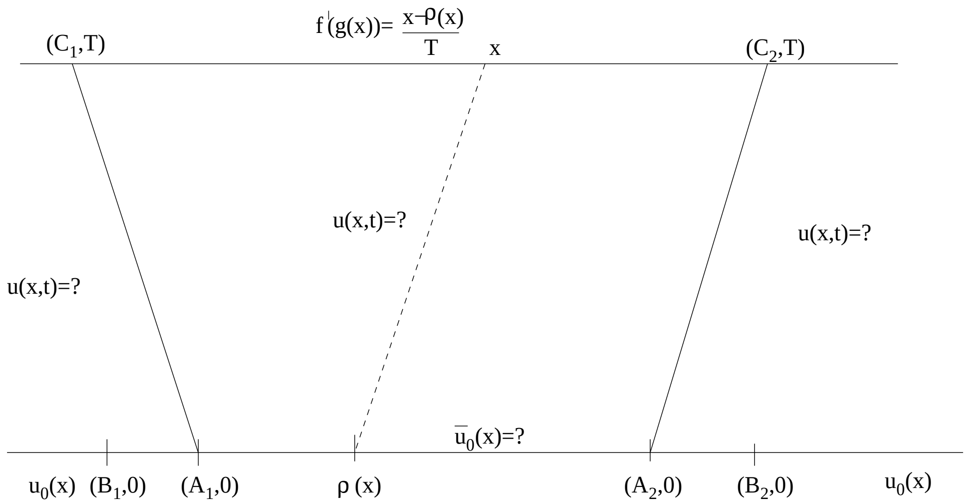

Figure 5:

THEOREM 1.2 *Let $\wedge > 0, C > 0, \delta > 0, J = (0, C)$. Let $g \in LA(J)$ given by $f'(g(x)) = \frac{x}{T-\rho(x)}$ for $x \in J$ and satisfying*

$$\delta \leq \rho(x) \leq T, \tag{1.36}$$

$$\left|\frac{x}{T-\rho(x)}\right| \leq \wedge. \tag{1.37}$$

Then there exist a $b \in L^\infty(0,T), u \in L^\infty(\Omega)$ such that (u, b) is a solution to Problem II (see Figure 6).

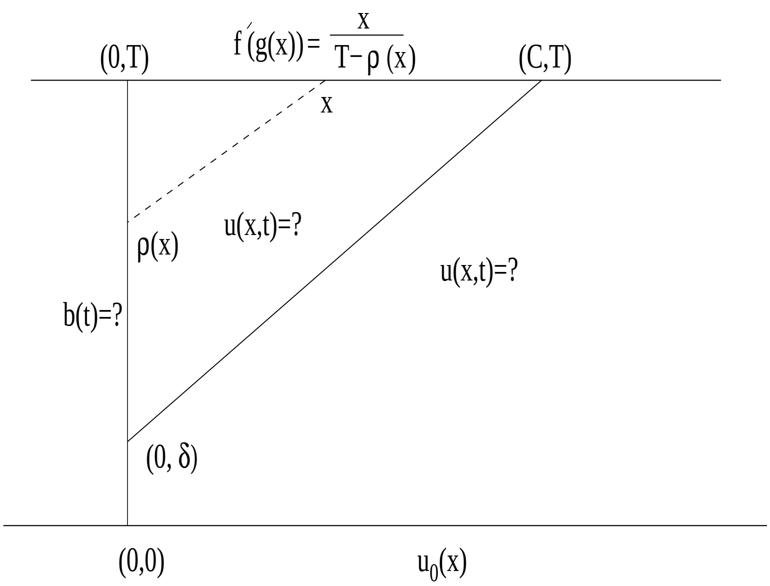

Figure 6:

THEOREM 1.3 *Let $I_1 = (B_1, B_2), B_1 < C < B_2, J_1 = (B_1, C), J_2 = (C, B_2)$, then*

(a). Let $A_1 < B_1 < B_2 < A_2$ and $g_1 \in IA(J_1), g_2 \in IA(J_2)$ given by $f'(g_1(x)) = \frac{x-\rho_1(x)}{T}$, $f'(g_2(x)) = \frac{x-\rho_2(x)}{T}$, satisfying

$$\rho_1(x) \leq A_1 \quad \text{if } x \in J_1, \tag{1.38}$$

$$\rho_2(x) \geq A_2 \quad \text{if } x \in J_2. \tag{1.39}$$

Then there exists $\bar{u_0} \in L^\infty((I\!R \setminus I_1)), u \in L^\infty(\Omega)$ such that $(u, \bar{u_0})$ is a solution to problem (a) of III (see Figure 7).

(b). Let $\wedge > 0, 0 < \delta < T, g_1 \in LA(J_1), g_2 \in RA(J_2)$, given by $f'(g_1(x)) = \frac{x-B_1}{T-\rho_1(x)}, f'(g_2(x)) = \frac{x-B_2}{T-\rho_2(x)}$ satisfying for $i = 1, 2,\ x \in J_i$

$$\delta \leq \rho_i(x) \leq T, \tag{1.40}$$

$$\left|\frac{x-B_i}{T-\rho_i(x)}\right| \leq \wedge. \tag{1.41}$$

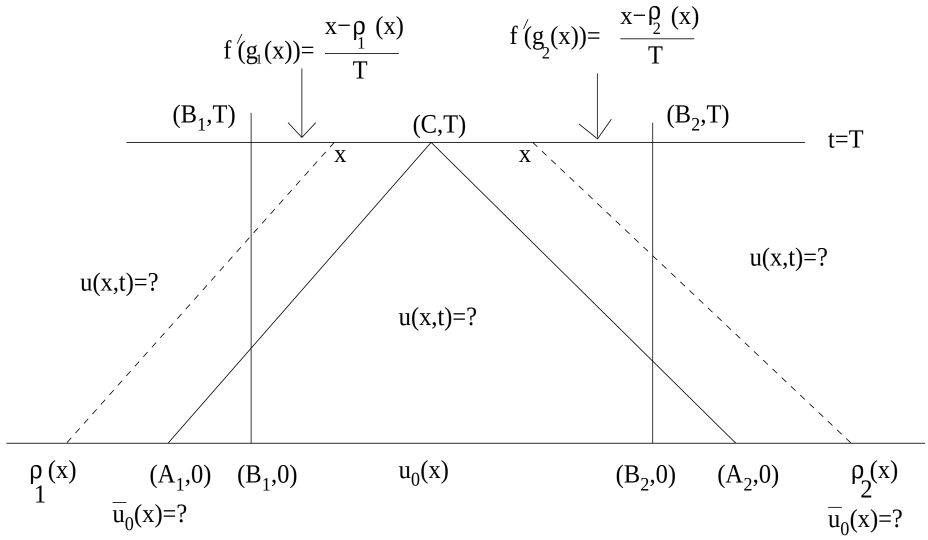

Figure 7:

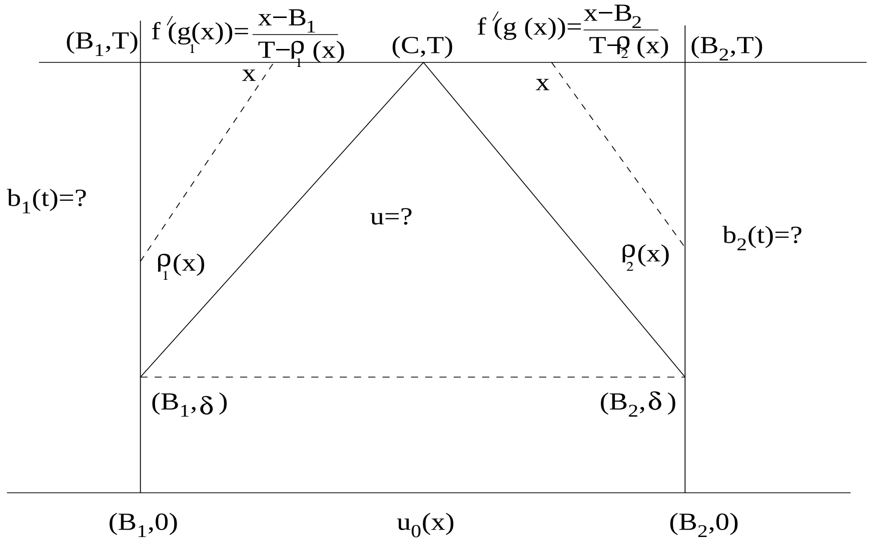

Figure 8:

Then there exists $b_0, b_1 \in L^\infty((0,T))$ *and* $u \in L^\infty(\Omega)$ *such that* (u, b_0, b_1) *is a solution to problem (b) of III (see Figure 8).*

Before going for further results, let us recall some of the earlier works in this direction.

Problem (a) in III was considered by Horsin [15] for the Burger's equation under similar assumptions on g_1 and g_2 as in (a) of Theorem (1.3). He proves that there exists $T_c > 2$, such that (a) of problem III has an approximate controllability solution. That is given $\epsilon > 0$, there exist $(u, \bar{u}_0)$ such that

$$\int_{B_1}^{B_2} |u(x,T_c) - g(x)|dx = O(\epsilon),$$

and $u(x,T_c) = g(x) = \chi_{(B_1,C)}g_1(x) + \chi_{(C,B_2)}g_2(x)$, outside an interval of length ϵ.

In the viscous case the same problem was considered by Glass-Guerrero [11] for the control $u(x,T) = M$ is constant. Using the Cole-Hopf transformation, they show that there exist $T_0 > 0$ such that for all time $T > T_0$ and small viscosity, they prove the exact controllability. Also Guerrero-Imanuvilov [13] proves a negative result by showing that $M = 0$ cannot be controllable.

Theorem (1.3) is stronger and much more precise result in the non viscous case because

(i). It removes the condition on time T_c and obtains exact controllability.

(ii). It deals with general convex flux instead of Burger's equation.

(iii). In section (5) we give a criterion when the constants are controllable.

In the case of problem (II), Fabio-Ancona and Andrea-Marson [3],[4] studied the problem from the point of view of Hamilton-Jacobi equations (Earlier similar method was developed by Joseph-Gowda [16]) and studies the compactness properties of $\{u(\cdot,T)\}$ when $u(x,0) = 0$ and $u(\cdot,0) \in \mathcal{U}$, here $\mathcal{U}$ is a set of controls satisfying some properties. But they do not address the exact controllability question and Theorem (1.2) gives a precise solution for control problem (II).

In our results on controllability, superlinearity of f plays an important role in removing the condition on T_c and obtain a free region (see Lemmas (2.9) and (2.10)). Next using convexity, we explicitly construct solutions in these free regions for particular data which allow to obtain solutions for control problems(see Lemmas (4.1) and (4.2)).

(3) Optimal controllability: Let $g \in L^\infty(I\!\!R)$ with compact support be given. For $u_0 \in L^\infty(I\!\!R)$, let u be the solution of (1.2) in $\Omega = I\!\!R \times I\!\!R_+$ with initial data $u_0(x)$. Define the cost functional

$$J(u_0) = \int\limits_{I\!\!R} |u(x,T) - g(x)|^2 dx, \tag{1.42}$$

and consider the minimization problem

$$C = \inf_{u_0 \in L^\infty(I\!\!R)} J(u_0).$$

This problem was considered by Castro-Palacious-Zuazua [6] and proved that there exists a minimizer. Since the functional is neither convex nor differentiable, it is quite hard to give a numerical scheme to capture a minimizer.

In section (5),we tackle this optimal controllability in a different way. From Lax-Oleinik formula, we first reduce the problem to a standard optimization in a Hilbert space and then using the explicit construction of a solution (see Lemma (4.1)) to obtain a minimizer. This construction turns out to be far **simpler**.

In Section (6), we generalize some of the results on control and optimal controllability.

Guidelines for the readers: Lemmas 2.2 and 2.4 are required to understand the shock profile. We use Lemmas 2.9, 2.10, 2.15, 4.1, 4.2 to prove the controllability results.

All this results in this paper has been extended to discontinuous flux [1].

Acknowledgement: Authors would like to thank Enric Zuazua for suggesting this problem and providing some valuable references.

2 Finer Analysis of Characteristics:

In a beautiful paper, Dafermos [8] had extensively studied the properties of characteristic curves. Here we make a finer analysis of these characteristics curves and then use them to obtain our results. In order to do this, first we recollect the results of Lax-Olenik explicit formula and a good reference for this, is third chapter in [10].

Let $f^*(p) = \sup\limits_q \{pq - f(q)\}$ denote the Legendre transform of f.

Then f^* is in C^1, strictly convex, super linear growth and satisfies

$$\begin{array}{rcl} f & = & f^{**}, \\ f^{*'}(p) & = & (f')^{-1}(p), \\ f^*(f'(p)) & = & pf'(p) - f(p), \\ f(f^{*'}(p)) & = & pf^{*'}(p) - f^*(p). \end{array} \tag{2.1}$$

Controlled Curves: Let $x \in I\!R, 0 \le s < t$ and define the controlled curves $\Gamma(x,s,t)$ by

$$\Gamma(x,s,t) = \{r : [s,t] \to I\!R;\ r \text{ is linear and } \quad r(t) = x\}, \tag{2.2}$$

and denote $\Gamma(x,t) = \Gamma(x,0,t)$.

Value function: Let $u_0 \in L^\infty(I\!R), x_0 \in I\!R$, define

$$v_0(x) = \int_{x_0}^{x} u_0(\theta)d\theta, \tag{2.3}$$

be its primitive. Define the value function $v(x,t)$ by

$$\begin{array}{rcl} v(x,t) & = & \min\limits_{r\in\Gamma(x,t)} \left\{ v_0(r(0)) + tf^*\left(\frac{x-r(0)}{t}\right)\right\} \\ & = & \min\limits_{\beta\in I\!R} \left\{ v_0(\beta) + tf^*\left(\frac{x-\beta}{t}\right)\right\}. \end{array} \tag{2.4}$$

Then v satisfies the

Dynamic Programming principle: For $0 \le s < t$,

$$v(x,t) = \min_{r\in\Gamma(x,s,t)} \left\{ v(r(s),s) + (t-s)f^*\left(\frac{x-r(s)}{t-s}\right)\right\}. \tag{2.5}$$

Define the characteristic set $ch(x,s,t,u_0)$ and extreme characteristics $y_\pm(x,s,t,u_0)$ by

$$ch(x,s,t,u_0) = \{r \in \Gamma(x,s,t);\ r \text{ is a minimizer in (2.5)}\}, \tag{2.6}$$

$$y_-(x,s,t,u_0) = \min\{r(s) : r \in ch(x,s,t,u_0)\}, \tag{2.7}$$

$$y_+(x,s,t,u_0) = \max\{r(s); r \in ch(x,s,t,u_0)\}, \tag{2.8}$$

Denote $ch(x,t,u_0) = ch(x,0,t,u_0)$, $y_\pm(x,t,u_0) = y_\pm(x,0,t,u_0)$. Then we have the following result due to Hopf, Lax -Oleinik:

THEOREM 2.1 *Let $0 \le s < t, u_0, v_0, v$ be as above, then*

1. *v is a uniformly Lipschitz continuous function and is a unique viscosity solution of the Hamilton-Jacobi equation*

$$\begin{array}{rcll} v_t + f(v_x) & = & 0 & (x,t) \in I\!R \times (0,\infty), \\ v(x,0) & = & v_0(x) & x \in I\!R. \end{array} \tag{2.9}$$

2. *There exist $M > 0$, depending only on $\|u_0\|_\infty$ and Lipschitz constant of f restricted to the interval $[-\|u_0\|_\infty, \|u_0\|_\infty]$ such that for $(x,t) \in I\!R \times I\!R_+$, $ch(x,s,t,u_0) \neq \phi$ and for $r \in ch(x,s,t,u_0)$*

$$\left|\frac{x - r(s)}{t-s}\right| \leq M. \tag{2.10}$$

3. *NIP (Non intersecting property of characteristics): Let $x_1 \neq x_2$ and for $i = 1,2, r_i \in ch(x_i,s,t,u_0)$. Then $r_1(\theta) \neq r_2(\theta)$ for all $\theta \in (s,t)$.*

   From NIP, it follows that for $0 \leq s < t$,

   (a). $x \mapsto y_\pm(x,s,t,u_0)$ are non decreasing functions,

   (b). At the points of continuity of y_+,

   $$y_+(x,s,t,u_0) = y_-(x,s,t,u_0),$$

   and hence $ch(x,s,t,u_0) = \{r\}$, where r is given by

   $$r(\theta) = \frac{x - y_+(x,s,t,u_0)}{t-s}(\theta - t) + x.$$

   (c). Let $r \in ch(x,t,u_0)$, $z = r(s)$. Let $r_1(\theta) = r(\theta)$ for $0 \leq \theta \leq s, r_2(\theta) = r(\theta)$ for $s \leq \theta \leq t$. Then $r_1 \in ch(z,s,u_0), r_2 \in ch(x,s,t,u_0)$.

4. *Let $u(x,t) = \frac{\partial v}{\partial x}(x,t)$. Then u is the unique solution of (1.2) in $\Omega = I\!R \times I\!R_+$ with initial data u_0 and satisfying*

   $$|u(x,t)| \leq \|u_0\|_\infty. \tag{2.11}$$

   For a.e x, $y_-(x,t) = y_+(x,t)$ and u is given by

   $$f^{'}(u(x,t)) = \frac{x - y_+(x,t,u_0)}{t} = \frac{x - y_-(x,t,u_0)}{t}. \tag{2.12}$$

   Let x be a point of differentiability of $y_\pm(x,t,u_0)$ and $y_\pm(x,t,u_0)$ is a point of differentiability of v_0, then

   $$u(x,t) = u_0(y_\pm(x,t,u_0)). \tag{2.13}$$

5. *Let $u_0, w_0 \in L^\infty(I\!R)$ and u,w be the solutions given in (4) with initial data u_0, w_0 respectively. Then*

   (a). Monotonicity: Let $u_0(x) \leq w_0(x)$ for $x \in I\!R$, then for a.e.$x \in I\!R, t > 0$,

   $$u(x,t) \leq w(x,t). \tag{2.14}$$

(b). L^1_{loc} *contractivity: Let* $c = \max(\|u_0\|_\infty, \|w_0\|_\infty)$ *and* $I = [-c, c]$. *Then there exist a* $M > 0$, *depending on Lipschitz constant* f *restricted to* I *such that for all* $t > 0,\ a < b$,

$$\int_a^b |u(x,t) - w(x,t)|dx \leq \int_{a-Mt}^{b+Mt} |u_0(x) - w_0(x)|dx. \tag{2.15}$$

For the proofs of (1) to (4) see chapter (3) of [10] and for (5), see chapter (3) of [14].

In this sequal we follow the notations of charactericitic curves as in [2]. From now onwards, we assume that $\Omega = I\!R \times (0, \infty), u_0 \in L^\infty(I\!R)$.

Left and right characteristic curves: Let $0 \leq s < t, u$ be a solution of (1.2) with initial data u_0 and $\alpha \in I\!R$. Define the left characteristic curve $R_-(t, s, \alpha, u_0)$ and right characteristic curve $R_+(t, s, \alpha, u_0)$ and denote $R_\pm(t, \alpha, u_0) = R_\pm(t, 0, \alpha, u_0)$ by

$$R_-(t, s, \alpha, u_0) = \inf\{x; \alpha \leq y_-(x, s, t, u_0)\}, \tag{2.16}$$

$$R_+(t, s, \alpha, u_0) = \sup\{x : y_+(x, s, t, u_0) \leq \alpha\}. \tag{2.17}$$

In view of (2.10), $y_-(x, s, t, u_0) \to -\infty$ as $x \to -\infty$, $y_+(x, s, t, u_0) \to +\infty$ as $x \to +\infty$. Hence (2.16) and (2.17) are well defined. Our aim is to study the continuous dependence of $R_\pm$ on their arguments (t, α, u_0).

For $x, y, \in I\!R, t > 0$, let $r(\theta, t, x, y) \in \Gamma(x, t)$ be the line joining $(x, t), (y, 0)$ given by

$$r(\theta, t, x, y) = \left(\frac{x - y}{t}\right)(\theta - t) + x. \tag{2.18}$$

Observe that $r(0, t, x, y) = y$ and hence $r \in ch(x, t, u_0)$ if and only if y is a minimizer in (2.4). Hence define the extreme characteristic lines by

$$r_\pm(\theta, t, x) = r(\theta, t, y_\pm(x, t, u_0)). \tag{2.19}$$

Since $r_\pm(0, t, x) = y_\pm(x, t.u_0)$ and $y_-(x, t, u_0) \leq y_+(x, t, u_0)$, hence for all $\theta \in [0, t]$,

$$r_-(\theta, t, x) \leq r_+(\theta, t, x). \tag{2.20}$$

Then we have the following

Lemma 2.2 *Let* $u_0, w_0, \{u_0^k\}$ *are in* $L^\infty(I\!R)$ *and* $\alpha, \{\alpha_k\}$ *are in* $I\!R$. *Let* $v, W, \{v_k\}$ *be the value functions defined in (2.4) with respect to the data* $u_0, w_0, \{u_0^k\}$ *respectively. Let* $u = \frac{\partial v}{\partial x}, w = \frac{\partial W}{\partial x}, u_k = \frac{\partial v_k}{\partial x}$ *be the solutions of (1.2). Then*

1. *Let* $x_1 < x_2, 0 \leq s < t$ *and* $\beta \in I\!R$ *be a minimizer for* $v(x_1, t)$ *and* $v(x_2, t)$ *in (2.5). Then for* $x_1 < x < x_2$, β *is the unique minimizer for* $v(x, t)$ *and satisfies*

$$f'(u(x, t)) = \frac{x - \beta}{t - s}. \tag{2.21}$$

2. *Let* $x_k \in I\!R, r_k \in ch(x_k, t, u_0)$ *such that* $\lim\limits_{k\to\infty}(x_k, r_k(0)) = (x, \beta)$*. Then* $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$*. Furthermore*

$$\lim_{x_k \uparrow x} y_+(x_k, t, u_0) = y_-(x, t, u_0), \tag{2.22}$$

$$\lim_{x_k \downarrow x} y_-(x_k, t, u_0) = y_+(x, t, u_0). \tag{2.23}$$

In particular, y_- *is left continuous and* y_+ *is right continuous.*

3. *(i). For all* $t > 0$,

$$R_-(t, \alpha, u_0) \le R_+(t, \alpha, u_0), \tag{2.24}$$

$$\begin{cases} y_-(R_-(t,\alpha,u_0),t,u_0) \le \alpha \le y_+(R_-(t,\alpha,u_0),t,u_0), \\ y_-(R_+(t,\alpha,u_0),t,u_0) \le \alpha \le y_+(R_+(t,\alpha,u_0),t,u_0). \end{cases} \tag{2.25}$$

Further more if $R_-(t, \alpha, u_0) < R_+(t, \alpha, u_0)$*, then for all* $x \in (R_-(t, \alpha, u_0), R_+(t, \alpha, u_0))$ *(see Figure 9)*

$$y_\pm(x, t, \alpha) = \alpha, f'(u(x,t)) = \frac{x - \alpha}{t}. \tag{2.26}$$

(ii). Let $0 < s < t$*, then*

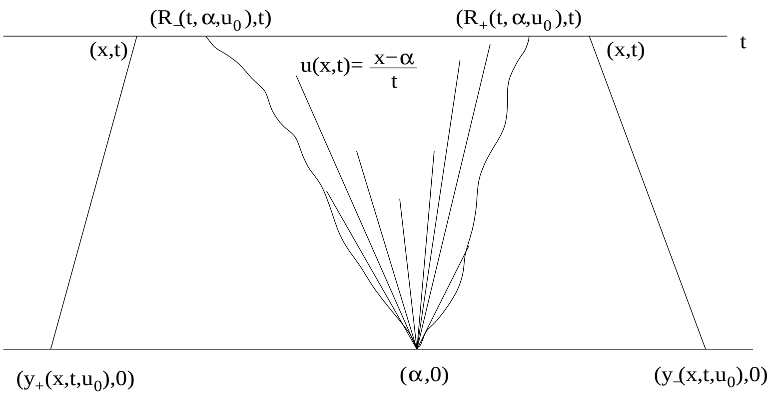

Figure 9:

$$R_-(t, s, \alpha, u_0) = R_+(t, s, \alpha, u_0). \tag{2.27}$$

4. *Let* $0 \le s < t$*. Then* $t \mapsto R_\pm(t, \alpha, u_0)$ *are Lipschitz continuous function with Lipschitz norm depends only on* α *and* $\|u_0\|_\infty$ *and satisfying*

$$\lim_{t\to 0} R_\pm(t, \alpha, u_0) = \alpha, \tag{2.28}$$

$$R_\pm(t, \alpha, u_0) = R_\pm(t, s, R_\pm(s, \alpha, u_0), u_0). \tag{2.29}$$

5. *Monotonicity: Let* $u_0 \le w_0, \alpha \le \beta$, *then*

$$R_\pm(t,\alpha,u_0) \le R_\pm(t,\alpha,w_0), \tag{2.30}$$

$$R_\pm(t,\alpha,u_0) \le R_\pm(t,\beta,u_0). \tag{2.31}$$

6. *Continuity with respect to data: Let* $\alpha_k \to \alpha, u_0^k \to u_0$ *in* $L^1_{loc}(I\!R)$. *Then for* $t > 0$, *(see counter example 2.16).*

   (a). Suppose for all $k, R_-(t,\alpha_k,u_0^k) \le R_-(t,\alpha,u_0)$, *then*

$$\lim_{k\to\infty} R_-(t,\alpha_k,u_0^k) = R_-(t,\alpha,u_0). \tag{2.32}$$

   (b). Suppose for all $k, R_+(t,\alpha_k,u_0^k) \ge R_+(t,\alpha,u_0)$, *then*

$$\lim_{k\to\infty} R_+(t,\alpha_k,u_0^k) = R_+(t,\alpha,u_0). \tag{2.33}$$

   (c). Suppose $R_-(t,\alpha,u_0) < \bar{R} = \overline{\lim_{k\to\infty}} R_-(t,\alpha_k,u_0^k)$, *then for all* $x \in (R_-(t,\alpha,u_0),\bar{R}), y_\pm(x,t,u_0) = \alpha$ *and*

$$f'(u(x,t)) = \frac{x-\alpha}{t}. \tag{2.34}$$

   (d). Suppose $\underline{\lim_{k\to\infty}} R_+(t,\alpha_k,u_0^k) = \bar{R} < R_+(t,\alpha,u_0)$, *then for all* $x \in (\bar{R}, R_+(t,\alpha,u_0)), y_\pm(x,t,u_0) = \alpha$ *and*

$$f'(u(x,t)) = \frac{x-\alpha}{t}. \tag{2.35}$$

As an immediate consequence of this, if $R_-(t,\alpha,u_0) = R_+(t,\alpha,u_0)$ *for* $t > 0$, *then* $R_\pm(t,\alpha,u_0)$ *is continuous at* (α,u_0).

Proof. (1). Let $x \in (x_1,x_2)$ and $r \in ch(x,s,t,u_0)$. Suppose $r(s) \neq \beta$. then r intersets one of the characteristics $\left(\frac{x_i-\beta}{t-s}\right)(\theta - t) + x_i,\ i = 1,2$, which contradicts NIP of Theorem (2.1). Hence $\beta = r(s) = y_\pm(x,s,t,u_0)$. Furthermore

$$v(x,t) = v(\beta,s) + (t-s)f^*\left(\frac{x-\beta}{t-s}\right),$$

and for a.e x,

$$u(x,t) = \frac{\partial v}{\partial x} = f^{*'}\left(\frac{x-\beta}{t-s}\right) = (f')^{-1}\left(\frac{x-\beta}{t-s}\right).$$

This proves (1).

(2). From the continuity of v and f^*, we have

$$\begin{aligned} v(x,t) &= \lim_{k\to\infty} v(x_k,t) \\ &= \lim_{k\to\infty} \left\{ v_0(r_k(0)) + tf^*\left(\frac{x_k - r_k(0)}{t}\right)\right\} \\ &= v_0(\beta) + tf^*\left(\frac{x-\beta}{t}\right), \end{aligned}$$

and hence $r(\cdot,t,x,\beta) \in ch(x,t,u_0)$. Let $x_1 < x_2$, then from NIC, $y_+(x_1,t,u_0) \leq y_-(x_2,t,u_0)$. From monotonicity of $y_\pm$, we have

$$y_-(x_1,t,u_0) \leq y_+(x_1,t,u_0) \leq y_-(x_2,t,u_0) \leq y_+(x_2,t,u_0).$$

Let $x_k \uparrow x$, then from above inequality,

$$\beta = \underline{\lim}_{k\to\infty} y_+(x_k,t,u_0) \leq y_-(x,t,u_0).$$

Since a subsequence of $y_+(x_k,t,u_0)$ converges to β, hence $r(\cdot,t,x,\beta) \in ch(x,t,u_0)$. Therefore $\beta \leq y_-(x,t,u_0) \leq r(0,t,x,\beta) = \beta$. This proves (2.22). Similarly (2.23) follows. This proves (2).

(3). (i). Suppose $y_-(R_-(t,\alpha,u_0),t,u_0) > \alpha$. Then from (2.22) there exist $x_0 < R_-(t,\alpha,u_0)$. such that for all $x \in (x_0, R_-(t,\alpha,u_0))$, $y_+(x,t,u_0) > \alpha$. Let x be a point of continuity of y_+, then from (3) of theorem (2.1), $y_-(x,t,u_0) = y_+(x,t,u_0) > \alpha$ and hence $R_-(t,\alpha,u_0) \leq x < R_-(t,\alpha,u_0)$ which is a contradiction. Suppose $y_+(R_-(t,\alpha,u_0),t,u_0) < \alpha$, again from (2.23) there exist $x_0 > R_-(t,\alpha,u_0)$ such that for all $x \in (R_-(t,\alpha,u_0),x_0)$, $y_-(x,t,u_0) < \alpha$. Therefore at points x of continuity, $\alpha \leq y_+(x,t,u_0) = y_-(x,t,u_0) < \alpha$, which is a contradiction. This proves (2.25) and (2.26) follows similarly.

Suppose $R_+(t,\alpha,u_0) < R_-(t,\alpha,u_0)$, then from (2.25), (2.26), $y_-(R_-(t,\alpha,u_0),\alpha, u_0) \leq \alpha \leq y_+(R_+(t,\alpha,u_0),t,u_0)$, therefore from NIC, $y_-(R_-(t,\alpha,u_0),t,u_0) = \alpha = y_+(R_+(t,\alpha,u_0),t,u_0)$. Hence from (2.21), for all $x \in (R_+(t,\alpha,u_0), R_-(t,\alpha,u_0))$, α is a minimizer for $v(x,t)$ which implies that $R_-(t,\alpha,u_0) \leq x < R_-(t,\alpha,u_0)$ which is a contradiction. This proves (2.24).

Suppose $R_-(t,\alpha,u_0) < R_+(t,\alpha,u_0)$. then from (2.24), (2.25), (2.26) we have

$$\alpha \leq y_+(R_-(t,\alpha,u_0),t,u_0) \leq y_-(R_+(t,\alpha,u_0),t,u_0) \leq \alpha.$$

Therefore from (1), for all $x \in (R_-(t,\alpha,u_0), R_+(t,\alpha,u_0))$, $y_\pm(x,t,u_0) = \alpha$ and $f'(u(x,t)) = \frac{x-\alpha}{t}$. This proves (2.26).

(3). (ii). Let $0 < s < t$, then as in (2.24) we have $R_-(t,s,\alpha,u_0) \leq R_+(t,s,\alpha,u_0)$. Suppose $R_-(t,s,\alpha,u_0) < R_+(t,s,\alpha,u_0)$, then as in (2.26), we have for all $x \in (R_-(t,s,\alpha,u_0), R_+(t,s,\alpha,u_0))$, $f'(u(x,t)) = \frac{x-\alpha}{t-s}$. Let $R_-(t,s,\alpha,u_0) < x_1 < x_2 < R_+ (t,s,\alpha,u_0)$ and $r_\pm(.,t,x_1), r_\pm(.,t,x_2)$ be the extreme characteristics at x_1, x_2. Hence $r_\pm(s,t,x_1) = r_\pm(s,t,x_2) = \alpha$ contradicting NIP. This proves (ii) and hence (3).

(4). Let $0 \leq s < t, R_- = R_-(t,\alpha,u_0), y_\pm = y_\pm(R_-,t,u_0)$ and $r_\pm(\theta) = r(\theta,t,R_-,y_\pm) \in ch(R_-,t,u_0)$. Then from (3) of theorem 2.1, $r_\pm|_{(0,s)} \in ch(r_\pm(s),s,u_0)$.
Claim : $r_-(s) \leq R_-(s,\alpha,u_0) \leq r_+(s)$.

Suppose $R_-(s,\alpha,u_0) < r_-(s)$. For $x \in (R_-(s,\alpha,u_0), r_-(s))$, $y_-(x,s,\alpha) \geq \alpha$. Hence if $y_- < \alpha$ or $y_-(x,s,\alpha) > \alpha$, then the characteristics $r_-(\theta), r_-(\theta,s,x)$ intersect for some $\theta \in (0,s)$ which contradicts NIC. Therefore $\alpha = y_- = y_-(x,s,\alpha)$ and from (2) $\tilde{r}(\theta) = \tilde{r}(\theta,s,R_-(s,\alpha,u_0),\alpha) \in ch(R_-(s,\alpha,u_0),s,u_0)$. From (2.22) choose a $\xi < R_-$, $y_-(\xi,t,u_0) < \alpha$ such that the characteristic $\tilde{r}(\theta)$ and $r(\theta,t,\xi,y_+(\xi,t,u_0))$ intersect for some $\theta \in (0,s)$ which contradicts NIC.

Suppose $r_+(s) < R_-(s,\alpha,u_0)$, then for $x \in (r_+(s), R_-(s,\alpha,u_0))$, $y_-(x,s,u_0) < \alpha \leq r_+(0) = y_+$ and therefore the characteristic at (x,s) with end point $(y_-(x,s,u_0), 0)$ intersects $r_+(\theta)$ for some $\theta \in (0,s)$ contradicting NIC. This proves the claim.

From (2.10) and the claim, we have

$$R_- + \left(\frac{R_- - y_-}{t}\right)(s-t) \leq R_-(s,\alpha,u_0) \leq R_- + \left(\frac{R_- - y_+}{t}\right)(s-t)$$

that is

$$\begin{aligned} |R_- - R_-(s,\alpha,u_0)| &\leq \left(\left|\frac{R_- - y_-}{t}\right| + \left|\frac{R_- - y_+}{t}\right|\right)|s-t| \\ &\leq 2M|s-t|. \end{aligned}$$

Also from (2.10) and (2.25), we have $|R_- - y_\pm| = |R_- - r_\pm(0)| \leq Mt$, hence $\lim\limits_{t\to 0} R_-(t,\alpha,u_0) = \alpha$. Similarly for $R_+(t,\alpha,u_0)$.

From (c) of (3) in theorem 2.1, we have $r_\pm|_{[s,t]} \in ch(R_-(t,\alpha,u_0),s,t,u_0)$, hence from NIC and from the above claim we have for any $x < R_-(t,\alpha,u_0) < z$, $y_+(x,s,t,u_0) \leq r_-(s) \leq R_-(s,\alpha,u_0) \leq r_+(s) \leq y_-(z,s,t,u_0)$. Therefore from the definitions it follows that $R_-(t,\alpha,u_0)) = R_-(t,s,R_-(t,s,u_0),u_0)$. Similarly for R_+ and this proves (4).

(5). From (5) of theorem (2.1), for $t > 0$, a.e. x, $u(x,t) \leq w(x,t)$. Let $y_{1,\pm}(x) = y_\pm(x,t,u_0)$, $y_{2,\pm}(x) = y_\pm(x,t,w_0)$. Choose a dense set $D \subset I\!R$ such that for $i = 1,2, x \in D, u(x,t) \leq v(x,t), y_{i,+}(x) = y_{i,-}(x)$. Hence from (2.12) we have for $x \in D$,

$$\frac{x - y_{1,\pm}(x)}{t} = f'(u(x,t)) \leq f'(w(x,t)) = \frac{x - y_{2,\pm}(x)}{t}.$$

This implies $y_{2,\pm}(x) \leq y_{1,\pm}(x)$. Therefore from (2.22) and (2.23),

$$\begin{aligned} R_-(t,\alpha,u_0) &= \inf\{x \in D : y_{1,-}(x) \geq \alpha\} \\ &\leq \inf\{x \in D : y_{2,-}(x) \geq \alpha\} \\ &= R_-(t,\alpha,w_0). \end{aligned}$$

$$
\begin{aligned}
R_+(t,\alpha,u_0) &= \sup\{x \in D, y_{1,+}(x) \le \alpha\} \\
&\le \sup\{x \in D : y_{2,+}(x) \le \alpha\} \\
&= R_+(t,\alpha,w_0).
\end{aligned}
$$

This proves (2.30).

$$
\begin{aligned}
R_-(t,\alpha,u_0) &= \inf\{x : y_-(x,t,u_0) \ge \alpha\} \\
&\le \inf\{x : y_-(x,t,u_0) \ge \beta\} \\
&= R_+(t,\beta,u_0),
\end{aligned}
$$

and similarly for R_+. This proves (5).

(6). Let $y^k_\pm(x) = y_\pm(x,t,u^k_0), R^k_\pm = R_\pm(t,\alpha_k,u^k_0)$. Since $\{y^k_\pm\}$ are monotone functions and $\{R^k_\pm\}$ are bounded, hence from Helly's theorem and L^1_{loc} - contractivity, there exist a subsequence still denoted by k such that for $a.e.$ x,

$$
\begin{aligned}
\lim_{k\to\infty} u_k(x,t) &= u(x,t) &(2.36)\\
\lim_{k\to\infty} y^k_\pm(x) &= y_\pm(x) &(2.37)\\
\left(\underline{\lim}_{k\to\infty} R^k_\pm, \bar{\lim}_{k+\infty} R^k_\pm\right) &= \left(\bar{R}_\pm, \tilde{R}_\pm\right), &(2.38)
\end{aligned}
$$

where u is the solution of (1.2) with $u(x,0) = u_0(x)$. Let $D \subset I\!R$ be a dense set such that for all $x \in D$, (2.36) to (2.38) holds and for all k.

$$
\begin{aligned}
y^k_+(x) &= y^k_-(x) &(2.39)\\
y_+(x,t,u_0) &= y_-(x,t,u_0) &(2.40)\\
f'(u_k(x,t)) &= \frac{x - y^k_\pm(x)}{t} &(2.41)\\
f'(u(x,t)) &= \frac{x - y_\pm(x,t,u_0)}{t}. &(2.42)
\end{aligned}
$$

Hence from (2.37) , (2.41) and (2.42), for $x \in D$,

$$
y_\pm(x) = \lim_{k\to\infty} y^k_\pm(x) = y_\pm(x,t,u_0). \tag{2.43}
$$

Case (i): Let for all $k, R^k_- \le R_-(t,\alpha,u_0)$, then $\bar{R}_- \le R_-(t,\alpha,u_0)$. Suppose $\bar{R}_- < R_-(t,\alpha,u_0)$. Let $I = (\bar{R}_-, R_-(t,\alpha,u_0))$, $x \in D \cap I$ and choose $k_0 = k_0(x) > 0$ such that for all $k \ge k_0, R^k_- < x$, then

$$
\alpha = \lim_{k\to\infty} \alpha_k \le \lim_{k\to\infty} y^k_-(x) = y_-(x,t,u_0) < \alpha,
$$

which is a contradiction. Hence $\bar{R}_- = R_-(t,\alpha,u_0)$.

Case (ii) : Let for all k, $R_-(t,\alpha,u_0) \le R_-^k$, then $R_-(t,\alpha,u_0) \le \tilde{R}_-$. Suppose $R_-(t,\alpha,u_0) < \tilde{R}_-$, then for $x \in D \cap (R_-(t,\alpha,u_0), \tilde{R}_-)$ choose $k_0 = k_0(x)$ such that for $k > k_0, x < R_-^k$. Hence $\alpha \le y_-(x,t,u_0) = \lim_{k\to\infty} y_-^k(x) \le \alpha$ and therefore $y_-(x,t,u_0) = \alpha$. Therefore from (2.12), $f'(u(x,t)) = \frac{x-\alpha}{t}$. Similarly for R_+ and this proves (6) and hence the Lemma.

Next we study the characterization of $R_\pm$ and some comparison properties. For this we need some well known results which will be proved in the following Lemma.

Let $\Omega \subset I\!R^n$ be an open set and $B(1)$ denote the unit ball in $I\!R^n$. Let $0 \le \chi \in C_c^\infty(B(1))$ with $\int_{I\!R^n} \chi(x)dx = 1$. Let $\varepsilon > 0$ and denote $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi(\frac{x}{\varepsilon})$ be the usual mollifiers. Let $u_0 \in L^1_{loc}(I\!R^n)$ and define

$$u_0^\varepsilon(x) = (\chi_\varepsilon * u_0)(x) = \int_{B(1)} \chi(y)u_0(x-\varepsilon y)dy \ ,$$

then

Lemma 2.3 *Denote ess inf and ess sup by inf and sup. Then*

1. *With the above notation, for $x \in \Omega$, there exists a $\varepsilon_0 = \varepsilon_0(x) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*
$$\inf_{y\in\Omega} u_0(y) \le u_0^\varepsilon(x) \le \sup_{y\in\Omega} u_0(y). \tag{2.44}$$

2. *Let $t_0, \varepsilon_0, \alpha \in I\!R$ and $\omega \in L^\infty((0,t_0))$. Let $R : (0,t_0] \to I\!R$ be a locally Lipschitz continuous function such that for a.e $t \in (0,t_0)$,*
$$\omega(t) \ \ge \ (f')^{-1}\left(\frac{R(t)-\alpha}{t}\right) + \varepsilon_0 \tag{2.45}$$
$$\frac{dR}{dt} \ = \ \frac{f(\omega(t)) - f((f')^{-1}(\frac{R(t)-\alpha}{t}))}{\omega(t) - (f')^{-1}(\frac{R(t)-\alpha}{t})}, \tag{2.46}$$
then
$$\overline{\lim_{t\to 0}} \left|\frac{R(t)-\alpha}{t}\right| = \infty. \tag{2.47}$$

Proof. (1). Let $\Omega_\varepsilon = \{x; d(x,\Omega^c) > \varepsilon\}$. Then for $x \in \Omega$, there exists an $\varepsilon_0 > 0$, such that $x \in \Omega_\varepsilon$, for all $\varepsilon < \varepsilon_0$. Hence $x - \varepsilon y \in \Omega$, for all $y \in B(1)$ and

$$\inf_{y\in\Omega} u_0(y) \le u_0(x-\varepsilon y) \le \sup_{y\in\Omega} u_0(y),$$

multiply this identity by χ and integrate over $B(1)$ gives (2.44).

(2). Suppose (2.47) is not true. That is

$$\sup_{t>0}\left|\frac{R(t)-\alpha}{t}\right|<\infty. \tag{2.48}$$

Let m be defined by

$$m=\inf_{t\in(0,t_0)}\int_0^1 f''\left((f')^{-1}\left(\frac{R(t)-\alpha}{t}\right)+\theta(\omega(t)-(f')^{-1}\left(\frac{R(t)-\alpha}{t}\right)\right)(1-\theta)d\theta.$$

Claim: $m>0$.
Suppose $m=0$, then there exists a sequence $t_k\to\tilde{t}$ in $[0,t_0]$ such that

$$\lim_{k\to\infty}\int_0^1 f'\left((f')^{-1}\left(\frac{R(t_k)-\alpha}{t_k}\right)+\theta\left(\omega(t_k)-(f')^{-1}\left(\frac{R(t_k)-\alpha}{t_k}\right)\right)\right)(1-\theta)d\theta=0.$$

Since $\omega\in L^\infty((0,t_0))$ and from (2.48), choose a subsequence still denoted by $\{t_k\}$ such that $\omega(t_k)\to a, (f')^{-1}\left(\frac{R(t_k)-\alpha}{t_k}\right)\to b$ as $k\to\infty$. Hence

$$\int_0^1 f''(b+\theta(a-b))d\theta=0.$$

Since f is convex and hence $f''(a+\theta(a-b))=0,$ for $\theta\in[0,1]$. From (2.45), $a-b\geq\varepsilon_0$ and hence $f(b+\theta(a-b))$ is linear in θ, which contradicts the strict convexity of f. This proves the claim.

From Taylor's series and the claim we have

$$\begin{aligned}\frac{dR}{dt} &= \frac{R(t)-\alpha}{t}\\ &+ \int_0^1 f''\left((f')^{-1}\left(\frac{R(t)-\alpha}{t}\right)+\theta\left(\omega(t)-(f')^{-1}\left(\frac{R(t)-\alpha}{t}\right)\right)\right)(1-\theta)d\theta\\ &\times \left(\omega(t)-(f')^{-1}\left(\frac{R(t)-\alpha}{t}\right)\right)\\ &\geq \frac{R(t)-\alpha}{t}+\frac{m\epsilon_0}{2},\end{aligned}$$

or

$$t\frac{d}{dt}\left(\frac{R(t)-\alpha}{t}\right)\geq\frac{m\epsilon_0}{2}.$$

For $0<t_1<t_0$, integrating from t to t_1, to obtain

$$\frac{R(t)-\alpha}{t}\leq\frac{R(t_1)-\alpha}{t_1}-\frac{m\epsilon_0}{2}\log\frac{t_1}{t}\to-\infty\quad\text{as } t\to 0,$$

which contradicts (2.48). This proves (2) and hence the Lemma.

Lemma 2.4 *Let $T > 0, \alpha, \beta \in I\!R, u_0, v_0$ and v be as in (2.3) and (2.4). Then*

(1). Let $x_0 \in I\!R, t > 0$ such that

$$y_-(x_0, t, u_0) \le \alpha \le y_+(x_0, t, u_0), \tag{2.49}$$

then

(i). if $x_0 \le R_-(t, \alpha, u_0)$, then $x_0 = R_-(t, \alpha, u_0)$. If $R_-(t, \alpha, u_0) < x_0$, then for all $x \in (R_-(t, \alpha, u_0), x_0), f'(u(x,t)) = \frac{x-\alpha}{t}$.

(ii). if $x_0 \ge R_+(t, \alpha, u_0)$, then $x_0 = R_+(t, \alpha, u_0)$. If $x_0 < R_+(t, \alpha, u_0)$, then for all $x \in (x_0, R_+(t, \alpha, u_0)), f'(u(x,t)) = \frac{x-\alpha}{t}$.

(2). (i). Let $x \ge R_-(t, \alpha, u_0)$, then

$$v(x,t) = \inf_{y \ge \alpha} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\}. \tag{2.50}$$

(ii). Let $x \le R_+(t, \alpha, u_0)$, then

$$v(x,t) = \inf_{y \le \alpha} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\}. \tag{2.51}$$

(iii). Let $\alpha < \beta$ and for $0 < t < T$ assume that

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0),$$

then for $R_+(t, \alpha, u_0) < x < R_-(t, \beta, u_0)$,

$$v(x,t) = \inf_{\alpha \le y \le \beta} \left\{ v_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\}, \tag{2.52}$$

$$m = \inf_{y \in [\alpha, \beta]} u_0(y) \le u(x,t) \le \sup_{y \in [\alpha, \beta]} u_0(y) = M. \tag{2.53}$$

$$f'(m) \le \frac{x - y_+(x, t, u_0)}{t} \le f'(M). \tag{2.54}$$

(3). Let $L(t, \alpha, u_0) \in \{R_\pm(t, \alpha, u_0)\}, R(t, \beta, u_0) \in \{R_\pm(t, \beta, u_0)\}$. Suppose at $t = T$,

$$L(T, \alpha, u_0) = R(T, \beta, u_0), \tag{2.55}$$

then for all $t \geq T$, *(see Figure 10).*

$$L(t,\alpha,u_0) = R(t,\beta,u_0). \tag{2.56}$$

Furthermore, let $\{u_0^k\}$ *are in* $L^\infty(I\!R)$ *with* $\sup_k \|u_0^k\|_\infty < \infty$. *Let* $(\alpha_k,\beta_k,u_0^k) \to (\alpha,\beta,u_0)$ *as* $k\to\infty$ *in* $I\!R^2 \times L^1_{loc}(I\!R)$ *and* $T_k \to T$ *in* $I\!R$ *such that*

$$\begin{array}{rcl} R_-(T,\alpha,u_0) &=& R_+(T,\beta,u_0) \\ R_-(T_k,\alpha_k,u_0^k) &=& R_+(T_k,\beta_k,u_0^k). \end{array} \tag{2.57}$$

Then for $t > T$,

$$\begin{array}{rcl} \lim\limits_{k\to\infty} R_+(t,\alpha_k,u_0^k) &=& \lim\limits_{k\to\infty} R_-(t,\beta_k,u_0^k) \\ &=& R_+(t,\alpha,u_0) \\ &=& R_-(t,\alpha,u_0). \end{array} \tag{2.58}$$

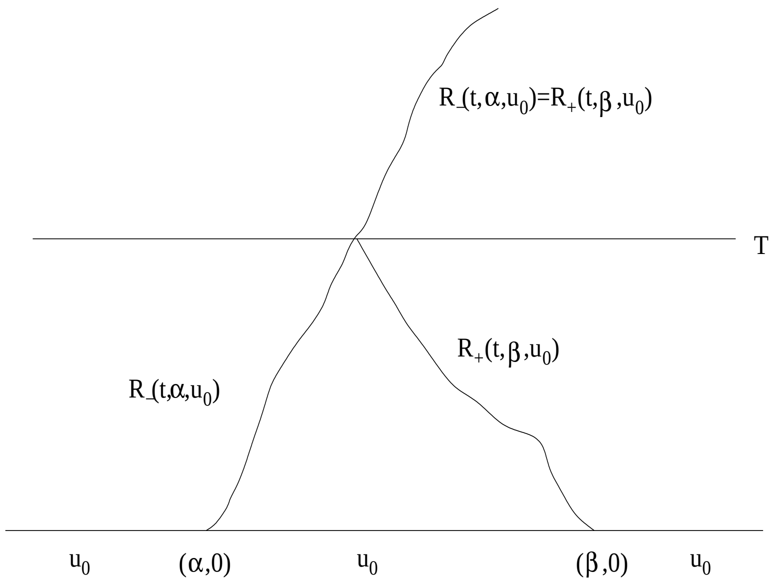

Figure 10:

Proof. (1). It is enough to prove (i) and (ii) follows similarly. Let $C = R_-(t,\alpha,u_0)$, then from (2.22) $y_-(C,t,u_0) \leq \alpha$. Suppose $x_0 < C$, then from (2.49), the characteristic line joining $(C,t),(y_-(C,t,u_0),0)$ and $(x_0,t),(y_+(x_0,t,u_0),0)$ intersect if $y_+(x_0,t,u_0) > \alpha$ or $y_-(C,t,u_0) < \alpha$, which contradicts NIC. Hence $y_+(x_0,t,u_0) = y_-(C,t,u_0) = \alpha$. Therefore from (2.21), for $x_0 < x < C, f'(u(x,t)) = \frac{x-\alpha}{t}$. This implies that $C = R_-(t,\alpha,u_0) < x < C$, which is a contradiction. Hence $x_0 = R_-(t,\alpha,u_0)$. Suppose $C < x_0$, then from the definition and (2.49), we have $y_-(x_0,t,u_0)$

$\leq \alpha \leq y_-(x_0, t, u_0)$ and hence $y_-(x_0, t, u_0) = \alpha$ and from (2.21), $f'(u(x,t)) = \frac{x-\alpha}{t}$ for all $C < x < x_0$. This proves (1).

(2). It is enough to prove (i) and (ii) follows similarly. Let $x \geq R_-(t, \alpha, u_0)$, then from (2.25), $y_+(x, t, u_0) \geq \alpha$. Hence

$$\begin{aligned} \inf\left\{\inf_{y\geq\alpha}\{v_0(y) + tf^*\left(\frac{x-y}{t}\right)\}, \inf_{y<\alpha}\{v_0(y) + tf^*\left(\frac{x-y}{t}\right)\}\right\} & \\ = \; & v(x,t) \\ = \; & v_0(y_+(x,t,u_0)) + tf^*\left(\frac{x - y_+(x,t,u_0)}{t}\right). \end{aligned}$$

Hence

$$v(x,t) = \inf_{y\geq\alpha}\{v_0(y) + tf^*\left(\frac{x-y}{t}\right)\}.$$

(iii). (2.52) follows from (2.50) and (2.51). Let $\varepsilon > 0$, $u_0^\varepsilon = \chi_\varepsilon * u_0$ and $v_0^\varepsilon, v_\varepsilon$ be as in (2.3) , (2.4) respectively. Let $u^\epsilon = \frac{\partial v^\epsilon}{\partial x}$ be the solution of (1.2) in $\Omega = I\!R \times I\!R_+$. Since v_0^ϵ is differentiable and hence for a.e x and from (2.13), $u^\varepsilon(x,t) = u_0^\varepsilon(y_+(x,t,u_0))$. Since $u_0^\varepsilon \to u_0$ in L^1_{loc} and hence $u^\varepsilon \to u$ in L^1_{loc}. Therefore from (2.32) to (2.35), we have for $0 < t < T$,

$$\overline{\lim_{\varepsilon\to 0}} R_+(t,\alpha,u_0^\varepsilon) \leq R_+(t,\alpha,u_0) < R_-(t,\beta,u_0) \leq \underline{\lim_{\varepsilon\to 0}} R_-(t,\beta,u_0^\epsilon).$$

Let $\varepsilon_k \to 0$ and choose a dense set $D \subset (R_+(t,\alpha,u_0), R_-(t,\beta,u_0))$ such that for all $x \in D$,

$$\begin{aligned} \lim_{k\to\infty} u^{\varepsilon_k}(x,t) &= u(x,t) \\ y(x) &= y_+(x,t,u_0) = y_-(x,t,u_0) \\ y_k(x) &= y_+(x,t,u_0^{\varepsilon_k}) = y_-(x,t,u_0^{\varepsilon_k}). \end{aligned}$$

For $x \in D$, choose $k_0(x)$ such that for all $k \geq k_0(x), x \in (R_+(t,\alpha,u_0^{\varepsilon_k}), R_-(t,\beta,u_0^{\epsilon_k}))$. Then from (4.52), $y_k \in [\alpha,\beta]$. Since $u^{\varepsilon_k}(x,t) = u_0^{\varepsilon_k}(y_k(x))$, hence from (2.44),

$$m \leq u_0^{\varepsilon_k}(y_k(x)) = u^{\varepsilon_k}(x,t) \leq M.$$

Letting $k \to \infty$ to obtain (2.53). From (2.12),

$$f'(u^{\varepsilon_k}(x,t)) = \frac{x - y_k(x)}{t},$$

letting $k \to \infty$ to obtain

$$\begin{aligned} \frac{x - y(x)}{t} &= f'(u(x,t)) = \lim_{k\to\infty} f'(u^{\varepsilon_k}(x,t)) \\ &= \lim_{k\to\infty} \frac{x - y_k(x)}{t}. \end{aligned}$$

Hence $\lim_{k\to\infty} y_k(x) = y(x)$,

$$f'(m) \le f'(u_0^{\varepsilon_k}(y_k(x))) = \frac{x - y_k(x)}{t} \le f'(M),$$

Now letting $k \to \infty$ to obtain

$$f'(m) \le \frac{x - y(x)}{t} \le f'(M),$$

For $x \notin D$, choose $x_k \uparrow x,\ y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$. Then from (2.22), $y_+(x_k, t, u_0) \to y_-(x, t, u_0)$. Now apply the inequalities for x_k and let $k \to \infty$ to obtain (2.53), (2.54). This proves (2).

(3). Without loss of generality we can take $L(t, \alpha, u_0) = R_-(t, \alpha, u_0)$ and $R(t, \beta, u_0) = R_+(t, \beta, u_0)$. Similar proof follows in all other cases. Let $C = R_-(T, \alpha, u_0) = R_+(T, \beta, u_0)$ and $t > T$. Then from (2.27) and (2.29) we have

$$\begin{aligned} R_-(t, \alpha, u_0) &= R_-(t, T, C, u_0) \\ &= R_+(t, T, C, u_0) \\ &= R_+(t, \beta, u_0). \end{aligned} \tag{2.59}$$

This proves (2.56).

Let $t > T$, then choose $k_0 = k_0(t)$ such that for all $k > k_0, t > T_k$. Then from (2.56) we have

$$R_k(t) = R_-(t, \alpha_k, u_0^k) = R_+(t, \alpha_k, u_0^k),$$
$$R(t) = R_-(t, \alpha, u_0) = R_+(t, \alpha, u_0).$$

Hence from (6) of Lemma (2.2),

$$\overline{\lim_{k\to\infty}}\, R_k(t) \le R_+(t, \alpha, u_0) = R_-(t, \alpha, u_0) \le \underline{\lim_{k\to\infty}}\, R_k(t). \tag{2.60}$$

This proves (2.58) and hence the Lemma.

Next we give a criteria under which $R_+ = R_-$. Let $\beta < \gamma$ and $I_1 = [\beta, \gamma]$, Define

$$m = \inf_{y\in I_1} u_0(y),\ \ M = \sup_{y\in I_1} u_0(y),\ \ I_2 = [f'(m), f'(M)].$$

Let

$$a_0 = \max\{f^*(q) - Mq; q \in I_2\},\ \ f'(q_0) = \max\{q; f^*(q) - Mq \le a_0\}.$$

Then we have the following.

Lemma 2.5 *Let $\alpha < \beta < \gamma, \varepsilon_0 > 0$. Let $u_0 \in L^\infty(I\!R)$ such that*

$$\inf_{[\alpha,\beta]} u_0(y) \ge q_0 + \varepsilon_0, \tag{2.61}$$

then for all $t > 0$,

$$R_+(t, \beta, u_0) = R_-(t, \beta, u_0).$$

Proof. Suppose for some $T > 0$, $R_+(T, \beta, u_0) > R_-(T, \beta, u_0)$, then from (2.56) , for $t < T$,

$$R_-(t, \beta, u_0) < R_+(t, \beta, u_0)$$

and from (2.26) for $R_-(t, \beta, u_0) < x < R_+(t, \beta, u_0)$.

$$f'(u(x,t)) = \frac{x-\beta}{t}. \tag{2.62}$$

From (2.28) we can choose T sufficiently small such that for all $0 < t \le T$,

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0) < R_+(t, \beta, u_0) < R_-(t, \gamma, u_0). \tag{2.63}$$

Claim: For $t \le T$,

$$\frac{R_+(t, \beta, u_0) - \beta}{t} \le f'(q_0). \tag{2.64}$$

Let $x_k > R_+(t, \beta, u_0)$ be such that $y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$ and $\lim\limits_{k\to\infty} x_k = R_+(t, \beta, u_0)$. Then from (2.54)

$$f'(m) \le \frac{x - y_-(x_k, t, u_0)}{t} \le f'(M).$$

Letting $k \to \infty$ and from (2.23) we have

$$f'(m) \quad \le \quad \frac{R_+(t, \beta, u_0) - y_+(R_+(t, \beta, u_0), t, u_0)}{t} \le f'(M). \tag{2.65}$$

Let $v_0(y) = \int\limits_\beta^y u_0(\theta) d\theta$, hence $v_0(\beta) = 0$. Denote

$$R(t) = R_+(t, \beta, u_0),\ y_\pm(t) = y_\pm(R_+(t, \beta, u_0), t, u_0),$$

then from (2.62), $y_-(t) = \beta$ and from (2.4) we have

$$\begin{aligned}
tf^*\left(\frac{R(t)-\beta}{t}\right) &= v_0(y_-(t)) + tf^*\left(\frac{R(t)-y_-(t)}{t}\right)\\
&= v_0(y_+(t)) + tf^*\left(\frac{R(t)-y_+(t)}{t}\right)\\
&\le M(y_+(t)-\beta) + tf^*\left(\frac{R(t)-y_+(t)}{t}\right)\\
&\le M(y_+(t)-R(t)) + M(R(t)-\beta) + tf^*\left(\frac{R(t)-y_+(t)}{t}\right),
\end{aligned}$$

and hence

$$f^*\left(\frac{R(t)-\beta}{t}\right) - M\left(\frac{R(t)-\beta}{t}\right) \le f^*\left(\frac{R(t)-y_+(t)}{t}\right) - M\left(\frac{R(t)-y_+(t)}{t}\right).$$

From (2.65) it follows that

$$f^*\left(\frac{R(t)-\beta}{t}\right) - M\left(\frac{R(t)-\beta}{t}\right) \le a_0,$$

and this proves (2.64) and hence the claim.

Let $L(t) = R_-(t,\beta,u_0)$, then letting x tends to $L(t)$ in (2.62) and (2.64) to obtain

$$f'(u(L(t)+,t)) = \frac{L(t)-\beta}{t} \le \frac{R_+(t,\beta,u_0)-\beta}{t} \le f'(q_0). \tag{2.66}$$

From (2.53), for $R_+(t,\alpha,u_0) < x < R_-(t,\beta,u_0) = L(t)$, $u(x,t) \ge \inf_{y\in[\alpha,\beta]} u_0(y)$, hence from (2.61) and (2.66), we have

$$\begin{aligned} u(L(t)-,t) &\ge \inf_{y\in[\alpha,\beta]} u_0(y) \ge q_0 + \varepsilon_0 \\ &\ge u(L(t)+,t) + \varepsilon_0 \\ &= f^{*\prime}\left(\tfrac{L(t)-\beta}{t}\right) + \varepsilon_0. \end{aligned} \tag{2.67}$$

From RH condition across $L(t)$ gives

$$\frac{dL}{dt} = \frac{f(u(L(t)-,t)) - f\left(f^{*\prime}\left(\frac{L(t)-\beta}{t}\right)\right)}{u(L(t)-,t) - f^{*\prime}\left(\frac{L(t)-\beta}{t}\right)}. \tag{2.68}$$

Therefore $L(t)$ satisfies the hypothesis (2) of Lemma (2.3) and hence from (2.47)

$$\overline{\lim_{t\to 0}}\left|\frac{L(t)-\beta}{t}\right| = \infty,$$

which contradicts the uniform Lipschitz continuity of L from (4) of Lemma 2.2. Hence $R_-(t,\beta,u_0) = R_+(t,\beta,u_0)$, for all t, and this proves the Lemma.

Remark 2.6 *Observe that q_0 entirely depends on the bounds of u_0 in $[\beta,\gamma]$.*

Lemma 2.7 *Let u be the solution of (1.2) with*

$$\bar{u}_0(x) = u(x,0) = \begin{cases} a & \text{if } x < \alpha, \\ u_0(x) & \text{if } x > \alpha. \end{cases}$$

Then for $x < R_-(t,\alpha,\bar{u}_0)$,

$$u(x,t) = a, \tag{2.69}$$

$$f'(a) = \frac{R_-(t,\alpha,\bar{u}_0) - y_-(R_-(t,\alpha,\bar{u}_0),t,\bar{u}_0)}{t}. \tag{2.70}$$

Proof. Since $\bar{v}_0(x) = \int\limits_{\alpha}^{x} \bar{u}_0(\theta)d\theta$ is differentiable for $x < \alpha$ and hence from (2.13), for a.e, $x < \alpha, u(x,t) = \bar{u}_0(y_+(x,t,\bar{u}_0)) = a$ and

$$f'(a) = \frac{x - y_-(x,t,\bar{u}_0)}{t}.$$

From (2.22) and letting $x \uparrow R_-(t,\alpha,\bar{u}_0)$ to obtain (2.70). This proves the Lemma.

Next we consider the following initial value problem taking three values. Let $a, \lambda, m \in I\!R, \alpha < \beta$ and consider

$$u_0^\lambda(x) = \begin{cases} a & \text{if} \quad x < \alpha, \\ \lambda & \text{if} \quad \alpha < x < \beta, \\ m & \text{if} \quad x > \beta. \end{cases} \tag{2.71}$$

and denote

$$v_0^\lambda(x) = \int\limits_{\beta}^{x} u_0^\lambda(\theta)d\theta, \tag{2.72}$$

and v^λ be as in (2.4). Let $u^\lambda = \frac{\partial v^\lambda}{\partial x}$ be the entropy solution of (1.2) in $\Omega = I\!R \times I\!R_+$ with initial data u_0^λ. Assume that

$$\lambda > \max(a,m), \tag{2.73}$$

then α is a point of rarefaction and β is the shock point.

Let

$$\begin{aligned} L_1(t) &= \alpha + f'(a)t, \\ L_2^\lambda(t) &= \alpha + f'(\lambda)t, \\ S^\lambda(t) &= \beta + \left(\tfrac{f(\lambda)-f(m)}{\lambda - m}\right)t. \end{aligned}$$

Let $(x_0(\lambda), T_0(\lambda))$ be the point of intersection of L_2^λ and S^λ given by

$$T_0(\lambda) = \frac{\beta - \alpha}{f'(\lambda) - \left(\frac{f(\lambda)-f(m)}{\lambda-m}\right)},$$

$$x_0(\lambda) = \alpha + \frac{(\beta-\alpha)f'(\lambda)}{f'(\lambda) - \left(\frac{f'(\lambda)-f(m)}{\lambda-m}\right)}.$$

Since β is the point of shock and hence from (2.26) we have

$$R^\lambda(t) = R_+(t,\beta,u_0^\lambda) = R_-(t,\beta,u_0^\lambda). \tag{2.74}$$

Then the solution u^λ for $t \leq T_0(\lambda)$ is given by

$$R^\lambda(t) = S^\lambda(t). \tag{2.75}$$

$$u^\lambda(x,t) = \begin{cases} m & \text{if } \; x > S^\lambda(t), \\ \lambda & \text{if } \; L_2^\lambda(t) < x < S^\lambda(t), \\ (f')^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } \; L_1(t) < x < L_2^\lambda(t), \\ a & \text{if } \; x < L_1(t). \end{cases} \tag{2.76}$$

Define $T_1(\lambda) > T_0(\lambda)$ be the first point of intersection of L_2^λ and R^λ. If they do not meet, then define $T_1(\lambda) = \infty$. Next Lemma describes the behavior of u^λ for $t > T_0(\lambda)$.

Lemma 2.8 *Let λ satisfies (2.73). Then u^λ is given by (see Figure 11).*

(i). For $T_0(\lambda) < t < T_1(\lambda), y_\pm(L_1(t), t, u_0^\lambda) = a$ and

$$u^\lambda(x,t) = \begin{cases} m & \text{if } \; x > R^\lambda(t), \\ {f'}^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } \; L_1(t) < x < R^\lambda(t), \\ a & \text{if } \; x < L_1(t). \end{cases} \tag{2.77}$$

(ii). $t > T_1(\lambda)$, then u^λ is the solution of (1.2) with initial data

$$u^\lambda(x, T_1(\lambda)) = \begin{cases} a & \text{if } \; x < R^\lambda(T_1(\lambda)), \\ m & \text{if } \; x > R^\lambda(T_1(\lambda)). \end{cases} \tag{2.78}$$

Furthermore for any compact sets K_1 and K_2 of $\mathbb{R}$ with $K = K_1 \times K_2, \eta > 0, T \leq T_1(\lambda)$ be bounded, then

$$\lim_{\lambda\to\infty} \inf_{(a,m)\in K} T_1(\lambda) = \infty, \tag{2.79}$$

$$f'(u^\lambda(R^\lambda(t)-,t) = \begin{cases} f'(\lambda) & \text{if } \; 0 < t < T_0(\lambda), \\ \frac{R^\lambda(t)-\alpha}{t} & \text{if } \; T_0(\lambda) < t < T_1(\lambda). \end{cases} \tag{2.80}$$

$$\lim_{\lambda\to\infty} \inf_{\substack{(a,m)\in K \\ T_0(\lambda)\leq t\leq T}} u^\lambda(R^\lambda(t)-,t) = \infty. \tag{2.81}$$

$$\lim_{\lambda\to\infty} \inf_{\eta\leq t\leq T} R^\lambda(t) = \infty, \tag{2.82}$$

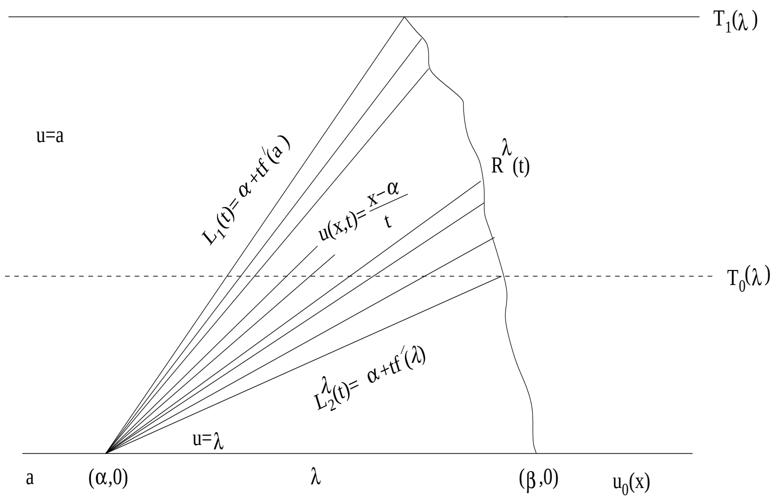

Figure 11:

Proof. Let $T_0(\lambda) < t \leq T_1(\lambda)$. Since $v_0^\lambda(x)$ is differentiable for $x > \beta$ and hence from (2.13) and (2.50), $u^\lambda(x,t) = u_0(y_+(x,t,u_0^\lambda)) = m$ if $x > R_+(t,\beta,u_0^\lambda) = R^\lambda(t)$.

Let $L_1(t) < x < R^\lambda(t)$, then $y_+(x,t,u_0^\lambda) < \beta$. Suppose for some $x_0 \in (L_1(t), R^\lambda(t)), y_-(x,t,u_0^\lambda) < \alpha$. Then for all $x \in (L_1(t), x_0),\ y_-(x,t,u_0^\lambda)) < \alpha,\ u^\lambda(x,t) = u_0(y_-(x,t,u_0^\lambda)) = a$ and

$$\frac{L_1(t)-\alpha}{t} = f'(a) = f'(u^\lambda(x,t)) = \frac{x - y_-(x,t,u_0^\lambda)}{t} > \frac{L_1(t)-\alpha}{t},$$

which is a contradiction. Suppose $y_+(x_0,t,u_0^\lambda) > \alpha$, then for all $x_0 < x < R^\lambda(t), \alpha < y_+(x,t,u_0^\lambda) < \beta$. Since u_0^λ is differentiable in (α,β) and hence from (2.13), for a.e $x \in (x_0, R^\lambda(t))$,

$$u^\lambda(x,t) = u_0^\lambda(y_+(x,t,u_0^\lambda)) = \lambda, f'(\lambda) = f'(u^\lambda(x,t)) = \frac{x - y_+(x,t,u_0^\lambda)}{t}.$$

Suppose $x_0 < L_2^\lambda(t)$, then for $x_0 < x < \min(L_2^\lambda(t), R^\lambda(t))$,

$$f'(\lambda) = \frac{x - y_+(x,t,u_0^\lambda)}{t} < \frac{L_2^\lambda(t)-\alpha}{t} = f'(\lambda),$$

which is a contradiction. Suppose $L_2^\lambda(t) < x_0 < R^\lambda(t)$, then for $x \in (x_0, R^\lambda(t))$, characteristic γ at (x,t) given by $\gamma(\theta) = y_+(x,t,u_0^\lambda) + f'(\lambda)\theta$ intersects S^λ at t_0, where

$$t_0 = \frac{\beta - y_+(x,t,u_0^\lambda)}{f'(\lambda) - \frac{f(\lambda)-f(m)}{\lambda-m}} < \frac{\beta-\alpha}{f'(\lambda) - \frac{f(\lambda)-f(m)}{\lambda-m}} = T_0(\lambda),$$

which contradicts NIP, since $S^\lambda(t)$ is a characteristic for $0 < t < T_0(\lambda)$. Hence for $L_1(t) < x < R^\lambda(t),\ y_+(x,t,u_0^\lambda) = y_-(x,t,u_0^\lambda) = \alpha$ and from (2.21), we have

$$f'(u^\lambda(x,t)) = \frac{x-\alpha}{t}. \tag{2.83}$$

Now letting $x \downarrow L_1(t)$ and from (2.22) to obtain $y_+(L_1(t), t, u_0^\lambda) = \alpha$ and $f'(u^\lambda(L_1(t) +, t)) = \frac{L_1(t)-\alpha}{t} = f'(a)$. This implies $u^\lambda(L_1(t)+, t) = a$. From RH condition across $L_1(t)$ implies that $u^\lambda(L_1(t)-, t) = a$. Therefore from (2.12),(2.22),(2.23) $y_\pm(L_1(t), t, u_0^\lambda) = a$. This implies for $x < L_1(t), y_+(x, t, u_0^\lambda) < \alpha$ and hence from (2.13), $u^\lambda(x, t) = u_0(y_+(x, t, u_0^\lambda)) = a$. This proves (2.77) and hence (2.78).

Let

$$\begin{aligned} y_\pm(t, \lambda) &= y_\pm(R^\lambda(t), t, u_0^\lambda), \\ y_\pm(\lambda) &= y_\pm(R^\lambda(T_1(\lambda)), T_1(\lambda), u_0^\lambda), \\ R^\lambda &= R^\lambda(T_1(\lambda)). \end{aligned}$$

Let $T_0(\lambda) < t \le T_1(\lambda)$ and letting $x \uparrow R^\lambda(t)$ in (2.83) to obtain

$$\frac{R^\lambda(t) - y_-(t, \lambda)}{t} = f'(u^\lambda(R^\lambda(t)-, t)) = \frac{R^\lambda(t) - \alpha}{t}. \tag{2.84}$$

Hence $y_-(t, \lambda) = \alpha$. Also at $t = T_1(\lambda)$,

$$f'(a) = \frac{R^\lambda - \alpha}{T_1(\lambda)} = \frac{R^\lambda - y_-}{T_1(\lambda)}. \tag{2.85}$$

$$\begin{aligned} \frac{R^\lambda(t) - y_+(t, \lambda)}{t} &= \lim_{x \downarrow R^\lambda(t)} \frac{x - y_-(x, t, u_0^\lambda)}{t} && (2.86) \\ &= \lim_{x \downarrow R^\lambda(t)} f'(u^\lambda(x, t)) = f'(m). && (2.87) \end{aligned}$$

From (2.84) to (2.86) we can evaluate $v^\lambda(R^\lambda(t), t)$ by

$$\begin{aligned} -(\beta - \alpha)\lambda + t f^*\left(\frac{R^\lambda(t) - \alpha}{t}\right) &= (y_+(t, \lambda) - \beta)m + t f^*\left(\frac{R^\lambda(t) - y_+(t, \lambda)}{t}\right) \\ -(\beta - \alpha)\lambda + t f^*\left(\frac{R^\lambda(t) - \alpha}{t}\right) &= m\left(\frac{y_+(t, \lambda) - R^\lambda(t)}{t} t + R^\lambda(t) - \beta\right) \\ &+ t f^*(f'(m)) \\ &= -t m f'(m) + m(R^\lambda(t) - \alpha) + m(\alpha - \beta) \\ &+ t f^*(f'(m)). \end{aligned}$$

$$\frac{(\beta - \alpha)(\lambda - m)}{t} = f^*\left(\frac{R^\lambda(t) - \alpha}{t}\right) - \frac{R^\lambda(t) - \alpha}{t} - f^*(f'(m) + m f'(m). \tag{2.88}$$

Let $t = T_1(\lambda)$ then $\frac{R^\lambda - \alpha}{T_1(\lambda)} = f'(a)$ and hence the right hand side of (2.88) is bounded uniformly for $(a, m) \in K$ and hence as $\lambda \to \infty, T_1(\lambda) \to \infty$. This proves (2.79).

Observe that $R_+(t, \alpha, u_0^\lambda) = L_2^\lambda(t)$ and $L_2^\lambda(t) < R^\lambda(t)$ for $0 < t < T_0(\lambda)$. Hence for a.e $x \in (L_2^\lambda(t), R^\lambda(t)), y_+(x, t, u_0^\lambda) = y_-(x, t, u_0^\lambda) \in (\alpha, \beta)$ and from (2.13),

$$u^\lambda(x, t) = u_0^\lambda(y_+(x, t, u_0^\lambda)) = \lambda.$$

From this and (2.84) , (2.80) follows. Let $T_0(\lambda) < t \leq T$, then from superlinearity of f^*, (2.81) follows from (2.84),(2.88). Suppose $\lim\limits_{\lambda\to\infty} T_0(\lambda) = 0$, and then (2.82) follows from (2.80) , (2.81). Hence assume that $\varliminf\limits_{\lambda\to\infty} T_0(\lambda) > 0$, then if $\eta < T_0(\lambda)$, then (2.82) follows from (2.88). This proves the Lemma.

Next we generalize the above Lemma by replacing m by u_0. More precisely let

$$u_0^\lambda(x) = \begin{cases} a & \text{if} \quad x < \alpha, \\ \lambda & \text{if} \quad \alpha < x < \beta, \\ u_0(x) & \text{if} \quad x > \beta, \end{cases} \tag{2.89}$$

and u^λ be the solution of (1.2) with initial data u_0^λ. Let

$$m_1 = \inf_{x\geq\alpha} u_0(x), m_2 = \sup_{x\geq\alpha} u_0(x). \tag{2.90}$$

For $i = 1, 2$, define $u_0^{i,\lambda}$ by

$$u_0^{i,\lambda}(x) = \begin{cases} a & \text{if} \quad x < \alpha, \\ \lambda & \text{if} \quad \alpha < x < \beta, \\ m_i & \text{if} \quad \beta < x, \end{cases} \tag{2.91}$$

and let u_i^λ be the solution of (1.2) with intial data $u_0^{i,\lambda}$. Let $L_1(t), L_2^\lambda(t)$ be as in Lemma (2.8), then

Lemma 2.9 *Let $T > 0$ be fixed, then there exist $\lambda_0 = \lambda_0(m_1, m_2, a, t)$ such that for $\lambda \geq \lambda_0, 0 < t \leq T$,*

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda). \tag{2.92}$$

and denote $R(\lambda, t) = R_-(t, \beta, u_0^\lambda)$, then

(i). $t \to R(\lambda, t)$ is a strictly increasing function.

(ii). $\lambda \to R(\lambda, t)$ is a strictly increasing function and

$$\lim_{\lambda\to\infty} R(\lambda, T) = \infty. \tag{2.93}$$

Let $T_0(\lambda)$ be the first point of intersection of $L_2^\lambda(t)$ and $R(\lambda, t)$. Then

$$u^\lambda(x, t) = \begin{cases} a & \textit{if} \quad x < L_1(t) \\ \lambda & \textit{if} \quad L_2^\lambda(t) < x < R(\lambda, t), \; 0 < t < T_0(\lambda). \end{cases} \tag{2.94}$$

Proof. Let q_0 be as in (2.61), then for $\lambda > q_0$, from Lemma (2.5) we have for $i = 1, 2$,

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda), \tag{2.95}$$

$$R_-(t, \beta, u_0^{i,\lambda}) = R_+(t, \beta, u_0^{i,\lambda}) \tag{2.96}$$

and denote $R_i(\lambda, t) = R_-(t, \beta, u_0^{i,\lambda})$. Since $u_0^{1,\lambda} \leq u_0^\lambda \leq u_0^{2,\lambda}$, hence from (2.30)

$$R_1(\lambda, t) \leq R(\lambda, t) \leq R_2(\lambda, t). \tag{2.97}$$

Next we obtain a bound on $u^\lambda(R(\lambda, t)+, t)$. For this let $\bar{u}(x, t)$ be the solution of (1.2) with initial data $\bar{u}_0(x)$ defined by

$$\bar{u}_0(x) = \begin{cases} \min(a, m_1) & \text{if} \quad x < \beta, \\ u_0(x) & \text{if} \quad x > \beta, \end{cases}$$

then for $\lambda > m, \bar{u}_0(x) \leq u_0^{1,\lambda}(x) \leq u_0^\lambda(x)$ and hence $\bar{u}(x, t) \leq u^\lambda(x, t)$ and $R_+(t, \beta, \bar{u}_0)$ $\leq R(\lambda, t)$. Since for $y > \beta, \int_\beta^y \bar{u}_0(\theta) d\theta = \int_\beta^y u_0^\lambda(\theta) d\theta$ and hence from (2.50) we have for $x > R(\lambda, t)$,

$$\begin{aligned} V^\lambda(x, t) &= \inf_{y \geq \beta} \{ \int_\beta^y u_0^\lambda(\theta) d\theta + t f^* \left(\frac{x - y}{t} \right) \} \\ &= \inf_{y \geq \beta} \{ \int_\beta^y \bar{u}_0(\theta) d\theta + t f^* \left(\frac{x - y}{t} \right) \} \\ &= V(x, t), \end{aligned}$$

where $u^\lambda = \frac{\partial V^\lambda}{\partial x}$ and $\bar{u} = \frac{\partial V}{\partial x}$. Hence for $x > R(\lambda, t)$,

$$u^\lambda(x, t) = \bar{u}(x, t). \tag{2.98}$$

Therefore

$$\begin{aligned} |u^\lambda(R(\lambda, t)+, t)| &\leq \|\bar{u}\|_\infty \\ &\leq \max(m_2, a). \end{aligned} \tag{2.99}$$

For $i = 1, 2$, let $T_{i,0}(\lambda)$ be the first intersection point of $L_2^\lambda(t)$ and $R_i(\lambda, t)$ and $T_{i,1}(\lambda) > T_{i,0}(\lambda)$ be the points of intersections of $L_1(t)$ and $R_i(\lambda, t)$.Then from Lemma (2.8), we can choose $\lambda_0 \geq q_0 + \|\bar{u}\|_\infty$ such that for all $\lambda \geq \lambda_0,\ f'(\lambda) > 0,\ f(\lambda) > f(\|\bar{u}\|_\infty)$ and

$$R_1(\lambda, T) > L_1(t), T_{1,1}(\lambda) > T \tag{2.100}$$

$$\inf_{T_{1,0}(\lambda)\le t\le T} f^{*'}\left(\frac{R_1(\lambda,t)-\alpha}{t}\right) = \inf_{T_{1,0}(\lambda)\le t\le T} u^{1,\lambda}(R_1(\lambda,t)-,t) > \lambda_0. \tag{2.101}$$

From (2.82) and (2.97) we have

$$\lim_{\lambda\to\infty} R(\lambda,T) \ge \lim_{\lambda\to\infty} R_1(\lambda,T) = \infty.$$

This proves (2.93).

Next imitating the proof as in Lemma (2.8) and from (2.98) we have for $0 < t < T$,

$$u^\lambda(x,t) = \begin{cases} \bar{u}(x,t) & \text{if } \; x > R(\lambda,t), \\ (f')^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } \; t > T_0(\lambda), L_1(t) < x < R(\lambda,t), \\ \lambda & \text{if } \; 0 < t < T_0(\lambda), L_2^\lambda(t) < x < R(\lambda), \\ a & \text{if } \; x < L_1(t). \end{cases} \tag{2.102}$$

Let $0 < t < T_0(\lambda)$ then from (2.99) and the choice of λ_0, we have for a.e. t,

$$\begin{aligned} \frac{d}{dt}R(\lambda,t) &= \frac{f(u^\lambda(R(\lambda,t)-,t)) - f(u^\lambda(R(\lambda,t)+,t))}{u^\lambda(R(\lambda,t)-,t) - u^\lambda(R(\lambda,t)+,t)} \\ &= \frac{f(\lambda) - f(u^\lambda(R(\lambda,t)+,t)}{\lambda - u^\lambda(R(\lambda,t)+,t)} > 0. \end{aligned}$$

Let $T_0(\lambda) < t \le T$, then from (2.97), $T_{1,0}(\lambda) \le T_0(\lambda)$. Hence from (2.102), (2.101)

$$\begin{aligned} u^\lambda(R(\lambda,t)-,t) &= f^{*'}\left(\frac{R(\lambda,t)-\alpha}{t}\right) \\ &\ge f^{*'}\left(\frac{R_1(\lambda,t)-\alpha}{t}\right) \\ &= u^{1,\lambda}(R_1(\lambda,t)-,t) \\ &> \lambda_0. \end{aligned}$$

Since $f'(\lambda) > 0$ for $\lambda \ge \lambda_0$, hence

$$f(u^\lambda(R(\lambda,t)-,t)) \ge f(\lambda_0) > f(\|\bar{u}\|_\infty).$$

Therefore from (2.98),(2.99) we have for $T_0(\lambda) < t \le T$.

$$\begin{aligned} \frac{d}{dt}R(\lambda,t) &= \frac{f(u^\lambda(R(\lambda,t)-,t) - f(u^\lambda(R(\lambda,t)+,t))}{u^\lambda(R(\lambda,t)-,t) - u^\lambda(R(\lambda,t)+,t)} \\ &= \frac{f(u^\lambda(R(\lambda,t)-,t) - f(\bar{u}(R(\lambda,t)+,t)}{u^\lambda(R(\lambda,t)-,t) - \bar{u}(R(\lambda,t)+,t)} \\ &> 0. \end{aligned}$$

This proves that $t \to R(\lambda, t)$ is a strictly increasing function.

Claim: $R(\lambda, t) \leq L_2^\lambda(t)$ for $t > T_0(\lambda)$.

Suppose for some $t_0 > T_0(\lambda), R(\lambda, t_0) > L_2^\lambda(t_0)$, then for a.e $x \in (L_2^\lambda(t_0), R(\lambda, t_0)$ $), y_+(x, t, u_0^\lambda) \in (\alpha, \beta)$ and hence from (2.13) and differentiability of u_0^λ in (α, β) gives $u(x, t_0) = \lambda$ and $f'(\lambda) = \frac{x - y_+(x, t_0, u_0^\lambda)}{t_0}$. Hence the characteristic line $r(\theta)$ at (x, t_0) is parallel to L_2^λ and $r(\theta) \geq L_2^\lambda(\theta)$ for $\theta \in [0, t_0]$. Since $t \to R(\lambda, t)$ is an increasing function for $t \in (0, T_1(\lambda))$ and $T_0(\lambda) < t_0$, hence $R(\lambda, T_0(\lambda)) < x$. Furthermore $y_+(R(\lambda, T_0(\lambda)), T_0(\lambda), u_0^\lambda) \geq \beta$. Hence the characteristic line at $(R(\lambda, T_0(\lambda)), T_0(\lambda))$ intersect r which contradicts NIP. This proves the claim. Hence for $t \geq T_0(\lambda)$,

$$\frac{R(\lambda, t) - \alpha}{t} \leq f'(\lambda). \tag{2.103}$$

Let $\lambda_0 \leq \lambda_1 < \lambda_2$, then $u_0^{\lambda_1} \leq u_0^{\lambda_2}$ and hence $R(\lambda_1, t) \leq R(\lambda_2, t)$ and for a.e. $x,\ y_\pm(x, t, u_0^{\lambda_1}) \geq y_\pm(x, t, u_0^{\lambda_2})$. Suppose for some $0 < t_0 < T, R = R(\lambda_1, t_0) = R(\lambda_2, t_0)$. From (2.98) at $x = R$, we have $\alpha \leq y_-(R, t_0, u_0^{\lambda_2}) \leq y_-(R, t_0, u_0^{\lambda_1}) < \beta$, and $u^{\lambda_1}(R+, t_0) = \bar{u}(R+, t_0) = u^{\lambda_2}(R+, t_0)$. Hence $y_+(R, t_0, u_0^{\lambda_1}) = y_+(R, t_0, u_0^{\lambda_2})$.

Let for $i = 1, 2,\ y = y_+(R, t_0, u_0^{\lambda_i}),\ y_i = y_-(R, t_0, u_0^{\lambda_i})$ and $V_0^{\lambda_i}(y) = \int\limits_\beta^y u_0^{\lambda_i}(\theta) d\theta$, then $V_0^{\lambda_1}(y) = V_0^{\lambda_2}(y)$ for $y \geq \beta$. Hence from (2.4) we have

$$\begin{aligned}
\lambda_2(y_2 - \beta) + t_0 f^*\left(\frac{R - y_2}{t_0}\right) &= V_0^{\lambda_2}(y_2) + t f^*\left(\frac{R - y_2}{t_0}\right) \\
&= V_0^{\lambda_2}(y) + t f^*\left(\frac{R - y}{t_0}\right) \\
&= V_0^{\lambda_1}(y) + t f^*\left(\frac{R - y}{t_0}\right) \\
&= V_0^{\lambda_1}(y_1) + t f^*\left(\frac{R - y_1}{t_0}\right) \\
&= \lambda_1(y_1 - \beta) + t_0 f^*\left(\frac{R - y_1}{t_0}\right).
\end{aligned}$$

Let $f'(\theta_i) = \frac{R - y_i}{t_0}$, then $R = f'(\theta_i) t_0 + y_i$ and since $y_2 \leq y_1$ implies that $f'(\theta_2) \geq f'(\theta_1)$, hence $\theta_2 \geq \theta_1$. Substituting this in the above expression and using $f^*(f'(p)) = p f'(p) - f(p)$ to obtain

$$(R - t_0 f'(\theta_2))\lambda_2 + t_0 f^*(f'(\theta_2)) = (R - t_0 f'(\theta_1))\lambda_1 + t_0 f^*(f'(\theta_1)) + \beta(\lambda_2 - \lambda_1)$$

$$R = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)}[(\lambda_2 - \theta_2) f'(\theta_2) - (\lambda_1 - \theta_1) f'(\theta_1)] + \left(\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1}\right) t_0.$$

That is for $i = 1, 2$,

$$y_i = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)}[(\lambda_2 - \theta_2)f'(\theta_2) - (\lambda_1 - \theta_1)f'(\theta_1)] + t_0\left[\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1} - f'(\theta_i)\right]. \tag{2.104}$$

Case (i) : Let $y_2 = y_1$. Then $\theta_2 = \theta_1$ and hence from (2.104), $\beta = y_1 < \beta$ which is a contradiction.
Case (ii): Let $\alpha < y_2 < y_1$.

Since $V_0^{\lambda_i}$ is differentiable for $y \in (\alpha, \beta)$ and hence from (2.13) , (2.23), we have $f'(\lambda_i) = \frac{R - y_i}{t_0}$. Therefore from (2.104) and from strict convexity of f we have

$$y_1 = \beta + t_0\left[\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} - f'(\lambda_1)\right] > \beta.$$

which is a contradiction.
Case(iii): Let $\alpha = y_2 < y_1$.

Since $y_1 > \alpha$, hence $f'(\theta_1) = \frac{R - y_1}{t_0} = f'(\lambda_1)$ and $\frac{R - \alpha}{t_0} = f'(\theta_2)$. From (2.103), $f'(\theta_2) \le f'(\lambda_2)$ and hence $\lambda_2 \ge \theta_2$. Since $\lambda_1 \ge \lambda_0$ and hence $f'(\theta_2) \ge f'(\lambda_1) > 0$. From (2.104), $\theta_1 = \lambda_1$ and convexity of f we have

$$\begin{aligned} 0 > \frac{(\lambda_2 - \lambda_1)(y_1 - \beta)}{t_0} &= (\lambda_2 - \theta_2)(f'(\theta_2) - f'(\lambda_1)) + f(\theta_2) - f(\lambda_1) \\ &- (\theta_2 - \lambda_1)f'(\lambda_1) \\ &> 0, \end{aligned}$$

which is a contradiction. This proves $\lambda \to R(\lambda, t)$ is a strictly increasing function for $\lambda \ge \lambda_0$ and $0 < t \le T$. This proves the Lemma.

Next we consider the variation from the right, Let u^μ be the solution of (1.2) with initial data u_0^μ given by

$$u_0^\mu = \begin{cases} u_0(x) & \text{if } x < \alpha, \\ \mu & \text{if } \alpha < x < \beta, \\ a & \text{if } x > \beta. \end{cases}$$

We state the following Lemma without proof since the proof follows exactly as that of Lemma 2.9.

Define

$$L_1(t) = \beta + f'(a)t, L_2^\mu(t) = \beta + f'(\mu)t.$$

Lemma 2.10 *There exist $\mu_1 = \mu_1(m_1, m_2, a)$ such that for $\mu < \mu_1, t > 0$,*

$$R_-(t, \alpha, u_0^\mu) = R_+(t, \alpha, u_0^\mu), \tag{2.105}$$

and denote $R(\mu,t) = R_-(t,\alpha,u_0^\mu)$. *Let* $T_0(\mu) > 0$ *be the first point of intersection of* $R(\mu,t)$ *and* $L_2^\mu(t)$ *and* $T_1(\mu) > T_0(\mu)$ *be the first point of intersection of* $R(\mu,t)$ *and* $L_1(t)$. *Then*

$$\lim_{\mu\to-\infty} T_1(\mu) = \infty. \tag{2.106}$$

For $0 < t < T_1(\mu)$,

$$u^\mu(x,t) = \begin{cases} a & \text{if } \ x > L_1(t), \\ (f')^{-1}\left(\frac{x-\beta}{t}\right) & \text{if } \ T_0(\mu) < t < T_1(\mu), \\ & \quad\ R(\mu,t) < x < L_1(t), \\ \mu & \text{if } \ 0 < t < T_0(\mu), \\ & \quad\ R(\mu,t) < x < L_2^\mu(t). \end{cases}$$

Furthermore let $T > 0$ *be fixed, then there exist* $\mu_0 = \mu_0(T,\mu_1) < \mu_1$ *such that*

(i). $\mu \to R(\mu,t)$ *is a strictly increasing function for* $0 < t \le T$ *and*

$$\lim_{\mu\to-\infty} R(\mu,t) = -\infty. \tag{2.107}$$

(ii). For $0 < t < T_1(\mu), t \to R(\mu,t)$ *is a strictly decreasing function of* t.

Next we study the interaction of $R_\pm$ with varying parameters in the data. For this first we need the following elementary results.
Let $B_1, B_2, \mu_0 < \lambda_0,\ L, R \in C([0,\infty])$, be given and for $\lambda \ge \lambda_0, \mu \le \mu_0,\ L$ and R satisfies the following hypothesis,
$(\mathbf{H_1})$. $\lambda \mapsto L(t,\lambda),\ \mu \mapsto R(t,\mu)$ are strictly increasing functions such that for all $\lambda \ge \lambda_0,\ \mu \le \mu_0$,

$$L(0,\lambda) = B_1,\ R(0,\mu) = B_2, \tag{2.108}$$

and for any $0 < \alpha < \beta$,

$$\lim_{\lambda\to\infty} \inf_{t\in[\alpha,\beta]} L(t,\lambda) = \infty,\ \lim_{\mu\to-\infty} \sup_{t\in[\alpha,\beta]} R(t,\mu) = -\infty. \tag{2.109}$$

$(\mathbf{H_2})$. For $\lambda \ge \lambda_0, \mu \le \mu_0, t \mapsto L(t,\lambda)$, is a strictly increasing function and $t \mapsto R(t,\mu)$ is a strictly decreasing function.

Let $I = [\lambda_0,\infty) \times (-\infty,\mu_0]$ and define $x_0(t), y_0(t), \lambda(x,t), \mu(y,t), \delta(\lambda,\mu), c(\lambda,\mu)$ as follows:

$$\begin{aligned} x_0(t) &= L(t,\lambda_0),\ y_0(t) = R(t,\mu_0) & (2.110)\\ L(t,\lambda(x,t)) &= x,\ R(t,\mu(y,t)) = y & (2.111)\\ L(\delta(\lambda,\mu),\lambda) &= R(\delta(\lambda,\mu),\ \mu) = c(\lambda,\mu), & (2.112) \end{aligned}$$

then we have the following

Lemma 2.11

1. *x_0 is a strictly increasing continuous and y_0 is a strictly decreasing function satisfying*
$$(x_0(0), y_0(0)) = (B_1, B_2). \tag{2.113}$$

2. *For $x \geq x_0(t), y \leq y_0(t), (\lambda(x,t), \mu(y,t)) \in I, x \mapsto \lambda(x,t), t \to \mu(y,t)$ are strictly increasing functions and $t \mapsto \lambda(x,t), y \mapsto \mu(y,t)$ are strictly decreasing continuous functions in $(0,\infty)$. Also for $x > B_1,\ y < B_2$*
$$\lim_{t\to 0}(\lambda(x,t), \mu(y,t)) = (\infty, -\infty). \tag{2.114}$$

3. *Let $B_1 < B_2$ and $(\lambda, \mu) \in I$. Then $\delta(\lambda,\mu)$ exist and is a continuous function. Furthermore $\lambda \to \delta(\lambda,\mu)$ is an decreasing function and $\mu \mapsto \delta(\lambda,\mu)$ is an increasing function and*
$$\lim_{\lambda\to\infty} \delta(\lambda,\mu) = \lim_{\mu\to-\infty} \delta(\lambda,\mu) = 0 \tag{2.115}$$
$$\lim_{\mu\to-\infty} c(\lambda,\mu) = B_1,\ \lim_{\lambda\to\infty} c(\lambda,\mu) = B_2. \tag{2.116}$$

Proof.

1. Follows from (H_1).

2. From (2.109) for $t > 0, L(t,\cdot) : [\lambda_0, \infty) \to [x_0(t), \infty)$ is a homeomorphism and hence $\lambda(x,t)$ exist and $x \mapsto \lambda(x,t)$ is a strictly increasing function. Let $t_1 < t_2$ and suppose $\lambda(x,t_1) \leq \lambda(x,t_2)$, then
$$x = L(t_1, \lambda(x,t_1)) \leq L(t_1, \lambda(x,t_2)) < L(t_2, \lambda(x,t_2)) = x,$$
which is a contradiction. Hence $t \mapsto \lambda(x,t)$ is a strictly decreasing function. Let $(x_n, t_n) \to (x,t),\ \lambda(x_n,t_n) \to \lambda$, then
$$x = \lim_{n\to\infty} L(t_n, \lambda(x_n,t_n)) = L(t,\lambda),$$
and hence $\lambda = \lambda(x,t)$. This proves the continuity of $\lambda(x,t)$. Suppose as $t_n \to 0, \{\lambda(x,t_n)\}$ is bounded. Then for a subsequence still denote by n such that $\lambda(x,,t_n) \to \lambda$ as $n \mapsto \infty$. Therefore by continuity of L and (2.108)
$$B_1 < x = \lim_{n\to\infty} L(t_n, \lambda(x,t_n)) = L(0,\lambda) = B_1,$$
which is a contradiction. Hence $\lambda(x,t) \to \infty$ as $t \to 0$. Similarly for $\mu(y,t)$ and this proves (2).

3. For $(\lambda,\mu) \in I, t \mapsto L(t,\lambda) \geq B_1$ and is a strictly increasing function and $t \mapsto R(t,\mu) \leq B_2$ is a strictly decreasing function. Hence there exist a unique $\delta(\lambda,\mu)$ satisfying (2.112) and $B_1 \leq c(\lambda,\mu) \leq B_2$ and continuity follows from the uniqueness of $\delta(\lambda,\mu)$.

Let $\lambda_1 < \lambda_2$ and $\delta(\lambda_1, \mu) \leq \delta(\lambda_2, \mu)$. Then

$$\begin{aligned} R(\delta(\lambda_1, \mu), \mu) &= L(\delta(\lambda_1, \mu), \lambda_1) \leq L(\delta(\lambda_2, \mu), \lambda_1) \\ &< L(\delta(\lambda_2, \mu), \lambda_2) \\ &= R(\delta(\lambda_2, \mu), \mu) \end{aligned}$$

and hence $\delta(\lambda_2, \mu) < \delta(\lambda_1, \mu)$ which is a contradiction. Suppose $\lim\limits_{\lambda \to \infty} \delta(\lambda, \mu) = \delta_0 > 0$, then from (2.109),

$$\infty = \lim_{\lambda \to \infty} L(\delta(\lambda, u), \lambda) = \lim_{\delta(\lambda,\mu) \to \delta_0} R(\delta(\lambda, \mu), \mu) = R(\delta_0, \mu) < \infty,$$

which is a contradiction hence $\delta_0 = 0$ and

$$\begin{aligned} \lim_{\lambda \to \infty} c(\lambda, \mu) &= \lim_{\lambda \to \infty} L(\delta(\lambda, \mu), \lambda) \\ &= \lim_{\delta(\lambda,\mu) \to 0} R(\delta(\lambda, \mu), \mu) \\ &= B_2, \end{aligned}$$

similarly for $\mu \to \delta(\lambda, \mu)$. This proves (3) and hence the Lemma.

Corollary 2.12 *Let $\delta_0 > 0$, then there exist $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1,\ \mu \leq \mu_1$,*

$$\delta(\lambda, \mu) \leq \delta_0. \tag{2.117}$$

Proof. Since $\delta(\lambda, \mu_0) \to 0$ as $\lambda \to \infty$, hence choose $\lambda_1 \geq \lambda_0$ such that $\delta(\lambda_1, \mu_0) \leq \delta_0$. Let $\mu_1 = \mu_0$, then for $\lambda \geq \lambda_1, \mu \leq \mu_1$, we have,

$$\delta(\lambda, \mu) \leq \delta(\lambda_1, \mu) \leq \delta(\lambda_1, \mu_1) \leq \delta_0.$$

This proves the Corollary.

Let $T > 0$ and $A_1 < B_1 \leq C \leq B_2 < A_2$ and for $i = 1, 2$, define $a_i, l_i, 0 < \delta_0 < T$ by

$$\begin{aligned} f'(a_i) &= \frac{C - A_i}{T} \\ l_i(t) &= A_i + t f'(a_i) \\ \delta_0 &= \min\{l_1(B_1), l_2(B_2)\}. \end{aligned}$$

Let u_1^λ and u_2^μ be solutions of (1.2) with respective initial data u_0^λ, u_0^μ given by

$$u_0^{1,\lambda}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \theta_f & \text{if } x > B_2. \end{cases} \tag{2.118}$$

$$u_0^{2,\mu}(x) = \begin{cases} \theta_f & \text{if} \quad x < B_1, \\ u_0(x) & \text{if} \quad B_1 < x < B_2, \\ \mu & \text{if} \quad B_2 < x < A_2, \\ a_2 & \text{if} \quad x > A_2. \end{cases} \tag{2.119}$$

From Lemma (2.9) and (2.10) we can choose $\lambda_0 = \lambda_0(\|u_0\|_\infty),\ \mu_0 = \mu_0(\|u_0\|_\infty)$ such that for all $\lambda \geq \lambda_0,\ \mu \leq \mu_0,\ t > 0$,

$$L(t,\lambda) = R_-(t, B_1, u_0^{1,\lambda}) = R_+(t, B_1, u_0^{1,\lambda}) \tag{2.120}$$

$$R(t,\mu) = R_-(t, B_2, u_0^{2,\mu}) = R_+(t, B_2, u_0^{2,\mu}), \tag{2.121}$$

and for $0 < t \leq T$, L and R satisfies the hypothesis $(H_1), (H_2)$ of Lemma (2.11). Let $(C(\lambda,\mu), \delta(\lambda,\mu))$ be the point of intersection of $L(t,\lambda)$ and $R(t,\mu)$ as defined in (2.112). From Corollary (2.12), choose $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1, \mu \leq \mu_1$

$$\delta(\lambda,\mu) < \delta_0. \tag{2.122}$$

Lemma 2.13 *With the above notation and $u(x,t,\lambda,\mu)$ be the solution of (1.2) with initial condition $u_0^{\lambda,\mu}$ given by*

$$u_0^{\lambda,\mu}(x) = \begin{cases} a_1 & \textit{if} \quad x < A_1, \\ \lambda & \textit{if} \quad A_1 < x < B_1, \\ u_0 & \textit{if} \quad B_1 < x < B_2, \\ \mu & \textit{if} \quad B_2 < x < A_2, \\ a_2 & \textit{if} \quad x > A_2. \end{cases} \tag{2.123}$$

then for $0 < t < \delta(\lambda,\mu)$,

$$u_1^\lambda(x,t) = u_2^\mu(x,t) \quad \textit{if}\ L(t,\lambda) < x < R(t,\mu), \tag{2.124}$$

$$u(x,t,\lambda,\mu) = \begin{cases} u_1^\lambda(x,t) & \textit{if} \quad x < L(t,\lambda), \\ u_1^\lambda(x,t) & \textit{if} \quad L(t,\lambda) < x < R(t,\mu), \\ u_2^\mu(x,t) & \textit{if} \quad x > R(t,\mu). \end{cases} \tag{2.125}$$

Proof. Let $\gamma = \frac{A_1+A_2}{2}$ and define $v_0^{1,\lambda}(x) = \int_\gamma^x u_0^{1,\lambda}(\theta)d\theta,\ v_0^{2,\mu}(x) = \int_\gamma^x u_0^{2,\mu}(\theta)d\theta,$ $V_0^{\lambda,\mu}(x) = \int_\gamma^x u_0^{\lambda,\mu}(\theta)d\theta$. Then for $x \in [B_1, B_2]$,

$$v_0^{1,\lambda}(x) = v_0^{2,\mu}(x) = \int_{\frac{A_1+A_2}{2}}^{x} u_0(\theta)d\theta. \tag{2.126}$$

$$v_0^{\lambda,\mu}(x) = \begin{cases} v_0^{1,\lambda}(x) & \text{if} \quad x < B_1, \\ v_0^{2,\mu}(x) & \text{if} \quad x > B_2. \end{cases} \tag{2.127}$$

Claim: Let $v^{1,\lambda}, v^{2,\mu}$ be the corresponding value functions defined in (2.4). Then

$$v^{1,\lambda}(x,t) = \inf_{y\in[B_1,B_2]} \left\{ v_0^{1,\lambda}(y) + tf^*\left(\frac{x-y}{t}\right)\right\}, \text{ if } \quad L(t,\lambda) < x < B_2 \qquad (2.128)$$

$$v^{2,\mu}(x,t) = \inf_{y\in[B_1,B_2]} \left\{ v_0^{2,\mu}(y) + tf^*\left(\frac{x-y}{t}\right)\right\}, \text{ if } \quad B_1 < x < R(t,\lambda). \qquad (2.129)$$

Let $L(t,\lambda) < x < B_2$, then $y_\pm(x,t,u_0^{1,\lambda}) > B_1$. Suppose for some $x_0 \in (L(t,\lambda), B_2)$, $y_+(x_0,t,u_0^{1,\lambda}) > B_2$. Since $v_0^{1,\lambda}$ is differentiable in (B_2,∞) and hence from (2.13) for a.e. $x \in (x_0, B_2)$, $u^{1,\lambda}(x,t) = \frac{\partial v^{1,\lambda}}{\partial x}(x,t) = \theta_f$ and $0 = f'(\theta_f) = \frac{x - y_+(x,t,u_0^{1,\lambda})}{t}$. Hence $B_2 > x = y_+(x,t,u_0^{1,\lambda}) > B_2$, which is a contradiction. Therefore $y_\pm(x,t,u_0^{1,\lambda}) \in [B_1, B_2]$ and hence (2.128) follows. Similarly (2.129) holds .This proves the claim.

From (2.126), (2.128), (2.129), for $L(t,\lambda) < x < R(t,\mu)$, $v^{1,\lambda}(x,t) = v^{2,\mu}(x,t)$ and hence for a.e. x, $u^{1,\lambda}(x,t) = \frac{\partial v^{1,\lambda}}{\partial x}(x,t) = \frac{\partial v^{2,\mu}}{\partial x}(x,t) = u^{2,\mu}(x,t)$. This proves (2.124). In view of (2.14), RHS of (2.125) is a solution of (1.2) with initial data $u_0^{\lambda,\mu}$. Hence from uniqueness of solutions (2.125) follows. This proves the Lemma.

As an immediate consequence of Lemma (2.13) and (2.27),(2.120),(2.121) we have

Corollary 2.14 *Let $\lambda \geq \lambda_1, \mu \leq \mu_1$, then*

$$\begin{aligned} R_\pm(t,B_1,u_0^{\lambda,\mu}) &= L(t,\lambda) \quad 0 < t < \delta(\lambda,\mu), \\ R_\pm(t,B_2,u_0^{\lambda,\mu}) &= R(t,\mu) \quad 0 < t < \delta(\lambda,\mu), \\ R_\pm(t,B_1,u_0^{\lambda,\mu}) &= R_\pm(t,B_2,u_0^{\lambda,\mu}), \quad t \geq \delta(\lambda,\mu). \end{aligned}$$

Furthermore, denote $S(t,\lambda,\mu) = R_+(t,B_1,u_0^{\lambda,\mu})$ for $t > \delta(\lambda,\mu)$,then

$$u(x,t,\lambda,\mu) = \begin{cases} u^{1,\lambda}(x,t) & \text{if } \ x < S(t,\lambda,\mu), \\ u^{2,\mu}(x,t) & \text{if } \ x > S(t,\lambda,\mu), \end{cases} \qquad (2.130)$$

and $(t,\lambda,\mu) \mapsto S(t,\lambda,\mu)$ is continuous.

Proof. Let $(t_k,\lambda_k,\mu_k) \to (t,\lambda,\mu)$. From lemma (2.11), $\delta(\lambda_k,\mu_k) \to \delta(\lambda,\mu)$ and hence for $t > \delta(\lambda,\mu)$

$$\begin{aligned} |S(t_k,\lambda_k,\mu_k) - S(t,\lambda,\mu)| &\leq |S(t_k,\lambda_k,\mu_k) - S(t,\lambda_k,\mu_k)| \\ &+ |S(t,\lambda_k,\mu_k) - S(t,\lambda,\mu)|. \end{aligned}$$

From (4) of Lemma (2.2) and from (3) of Lemma (2.4), the right hand side tends to zero as $k \to \infty$. For $x < S(t,\lambda,\mu), y_\pm(x,t,u_0^{\lambda,\mu}) < B_1$ and hence from (2.127), $v^{\lambda,\mu}(x,t) = v^{1,\lambda}(x,t)$, where $\frac{\partial v^{\lambda,\mu}}{\partial x} = u(x,t,\lambda,u)$, hence $u(x,t,\lambda,\mu) = \frac{\partial v^{1,\lambda}}{\partial x}(x,t) = u^{1,\lambda}(x,t)$. Similarly for $x > S(t,\lambda,\mu), u(x,t,\lambda,\mu) = u^{2,\mu}(x,t)$, this proves (2.125) and hence the Lemma.

Lemma 2.15 *Let $\lambda \geq \lambda_1, \mu \leq \mu_1$ and $\delta(\lambda,\mu) < t_0 \leq T$, then* *(i). Suppose $l_1(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,*

$$S(t, \lambda, \mu) < l_1(t). \tag{2.131}$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_2 & \text{if } \; 0 < t < T, \; x > l_2(t) \\ a_1 & \text{if } \; x < \min(l_1(t), S(t, \lambda, \mu)). \end{cases} \tag{2.132}$$

(ii). Suppose $l_2(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,

$$S(t, \lambda, \mu) > l_2(t). \tag{2.133}$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_1 & \text{if } \; 0 < t < T, \; x < l_1(t), \\ a_2 & \text{if } \; x > \max(l_2(t), S(t, \lambda, \mu)). \end{cases} \tag{2.134}$$

Furthermore there exist λ and μ such that $S(T, \lambda, \mu) = C$, for $0 < t < T$, u satisfies

$$u(x, t, \lambda, \mu) = \begin{cases} a_1 & \text{if } \; x < l_1(t), \\ a_2 & \text{if } \; x > l_2(t). \end{cases} \tag{2.135}$$

Proof. (See Figure 12) Let $g(t) = \min(l_1(t), S(t, \lambda, \mu))$. Then we claim that for all $x < g(t)$,

$$u(x, t, \lambda, \mu) = a_1. \tag{2.136}$$

Suppose $x < l_1(t) \leq S(t, \lambda, \mu)$, then from (2.130), (2.94) we have $u(x, t, \lambda, \mu) = u^{1,\lambda}(x, t) = a_1$. Hence assume that $S(t, \lambda, \mu) < l_1(t)$. Suppose there exist $x_0 < S(t, \lambda, \mu)$ such that $y_+(x_0, t, u_0^{\lambda,\mu}) > A_1$, then for all $x \in (x_0, S(t, \lambda, \mu)), A_1 < y_+(x, t, u_0^{\lambda,\mu}) < B_1$. Since $u_0^{\lambda,\mu}$ is differentiable in (A_1, B_1) and hence from (2.13), for a.e. $x \in (x_0, S(t, \lambda, \mu))$

$$\begin{aligned} f'(\lambda) = f'(u(x, t, \lambda, \mu)) &= \frac{x - y_+(x, t, u_0^{\lambda,\mu})}{t} \\ &< \frac{l_1(t) - A_1}{t} = f'(a_1), \end{aligned}$$

which is a contradiction since $\lambda > a_1$. Hence $y_+(x, t, u_0^{\lambda,\mu}) \leq A_1$ for all $x \in (x_0, S(t, \lambda, \mu))$. Suppose $y_+(x_0, t, u_0^{\lambda,\mu}) = A_1$. Then from (2.21) $f'(u(x, t)) = \frac{x - A_1}{t}$ for $x \in (x_0, S(t, \lambda, \mu))$. Let $\gamma_x(\theta) = A_1 + \theta\left(\frac{x - A_1}{t}\right) < l_1(\theta)$ be the characteristic at (x, t), then from (c) of (3) in theorem 2.1, γ_x is also a characteristic at $(\gamma_x(s), s)$ for $0 < s < t$ and $f'(u(\gamma_x(s), s, \lambda, \mu) = \frac{\gamma_x(s) - A_1}{s} < \frac{l_1(s) - A_1}{s} = f'(a_1)$. Let $s < \delta(\lambda, \mu)$, then $l_1(s) < L(s, \lambda)$ and hence $f'(a_1) > f'(u(\gamma_x(s), s)) = f'(a_1)$ which is a contradiction. This proves the claim.

Let $t_0 > \delta(\lambda,\mu)$ such that $l(t_0) = S(t_0,\lambda,\mu)$. Hence for $x > S(t_0,\lambda,\mu), y_-$ $(x, t_0, u_0^{\lambda,\mu}) \geq B_1$. Therefore

$$a_2 = \max(\mu, a_2) \geq u(x, t_0, \lambda, \mu). \tag{2.137}$$

Let $t_0 < t < T$ and w be the solution of (1.2) with initial data w_0 at t_0 is given by

$$w_0(x) = \begin{cases} a_1 & \text{if } \; x < S(t_0,\lambda,\mu) = l_1(t_0), \\ a_2 & \text{if } \; x > S(t_0,\lambda,\mu) = l_1(t_0). \end{cases}$$

Then w admits a shock at $l_1(t_0)$ and for $t > t_0$ is given by

$$\begin{aligned} \eta(t) &= l_1(t_0) + \tfrac{f(a_1)-f(a_2)}{a_1-a_2}(t-t_0) \\ &< l_1(t_0) + f'(a_1)(t-t_0) \\ &= A_1 + f'(a_1)t \\ &= l_1(t), \end{aligned} \tag{2.138}$$

since f is strictly convex and $f'(a_1) > 0 > f'(a_2)$. From the claim and (2.137),

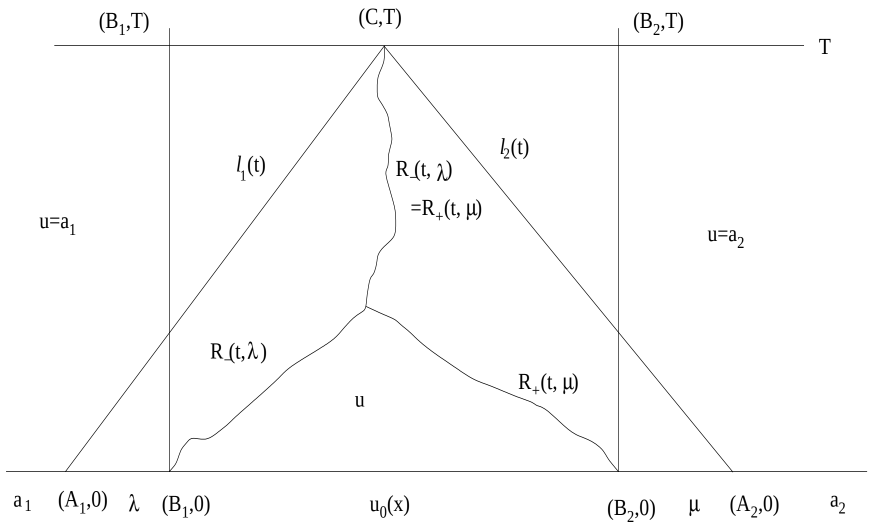

Figure 12:

$w_0(x) \geq u(x, t_0, \lambda, \mu)$ and therefore from (2.30) and (2.29) we have for $t > t_0$, $l_1(t) > \eta(t) \geq S(t,\lambda,\mu)$. This proves (2.131).

From (3) of Lemma (2.4) , $(\lambda,\mu) \to S(T,\lambda,\mu)$ is a continuous function for $\lambda \geq \lambda_1$ and $\mu \leq \mu_1$. From (2.93), choose a $\tilde{\lambda_1} > \lambda_1$ such that $S(T, \tilde{\lambda_1}, \mu_1) > T$ and from (2.107) choose $\tilde{\mu}_1 < \mu_1$ such that $S(T, \lambda_1, \tilde{\mu}_1) < T$. From Corrollary 2.14, S is continuous in $[\lambda_1, \tilde{\lambda_1}] \times [\mu_1, \tilde{\mu_1}]$ there exist a $(\lambda,\mu) \in [\lambda_1, \tilde{\lambda_1}] \times [\mu_1, \tilde{\mu_1}]$ such that $S(T,\lambda,\mu) = C$. Hence (2.135) follows from (2.134). This proves the Lemma.

Counter Example 2.16: Let $\alpha = 0, x_k < 0, \lim\limits_{k\to\infty} x_k = 0, \lambda > \theta_f$ and define u_0, u_0^k by

$$u_0(x) = \begin{cases} \theta_f & \text{if} \quad x < 0, \\ \lambda & \text{if} \quad x > 0. \end{cases}$$

$$u_0^k(x) = \begin{cases} \theta_f & \text{if} \quad x < x_k, \\ \lambda & \text{if} \quad x > x_k. \end{cases}$$

Then the solution u and u_k with respective initial datas u_0 and u_0^k are given by

$$u(x,t) = \begin{cases} \theta_f & \text{if} \quad x < 0, t > 0, \\ (f')^{-1}(\frac{x}{t}) & \text{if} \quad 0 < x \le f'(\lambda)t, \\ \lambda & \text{if} \quad x > f'(\lambda)t, \end{cases}$$

then

$$R_-(t,0,u_0) = 0.$$

$$u_k(x,t) = \begin{cases} \theta_f & \text{if} \quad x < x_k, t > 0, \\ (f')^{-1}(\frac{x-x_k}{t}) & \text{if} \quad x_k < x < f'(\lambda)t + x_k, \\ \lambda & \text{if} \quad x > f'(\lambda)t + x_k, \end{cases}$$

then

$$R_-(t,0,u_0^k) = f'(\lambda)t,$$

$$\int_{I\!R} |u_0(x) - u_0^k(x)|dx = \int_{x_k}^{0} (\lambda - \theta_f) = (\lambda - \theta_f)|x_k| \to 0 \text{ as } k \to \infty.$$

But

$$\lim_{k\to\infty} R_-(t,0,u_0^k) = f'(\lambda)t > 0 = R_-(t,0,u_0).$$

3 Shock Profile:

In this section we study the behavior of solutions of (1.2) in an explicit manner.

Let $-\infty \le A_1 < A_2 \le \infty,\ I = (A_1, A_2),\ u_\pm \in I\!R,\ u_0 \in L^\infty((A_1, A_2))$ and define m, M, k_1, k_2, γ_1 and γ_2 as follows:

$$m = \inf_{x\in I}\{u_0(x), u_+\},\ M = \sup_{x\in I}\{u_0(x), u_-\} \tag{3.1}$$

$$f'(k_1) = \begin{cases} f'(u_-) & \text{if} \quad u_- < m, \\ \frac{f(u_-)-f(m)}{u_- - m} & \text{if} \quad u_- \ge m. \end{cases} \tag{3.2}$$

$$f'(k_2) = \begin{cases} f'(u_+) & \text{if} \quad u_+ > M, \\ \frac{f(u_+)-f(M)}{u_+-M} & \text{if} \quad u_+ \le M. \end{cases} \tag{3.3}$$

$$\gamma_1(t) = A_1 + f'(k_1)t, \ \gamma_2(t) = A_2 + f'(k_2)t.$$

Let u_m, u_M be the solutions of (1.2) with respective initial data u_0^m, u_0^M given by

$$u_0^m(x) = \begin{cases} u_- & \text{if} \quad x < A_1, \\ m & \text{if} \quad x > A_1, \end{cases}$$

$$u_0^M(x) = \begin{cases} M & \text{if} \quad x < A_2, \\ u_+ & \text{if} \quad x > A_2. \end{cases} \tag{3.4}$$

Then u_m, u_M are given by

$$u_m(x,t) = \begin{cases} u_- & \text{if} \quad x < \gamma_1(t), \\ (f')^{-1}(\frac{x-A_1}{t}) & \text{if} \quad \gamma_1(t) < x < \text{Max}(\gamma_1(t), A_1 + tf'(m)), \\ m & \text{if} \quad x > \text{Max}(\gamma_1(t), A_1 + tf'(m)). \end{cases} \tag{3.5}$$

$$u_M(x,t) = \begin{cases} M & \text{if} \quad x < \text{Min}(\gamma_2(t), A_2 + tf'(M)), \\ (f')^{-1}(\frac{x-A_2}{t}) & \text{if} \quad \text{Min}(\gamma_2(t), A_2 + tf'(M)) < x < \gamma_2(t), \\ u_+ & \text{if} \quad x > \gamma_2(t). \end{cases} \tag{3.6}$$

Hence it follows that

$$\gamma_1(t) = R_-(t, A_1, u_0^m), \ \gamma_2(t) = R_+(t, A_2, u_0^M). \tag{3.7}$$

Let

$$\bar{u}_0(x) = \begin{cases} u_- & \text{if} \quad x < A_1, \\ u_0(x) & \text{if} \quad A_1 < x < A_2, \\ u_+ & \text{if} \quad x > A_2. \end{cases} \tag{3.8}$$

$$v_0(x) = \int_{A_1}^{x} \bar{u}_0(\theta) d\theta.$$

Let v be the corresponding value function defined in (2.4) and $u = \frac{\partial v}{\partial x}$ be the solution of (1.2) with initial data $\bar{u}_0$. Since $u_0^m \le \bar{u}_0 \le u_0^M$, hence from lemma (2.2) and for all $t > 0$,

$$\gamma_1(t) \le R_-(t, A_1, \bar{u}_0) \le R_+(t, A_2, \bar{u}_0) \le \gamma_2(t). \tag{3.9}$$

Furthermore if $x < R_-(t, A_1, \bar{u}_0)$, then $y_+(x,t,\bar{u}_0) < A_1$ and hence from (2.13) for a.e. x, $u(x,t) = u_0(y_+(x,t,\bar{u}_0)) = u_-$. Similarly for $x > R_+(t, A_2, \bar{u}_0)$, $u(x,t) = u_+$. Now letting $x \uparrow R_-(t, A_1, \bar{u}_0)$ and $x \downarrow R_+(t, A_2, \bar{u}_0)$ and from (2.12),(2.22) and (2.23) we have

$$\frac{R_-(t, A_1, \bar{u}_0) - y_-(R_-(t, A_1, \bar{u}_0), t, \bar{u}_0)}{t} = f'(u_-). \tag{3.10}$$

$$\frac{R_+(t, A_2, \bar{u}_0) - y_+(R_+(t, A_2, \bar{u}_0), t, \bar{u}_0)}{t} = f'(u_+). \tag{3.11}$$

$$u(x,t) = \begin{cases} u_- & \text{if} \quad x < R_-(t, A_1, u_0), \\ u_+ & \text{if} \quad x > R_+(t, A_2, u_0). \end{cases} \tag{3.12}$$

Now the main question is **how does the solution** u **behaves for** $R_-(t, A_1, \bar{u}_0) < x < R_+(t, A_2, \bar{u}_0)$**?**. In order to answer this question we have to consider two cases.
Case 1: For some $T > 0$, $R_-(T, A_1, \bar{u}_0) = R_+(T, A_2, \bar{u}_0)$.
Case 2: $R_-(t, A_1, \bar{u}_0) < R_+(t, A_2, \bar{u}_0)$, for all $t > 0$.
Case 1 is a single shock and we have the following

LEMMA 3.1 *Suppose for some* $T > 0$,

$$R_-(T, A_1, \bar{u}_0) = R_+(T, A_2, \bar{u}_0) \tag{3.13}$$

and denote $S(T) = R_-(T, A_1, \bar{u_0})$ *and for* $t > T$

$$S(t) = S(T) + \frac{f(u_+) - f(u_-)}{u_+ - u_-}(t - T), \tag{3.14}$$

Then $u_- > u_+$ *and for* $t > T$,

$$u(x,t) = \begin{cases} u_- & \textit{if} \quad x < S(t), \\ u_+ & \textit{if} \quad x > S(t). \end{cases} \tag{3.15}$$

Proof. From (3.12) and (3.13) we have

$$u(x,t) = \begin{cases} u_- & \text{if} \quad x < S(T), \\ u_+ & \text{if} \quad x > S(T). \end{cases} \tag{3.16}$$

From (2.27), for $t > T$, $R_-(t, A_1, \bar{u_0}) = R_+(t, A_2, u_0) = R_+(t, T, S(T), \bar{u}_0) = R_-(t, T, S(T), \bar{u}_0)$. Hence $S(T)$ is not a rarefaction point and therefore $u_- > u_+$ and (3.15) follows. This proves the theorem.

Case 2 is quite involved and we need few preliminary results. From now on assume that for all $t > 0$,

$$R_-(t, A_1, \bar{u}_0) < R_+(t, A_2, \bar{u}_0). \tag{3.17}$$

For α, x in $I\!R$, $t > 0$ denote

$$\begin{aligned} y_\pm(x,t) &= y_\pm(x,t,\bar{u}_0) \\ R_\pm(t,\alpha) &= R_\pm(t,\alpha,\bar{u}_0) \\ y_\pm(t,\alpha) &= y_\pm(R_-(t,\alpha),\alpha) \\ Y_\pm(t,\alpha) &= y_\pm(R_+(t,\alpha),\alpha). \end{aligned} \tag{3.18}$$

At the end points A_1 and A_2, let

$$\begin{aligned} L(t) &= R_-(t, A_1, \bar{u}_0) \\ R(t) &= R_+(t, A_2, \bar{u}_0) \\ y_\pm(t, A_1) &= y_\pm(t) = y_\pm(R_-(t, A_1), t) \\ Y_\pm(t, A_2) &= Y_\pm(t) = y_\pm(R_+(t, A_2), t). \end{aligned} \tag{3.19}$$

From (2.4) we have for all $x \in I\!R,\ t > 0$,

$$\int_{A_1}^{y_-(x,t)} \bar{u}_0(\theta)d\theta + tf^*\left(\frac{x - y_-(x,t)}{t}\right) = \int_{A_1}^{y_+(x,t)} \bar{u}_0(\theta)d\theta + tf^*\left(\frac{x - y_+(x,t)}{t}\right). \quad (3.20)$$

If $y_-(x,t) < A_1$, then (3.20) becomes

$$(y_-(x,t) - A_1)u_- + tf^*\left(\frac{x - y_-(x,t)}{t}\right) = \int_{A_1}^{y_+(x,t)} \bar{u}_0(\theta)d\theta + tf^*\left(\frac{x - y_+(x,t)}{t}\right).$$

If $y_+(x,t) > A_2$, then

$$\begin{aligned} \int_{A_1}^{y_-(x,t)} \bar{u}_0(\theta)d\theta + tf^*\left(\tfrac{x-y_-(x,t)}{t}\right) = \int_{A_1}^{A_2} \bar{u}_0(\theta)d\theta \ &+\ (y_+(x,t) - A_2)u_+ \\ &+\ tf^*\left(\tfrac{x-y_+(x,t)}{t}\right). \end{aligned} \quad (3.21)$$

LEMMA 3.2 *Let $r(t) = \alpha + tf'(p)$ be a straight line such that for all $t > 0$,*
(i). If $y_-(r(t),t) = \alpha$, then

$$y_+(r(t),t) = \alpha \quad (3.22)$$
$$u(r(t)-,t) = u(r(t)+,t) = p. \quad (3.23)$$

(ii). If $y_+(r(t),t) = \alpha$, then

$$y_-(r(t),t) = \alpha \quad (3.24)$$
$$u(r(t)-,t) = u(r(t)+,t) = p. \quad (3.25)$$

Proof. It is enough to prove (i) and (ii) follows similarly. From (2.12) and (2.22) choose a sequence $x_k < r(t)$ such that

$$\lim_{k\to\infty}(x_k, y_+(x_k,t)) = (r(t), y_-(r(t),t)), f'(u(x_k,t)) = \frac{x_k - y_+(x_k,t)}{t}. \quad (3.26)$$

Then

$$\begin{aligned} f'(u(r(t)-,t)) &= \frac{r(t) - y_-(r(t),t)}{t} \\ &= \frac{r(t) - \alpha}{t} \\ &= f'(p). \end{aligned}$$

Hence $u(r(t)-,t) = p$ and from Rankine-Hugoniot condition across $r(t)$, we obtain $u(r(t)+,t) = p$. Again from (2.12) and (2.22) choose a sequence $y_k > r(t)$ such that

$$\lim_{k\to\infty}(y_k, y_-(y_k,t)) = (r(t), y_+(r(t),t)), f'(u(y_k,t)) = \frac{y_k - y_-(y_k,t)}{t}$$

letting $k \to \infty$ to obtain

$$\frac{r(t)-\alpha}{t} = f'(p) = f'(u(r(t)+,t)) = \frac{r(t) - y_+(r(t),t)}{t},$$

and hence $y_+(r(t),t) = \alpha$. This proves the lemma.

Lemma 3.3 *1. Suppose for all $t > 0$, $y_-(t,\alpha) = \alpha$, then there exists p_α such that*

$$R_-(t,\alpha) = \alpha + tf'(p_\alpha). \tag{3.27}$$

2. Suppose for all $t > 0$, $Y_+(t,\alpha) = \alpha$, then there exists q_α such that

$$R_+(t,\alpha) = \alpha + tf'(q_\alpha). \tag{3.28}$$

3. $y_-(t,\alpha)$, $Y_-(t,\alpha)$, are non-increasing functions and $y_+(t,\alpha)$, $Y_+(t,\alpha)$ are non-decreasing functions of t. Moreover we have

Boundary estimates: *Here we take $\alpha = A_1$ or A_2. Let*

$$\lim_{t\to\infty}(y_+(t), Y_-(t)) = (B_1, B_2). \tag{3.29}$$

Then (i). $A_1 \le B_1 \le B_2 \le A_2$ and following limits exist.

$$f'(u_-) = \lim_{t\to\infty}\frac{L(t) - y_+(t)}{t} \tag{3.30}$$

$$f'(u_+) = \lim_{t\to\infty}\frac{R(t) - Y_-(t)}{t}. \tag{3.31}$$

(ii). Let $s_1 \ge 2,\ s_2 \ge 2,\ s = max(s_1, s_2)$ and $2 \le j \le s_1 - 1,\ 2 \le l \le s_2 - 1$. Suppose $f \in C^s(I\!R)$ and satisfies for all j and l,

$$f^{(j)}(u_-) = f^{(l)}(u_+) = 0, f^{(s_1)}(u_-) \neq 0, f^{(s_2)}(u_+) \neq 0,$$

then

$$\left|\frac{L(t) - y_+(t)}{t} - f'(u_-)\right| \le \frac{M}{t^{1/s_1}} \tag{3.32}$$

$$\left|\frac{R(t) - Y_-(t)}{t} - f'(u_+)\right| \le \frac{M}{t^{1/s_2}}, \tag{3.33}$$

where M depends only on $[A_1, A_2], \|\bar{u}_0\|_\infty$.

(iii). Define $\Gamma_1(t) = B_1 + tf'(u_-), \Gamma_2(t) = B_2 + tf'(u_+)$, *then for all* $t > 0$,

$$\Gamma_1(t) \leq \Gamma_2(t). \tag{3.34}$$

Furthermore

$$y_-(\Gamma_1(t), t) = B_1, Y_+(\Gamma_2(t), t) = B_2 \tag{3.35}$$

$$\Gamma_1(t) = R_-(t, B_1), \Gamma_2(t) = R_+(t, B_2). \tag{3.36}$$

Let $a \in (A_1, B_1), b \in (B_2, A_2)$ *then there exist* $T_a > 0, T_b > 0$ *such that for all* $t > T_a, s > T_b$

$$R_\pm(t, a) = L(t) \tag{3.37}$$

$$R_\pm(s, b) = R(s). \tag{3.38}$$

Proof. It is enough to prove (3.27) and (3.28) follows similarly. Let $t_0 > 0$ and since $y_-(t_0, \alpha) = \alpha$ and hence $\gamma(\theta) = \alpha + \left(\frac{R_-(t_0,\alpha)-\alpha}{t_0}\right)\theta$ is the left extreme characteristic at $(R_-(t_0, \alpha), t_0)$. Let $0 < t < t_0$, then $\gamma|_{[0,t]}$ is a characteristic and hence $R_-(t, \alpha) \leq \gamma(t)$. Suppose for some $0 < t_1 < t_0, R_-(t_1, \alpha) < \gamma(t_1)$. Then from (2.22) we can choose a sequence $x_k < R_-(t_0, \alpha)$ such that

$$y_+(x_k, t_0) = y_-(x_k, t_0) < \alpha, \lim_{k\to\infty}(x_k, y_+(x_k, t_0)) = (R_-(t_0, \alpha), \alpha).$$

Since $y_-(t_1, \alpha) = y_-(R_-(t_1, \alpha), t_1) = \alpha$ and hence for k large the left extreme characteristic at (x_k, t_0) intersect with extreme characteristics at $R((t_1, \alpha), t_1)$ which contradicts NIP. Hence $R_-(t, \alpha) = \gamma(t)$ for $0 < t < t_0$ and t_0 is arbitrary implies that $p_\alpha = \frac{R_-(t_0,\alpha)-\alpha}{t_0}$ is independent of t_0. This proves (3.27).

(3). It is enough to prove for $y_-(t, \alpha)$ and all other cases follow similarly. Suppose for $0 < t_1 < t_2,\ \gamma_-(0) = y_-(R_-(t_1, \alpha), t_1) < y_-(R_-(t_2, \alpha), t_2) \leq \alpha \leq \gamma_+(0)$ where $\gamma_\pm$ are the extreme characteristic at $(R_-(t_1, \alpha), t_1)$. From (2.22), choose $x_1 < x_2 < R_-(t_2, \alpha)$ such that $y(x_i, t_2) = y_\pm(x_i, t_2)$ for $i = 1, 2$ and

$$\gamma_-(0) < y(x_1, t_2) \leq y(x_2, t_2) < y_-(R_-(t_2, \alpha), t_2) \leq \gamma_+(0). \tag{3.39}$$

Let r_1, r_2 be the characteristics at $(x_1, t_2), (x_2, t_2)$ respectively. Then from (3.39), one of the characteristics r_1, r_2 intersect $\gamma_\pm(\theta)$ in $(0, t_1)$ contradicting NIP. Hence $y_-(t, \alpha)$ is a non increasing function.

First we claim that $y_+(t) \leq A_2$. Suppose not, then there exist $t_0 > 0$ such that $y_+(t_0) > A_2$. Since $Y_-(t_0) \leq A_2$ and $L(t_0) < R(t_0)$, therefore the characteristic joining $(L(t_0), t_0), (y_+(t_0), t_0)$ and $(R(t_0), t_0), (Y_-(t_0), t_0)$ intersects, which contradicts NIP. This proves the claim. Similarly $A_1 \leq Y_-(t) \leq A_2$. Again from NIP and $L(t) < R(t)$, we have $y_+(t) \leq Y_-(t)$ for all $t > 0$. Therefore $A_1 \leq B_1 \leq B_2 \leq A_2$.

From (3.10) and (3.21), we have at $x = L(t)$,

$$\begin{aligned} u_-(y_-(t) - A_1) + tf^*\left(\frac{L(t) - y_-(t)}{t}\right) &= \int_{A_1}^{y_+(t)} u_0(\theta)d\theta + tf^*\left(\frac{L(t) - y_+(t)}{t}\right) \\ &- \left(\frac{L(t) - y_-(t)}{t}\right)u_- + f^*\left(\frac{L(t) - y_-(t)}{t}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{t}\int_{A_1}^{y_+(t)} u_0(\theta)d\theta + \left(\frac{A_1 - y_+(t)}{t}\right)u_- - \left(\frac{L(t) - y_+(t)}{t}\right)u_- \\ &+ f^*\left(\frac{L(t) - y_+(t)}{t}\right) - f'(u_-)u_- + f^*(f'(u_-)) \\ &= O\left(\frac{1}{t}\right) - \left(\frac{L(t) - y_+(t)}{t}\right)u_- + f^*\left(\frac{L(t) - y_+(t)}{t}\right). \end{aligned}$$

Let $f'(p_t) = \frac{L(t)-y_+(t)}{t}$, then from (2.1) and above equality we have

$$f(u_-) - f(p_t) - (u_- - p_t)f'(p_t) = O(\frac{1}{t}). \tag{3.40}$$

That is

$$(p_t - u_-)^2 \int_0^1 (1 - \theta)f''(p_t + \theta(u_- - p_t))d\theta = O(\frac{1}{t}). \tag{3.41}$$

From (3.9) we have $A_1 + f'(k_1)t \leq L(t) \leq A_2 + f'(k_2)t$ and hence

$$\frac{A_1 - y_+(t)}{t} + f'(k_1) \leq f'(p_t) = \frac{L(t) - y_+(t)}{t} \leq \frac{A_2 - y_+(t)}{t} + f'(k_2).$$

Therefore $\{p_t\}$ is bounded. Let $t_k \to \infty$ such that $p_{t_k} \to p$, then from (3.41) if $p \neq u_-$, then

$$\int_0^1 (1 - \theta)f''(p + \theta(u_- - p))d\theta = 0,$$

and hence $f(p+\theta(u_- - p))$ is linear in $\theta \in [0, 1]$ which contradicts the strict convexity of f. Hence $p = u_-$ and

$$\lim_{t\to\infty} \frac{L(t) - y_+(t)}{t} = f'(u_-). \tag{3.42}$$

This proves (3.30) and similarly (3.31) follows.

Let $f^{(j)}(u_-) = 0$ for $2 \le j \le s_1 - 1$ and $f^{(s_1)}(u_-) \neq 0$. Let $T_0 > 0$ such that

$$0 < C = \sup_{t > T_0} \frac{1}{(s_1-1)!} \left| \int_0^1 (1-\theta)^{s_1-1} f^{(s_1)}(p_t + \theta(u_- - p_t)) d\theta \right|. \tag{3.43}$$

From (3.40) and by Taylor's series expansion

$$(u_- - p_t)^{s_1} \frac{1}{(s_1-1)!} \int_0^1 (1-\theta)^{s_1-1} f^{(s_1)}(p_t + \theta(u_- - p_-)) d\theta = O\left(\frac{1}{t}\right)$$

and therefore from (3.43) for $t \ge T_0$,

$$|p_t - u_-| = O\left(\frac{1}{t^{\frac{1}{s_1}}}\right)$$

or

$$|f'(p_t) - f'(u_-)| \le M/t^{\frac{1}{s_1}},$$

where M depends only on $\|\bar{u}_0\|_\infty$ and $[A_1, A_2]$. This proves (3.32). Similarly (3.33) follows.

Next consider the extreme right characteristic $r_+(\theta, t) = y_+(t) + \theta f'(p_t)$ at $(L(t), t)$ and for $0 < t_0 < t$, define $x(t_0, t) = r_+(t_0, t)$. Then $r_+(\cdot, t)|_{[0,t_0]}$ is a characteristic and is given by

$$x(t_0, t) = y_+(t) + t_0 f'(p_t)$$
$$r_+(\theta, t) = x(t_0, t) + (\theta - t_0) f'(p_t).$$

Let $t_2 > t_1 > t_0$, since $y_+(t)$ is a non decreasing function, hence from NIP we have $x(t_0, t_1) \le x(t_0, t_2)$ and therefore for $0 \le \theta \le t_0$

$$\begin{aligned} x(t_0) &= \lim_{t\to\infty} x(t_0, t) \\ &= B_1 + t_0 f'(u_-) = \Gamma_1(t_0). \end{aligned}$$

$$\begin{aligned} \lim_{t\to\infty} r_+(\theta, t) &= x(t_0) + (\theta - t_0) f'(u_-) \\ &= B_1 + \theta f'(u_-) \\ &= \Gamma_1(\theta). \end{aligned} \tag{3.44}$$

From (2) of Lemma (4.2), $\Gamma_1(\theta)$ is a characteristic at $(\Gamma_1(t_0), t_0)$ for all $t_0 > 0$ and $\Gamma_1(0) = B_1$. Hence $y_-(\Gamma_1(t_0), t_0) \le B_1$. If $y_-(\Gamma_1(t_0), t_0) < B_1$, then from (3.29) for large $t, r_+(\cdot, t)$ intersect the extreme left characteristic at $(\Gamma_1(t_0), t_0)$ contradicting NIP. Hence $y_-(\Gamma_1(t_0), t_0) = B_1$, and from (3.22),(3.24) $R_-(t_0, B_1) = \Gamma_1(t_0)$. This proves (3.35) and (3.36).

Let $a \in (A_1, B_1)$ and choose $T_a > 0$ such that for $t \geq T_a, y_+(t) \geq y_+(T_a) > a$. Suppose for some $t > T_a, R_-(t,a) > L(t)$, then $a \geq y_-(t,a) \geq y_+(t) > a$, which is a contradiction. Hence $R_-(t,a) = L(t)$ for all $t \geq T_a$. This proves (3.37) and (3.38) follows similarly. This proves the Lemma.

Behaviour of $\mathbf{R_\pm(t,\alpha)}$ for $\alpha \in (\mathbf{B_1, B_2})$: From Monotonicity of $y_\pm$ and from (3.34) we have

$$\begin{aligned} \Gamma_1(t) = R_-(t,B_1) \leq R_+(t,B_1) &\leq R_-(t,\alpha) \leq R_+(t,\alpha) \\ &\leq R_-(t,B_2) \leq R_+(t,B_2) = \Gamma_2(t). \end{aligned} \tag{3.45}$$

$$B_1 \leq y_-(t,\alpha) \leq Y_-(t,\alpha) \leq \alpha \leq y_+(t,\alpha) \leq Y_+(t,\alpha) \leq B_2. \tag{3.46}$$

Next define the regular points as follows:

Definition 3.4. *Let $\alpha \in [B_1, B_2]$, then*

(i) α is said to be a left regular point if for all $t > 0$,

$$y_-(R_-(t,\alpha),t) = \alpha.$$

(ii) α is said to be a right regular point if for all $t > 0$,

$$y_+(R_+(t,\alpha),t) = \alpha.$$

(iii) α is said to be a regular point if for all $t > 0$,

$$y_-(R_-(t,\alpha),\alpha) = y_+(R_+(t,\alpha),\alpha) = \alpha.$$

Definition 3.5. (characteristic Line): Let $r(t) = \alpha + tf'(p)$ be a straight line. r is called a characteristic line if $y_-(r(t),t) = \alpha$ for all $t > 0$ or $y_+(r(t),t) = \alpha$ for all $t > 0$.

Definition 3.6. *u_0 is said to be mildly oscillating at a if there exist $r_1 < a < r_2$ such that u_0 is monotone in $(r_1,a) \cup (a,r_2)$.*

Definition 3.7. Let $\alpha < \beta$ and u_0 is said to have same parity at α, β, if there exist $\alpha < r_1 < r_2 < \beta$ such that

$$u_0(\alpha) = u_0(\beta) \tag{3.47}$$

and u_0 is a non decreasing function in $(\alpha, r_1) \cup (r_2, \beta)$.

Lemma 3.8 *Let $\alpha \in I\!R$ be such that either $\{Y_\pm(t,\alpha)\}$ is bounded or $\{y_\pm(t,\alpha)\}$ is bounded for all $t > 0$. Then there exist $A_{1,\alpha} \leq A_{2,\alpha}$ and p_α and denote for $i = 1, 2$*

$$r_{i,\alpha}(t) = A_{i,\alpha} + tf'(p_\alpha) \tag{3.48}$$

$$D_\alpha = \{(x,t) : r_{1,\alpha}(t) < x < r_{2,\alpha}(t)\}, \tag{3.49}$$

then

1. *Following limit exist. If* $\{y_\pm(t,\alpha))$ *is bounded then*

$$(A_{1,\alpha}, A_{2,\alpha}) = \lim_{t\to\infty}(y_-(t,\alpha), y_+(t,\alpha)). \tag{3.50}$$

   If $\{Y_\pm(t,\alpha)\}$ *is bounded, then*

$$(A_{1,\alpha}, A_{2,\alpha}) = \lim_{t\to\infty}(Y_-(t,\alpha), Y_+(t,\alpha)). \tag{3.51}$$

2. $A_{1,\alpha} \le \alpha \le A_{2,\alpha}$ *and for* $i = 1, 2, r_{i,\alpha}$ *are characteristic lines.*

3. *Let* $A_{1,\alpha} < \beta_1 \le \beta_2 < A_{2,\alpha}$*, then* β_i *are neither a left nor right regular point and there exist* $T > 0$ *such that for all* $t > T$.

$$R_-(t,\alpha) = R_-(t,\beta_1) = R_+(t,\beta_2) \tag{3.52}$$

$$y_\pm(t,\beta_1) = Y_\pm(t,\beta_2). \tag{3.53}$$

$$\lim_{t\to\infty}(y_-(t,\beta_1), y_+(t,\beta_2)) = (A_{1,\alpha}, A_{2,\alpha}) \tag{3.54}$$

$$D_{\beta_1} = D_{\beta_2} = D_\alpha. \tag{3.55}$$

   Furthermore D_α *does not contain a characteristic line and for any* α, β *either* $D_\alpha = D_\beta$ *or* $D_\alpha \cap D_\beta = \phi$.

4. *If* α *is a left regular point then* $A_{1,\alpha} = \alpha$*. If* α *is a right regular point, then* $A_{2,\alpha} = \alpha$.

5. *For all* $t > 0$,

$$\int_{A_{1,\alpha}}^{A_{2,\alpha}} \bar{u}_0(x)dx = \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x,t)dx \tag{3.56}$$

$$\int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} |f'(u(x,t)) - f'(p_\alpha)| \le \frac{2(A_{2,\alpha} - A_{1,\alpha})}{t}, \tag{3.57}$$

   and

$$\int_{A_{1,\alpha}}^{A_{2,\alpha}} \bar{u}_0(x)dx = p_\alpha(A_{2,\alpha} - A_{1,\alpha}). \tag{3.58}$$

6. *If* $\bar{u}_0$ *is continuous in neighbourhoods of* $\{A_{1,\alpha}, A_{2,\alpha}\}$ *then*

$$\bar{u}_0(A_{1,\alpha}) = \bar{u}_0(A_{2,\alpha}) = p_\alpha. \tag{3.59}$$

7. *If* $\bar{u}_0$ *is mildly oscillatory at* $A_{1,\alpha}$ *and* $A_{2,\alpha}$ *then* $\bar{u}_0$ *have the same parity at* $A_{1,\alpha}, A_{2,\alpha}$.

8. *If* $\alpha \in [B_1, B_2]$, *then* $\{y_\pm(t,\alpha)\}, \{Y_\pm(t,\alpha)\}$ *are bounded and*

$$B_1 \leq A_{1,\alpha} \leq \alpha \leq A_{2,\alpha} \leq B_2 \tag{3.60}$$

$$u_- \leq p_\alpha \leq u_+. \tag{3.61}$$

Proof. Without loss of generality we can assume that $\{y_\pm(t,\alpha)\}$ are bounded. Similar arguments follow if $\{Y_\pm(t,\alpha)\}$ are bounded. Then from (3) of Lemma (3.2), $y_-(t,\alpha), y_+(t,\alpha)$ are non increasing and non decreasing functions of t and $y_-(t,\alpha) \leq \alpha \leq y_+(t,\alpha)$. Hence

$$(A_{1,\alpha}, A_{2,\alpha}) = \lim_{t\to\infty}(y_-(t,\alpha), y_+(t,\alpha))$$

exist and $A_{1,\alpha} \leq \alpha \leq A_{2,\alpha}$. If α is a left regular point then $y_-(t,\alpha) = \alpha$ and hence $A_{1,\alpha} = \alpha$. If α is a right regular point. Then one must, take $Y_\pm(t,\alpha)$ and $A_{2,\alpha} = Y_+(t,\alpha)]$.

Case (1): Let α be a regular point. Then $\alpha = y_-(t,\alpha) \leq Y_+(t,\alpha) = \alpha$. Hence $A_{1,\alpha} = A_{2,\alpha}$ and from Lemma (3.3), there exist p_α such that

$$r_{1,\alpha}(t) = r_{2,\alpha}(t) = R_-(t,\alpha) = R_+(t,\alpha) = \alpha + tf'(p_\alpha), \tag{3.62}$$

is a characteristic line.

Case (2): Let α not a regular point. Without loss of generality we can assume that α is not a right regular point. Hence there exist $T > 0$ such that for all $t > T, A_{1,\alpha} \leq y_-(t,\alpha) \leq \alpha < y_+(t,\alpha) \leq A_{2,\alpha}$. Let $r_{\pm,t}(\theta)$ be the extreme characteristics at $(R_-(t,\alpha), t)$ and

$$f'(p_{\pm,t}) = \frac{R_-(t,\alpha) - y_\pm(t,\alpha)}{t}$$

$$r_{\pm,t}(\theta) = y_\pm(t,\alpha) + \theta f'(p_{\pm,t}).$$

Since $\{y_\pm(t,\alpha)\}$ are bounded and from (4) of Lemma (2.1), there exist a $c > 0$ such that

$$\left|\frac{R_-(t,\alpha) - y_\pm(t,\alpha)}{t}\right| \leq c.$$

Hence $\{f'(p_{\pm,t}\}$ is bounded as $t \to \infty$.

Claim 1: Following limits exist and are equal

$$\lim_{t\to\infty} f'(p_{\pm,t}) = f'(p_{\pm,\alpha}) \tag{3.63}$$

$$p_{+,\alpha} = p_{-,\alpha} = p_\alpha \tag{3.64}$$

and $r_{1,\alpha}, r_{2,\alpha}$ are characteristic lines.

Let $0 < t_0 < t, x(t_0,t) = r_{+,t}(t_0)$, then for $0 < \theta < t_0, r_{+,t}(\theta)$ is a characteristic at $(x(t_0,t), t_0)$ and is given by

$$x(t_0,t) = y_+(t,\alpha) + t_0 f'(p_{+,t}) \tag{3.65}$$

$$r_{+,t}(\theta) = x(t_0,t) + (\theta - t_0) f'(p_{+,t}). \tag{3.66}$$

Let $t_2 > t_1 > t_0$, then from (3) of Lemma (3.3) we have $y_+(t_2,\alpha) \geq y_+(t_1,\alpha) \geq y_+(t_0,\alpha)$ and hence from NIP, $x(t_0,t_2) \geq x(t_0,t_1)$. Since $f'(p_{+,t})$ is bounded as $t \to \infty$ and hence from (3.65) $x(t_0,t)$ is bounded as $t \to \infty$. Let $x(t_0) = \lim\limits_{t\to\infty} x(t_0,t)$. Let $t_2 > t_1 > t_0$, then from (3.65)

$$\begin{aligned} f'(p_{+,t_2}) - f'(p_{-,t_2}) &= \frac{x(t_0,t_2) - y_+(t_2,\alpha)}{t_0} - \frac{x(t_0,t_1) - y_+(t_1,\alpha)}{t_0} \\ &= \frac{x(t_0,t_2) - x(t_0,t_1)}{t_0} + \frac{y_+(t_1,\alpha) - y_+(t_2,\alpha)}{t_0} \\ &\to 0 \text{ as } t_2, t_1 \to \infty. \end{aligned}$$

Therefore $p_{+,\alpha}$ exist and similarly $p_{-,\alpha}$ exist and this proves (3.63). From (2) of Lemma (2.1), limit of a characteristic is a characteristics, hence by taking $t \to \infty$ in (3.65) and (3.66) to obtain

$$x(t_0) = A_{2,\alpha} + t_0 f'(p_{+,\alpha}). \tag{3.67}$$

$$\begin{aligned} \lim_{t\to\infty} r_{+,t}(\theta) &= x(t_0) + (\theta - t_0) f'(p_{+,\alpha}) \\ &= A_{2,\alpha} + \theta f'(p_{+,\alpha}) \\ &= r_{2,\alpha}(\theta). \end{aligned} \tag{3.68}$$

Suppose $y_-(r_{2,\alpha}(t_0), t_0) < A_{2,\alpha}$, then choose $t_1 > t_0$ such that $y_-(r_{2,\alpha}(t_0), t_0) < y_+(t_1,\alpha) \leq A_{2,\alpha}$ and hence r_{+,t_1} intersects the extreme left characteristic at $(r_{2,\alpha}(t_0), t_0$) contradicting NIP. Hence $y_-(r_{2,\alpha}(t), t) = A_{2,\alpha}$ and therefore $r_{2,\alpha}$ is a characteristic line. Similarly $r_{1,\alpha}$ is a characteristic line. Since $y_-(t,\alpha) \leq \alpha \leq y_+(t,\alpha)$ and therefore

$$\begin{aligned} f'(p_{-,\alpha}) &= \lim_{t\to\infty} \frac{R_-(t,\alpha) - y_-(t,\alpha)}{t} \\ &\geq \lim_{t\to\infty} \frac{R_-(t,\alpha) - y_+(t,\alpha)}{t} \\ &= f'(p_{+,\alpha}). \end{aligned}$$

Suppose $f'(p_{+,\alpha}) < f'(p_{-,\alpha})$, then $r_{1,\alpha}$ and $r_{2,\alpha}$ intersects at t_0 given by

$$t_0 = \frac{A_{2,\alpha} - A_{1,\alpha}}{f'(p_{-,\alpha}) - f'(p_{+,\alpha})} > 0.$$

Let $t > t_0$, then $r_{1,\alpha}$ and $r_{2,\alpha}$ intersects at $t_0 \in [0,t]$ contradicting NIP, since $r_{i,\alpha}$ are characteristic lines for $i = 1, 2$. Hence $p_{+,\alpha} = p_{-,\alpha}$ and this proves the claim.

Claim 2: Let $A_{1,\alpha} < \beta < A_{2,\alpha}$, then there exist $T > 0$ such that for all $t > T$

$$R_-(t,\beta) = R_+(t,\beta) = R(t,\alpha). \tag{3.69}$$

Choose $T > 0$ such that for $t > T$

$$y_-(t,\alpha) \le y_-(T,\alpha) < \beta < y_+(T,\alpha) \le y_+(t,\alpha). \tag{3.70}$$

Suppose for some $t_0 > T, R_-(t_0,\beta) < R_-(t_0,\alpha)$, then from (2.22), (3.70), $\beta \le y_+(R_-(t_0,\beta),\beta) \le y_-(R_-(t_0,\alpha),\alpha) = y_-(t_0,\alpha) < \beta$ which is a contradiction. Suppose $R_+(t_0,\beta) > R_-(t_0,\alpha)$, then from (2.23) and (3.70) $\beta < y_+(t_0,\alpha) \le y_-(R_+(t_0,\beta),t_0) \le \beta$ which is a contradiction. Hence $R_-(t,\beta) = R_-(t,\alpha)$ for $t > T$. Similarly $R_+(t,\beta) = R_+(t,\alpha)$ for $t \ge T$ and this proves the claim.

Claim 3: D_α does not contain a characteristic line.

Suppose $r(\theta) = \beta + \theta f'(a)$ be a characteristic line in D_α. Then $A_{1,\alpha} + \theta f'(p_\alpha) <$ $r(\theta)$
$< A_{2,\alpha} + \theta f'(p_\alpha)$ for all θ. Hence $p_\alpha = p$ and $A_{1,\alpha} < \beta < A_{2,\alpha}$ and $R_-(t,\beta) \le r(t) \le R_+(t,\beta)$. From (3.69), there exist $T > 0$ such that for $t \ge T, r(t) = R_-(t,\alpha)$ and $\beta = y_-(r(t),t) = y_-(t,\alpha) \to A_{2,\alpha}$ as $t \to \infty$, which is a contradiction and hence the claim.

(2) and (4) follows from claim 1. Let $A_{1,\alpha} < \beta < A_{2,\alpha}$ and suppose β is left regular point, then from (3.27), D_α contains a characteristic line which contradicts claim 3. Similarly β cannot be right regular point. Let $D_\alpha \cap D_\beta \neq \phi$. Since ∂D_α consists of characteristic lines therefore D_β contains a characteristic line which is a contradiction. Hence $\partial D_\alpha \cap D_\alpha = \phi$ and similarly $\partial D_\beta \cap D_\alpha = \phi$. This shows that $D_\alpha = D_\beta$.

(3.52) to (3.55) follows from claim (2). This proves (3).

Let $t_0 > 0$, then from (2.22) and (2.12) choose a sequence $x_k \uparrow r_{2,\alpha}(t_0)$ such that

$$f'(u(r_{2,\alpha}(t_0)-,t_0) = \lim_{k\to\infty} \frac{x_k - y_+(x_k,t_0)}{t_0}. \tag{3.71}$$

From (3.65) to (3.68) choose a $t_k \to \infty$ such that $y_+(t_k, \alpha) \leq y_+(x_k, t_0) \leq y_+(t_{k+1}, \alpha)$ as $k \to \infty$. Hence from (3.71) we have

$$\begin{aligned} f'(u(r_{2,\alpha}(t_0)-, t_0) &= \lim_{k \to \infty} \frac{x_k - y_+(t_k, \alpha)}{t_0} \\ &= \frac{r_{2,\alpha}(t_0) - A_{2,\alpha}}{t_0} \\ &= f'(p_\alpha). \end{aligned}$$

Hence $u(r_{2,\alpha}(t_0)-, t_0) = p_\alpha$. Similarly at $r_{1,\alpha}(t_0)$. Hence for all $t_0 > 0$,

$$u(r_{1,\alpha}(t_0)+, t_0) = u(r_{2,\alpha}(t_0)-, t_0) = p_\alpha. \tag{3.72}$$

Let $D_\alpha(t_0) = \{(x, t) \in D_\alpha; 0 < t < t_0\}$, then from (3.72) we have

$$0 = \int_{D_\alpha(t_0)} (u_t + f(u)_x) dx dt = \int_{r_{1,\alpha}(t_0)}^{r_{2,\alpha}(t_0)} u(x, t_0) dx - \int_{A_{1,\alpha}}^{A_{2,\alpha}} \bar{u}_0(x) dx.$$

This proves (3.56).

From (2.12) we have

$$\begin{aligned} \int_{r_{1,\alpha}(t_0)}^{r_{2,\alpha}(t_0)} |f'(u(x,t)) - f'(p_\alpha)| dx &= \int_{r_{1,\alpha}(t_0)}^{r_{2,\alpha}(t_0)} \left| \frac{x - y_+(x, t_0)}{t_0} - f'(p_\alpha) \right| \\ &= \int_{A_{1,\alpha} + t_0 f'(p_\alpha)}^{A_{2,\alpha} + t_0 f'(p_\alpha)} \left| \frac{x - t_0 f'(p_\alpha)}{t_0} - \frac{y_+(x, t_0)}{t_0} \right| dx. \end{aligned}$$

By change of variables $x \to x - t_0 f'(p_\alpha)$ and observing $A_{1,\alpha} \leq y_+(x, t_0) \leq A_{2,\alpha}$ to obtain

$$\int_{r_{1,\alpha}(t_0)}^{r_{2,\alpha}(t_0)} |f'(u(x,t) - f'(p_\alpha)| dx \leq \frac{2(A_{2,\alpha} - A_{1,\alpha})}{t_0}.$$

This proves (3.57).

From (3.56) to (3.57) we have for $t > 0$,

$$\begin{aligned}
\left|\int_{A_{1,\alpha}}^{A_{2,\alpha}} \bar{u}_0(x)dx\right| &= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x,t)dx \\
&= \int_{A_{1,\alpha}+tf'(p_\alpha)}^{A_{2,\alpha}+tf'(p_\alpha)} (f')^{-1}\left(\frac{x-y_+(x,t)}{t}\right)dx \\
&= \int_{A_{1,\alpha}}^{A_{2,\alpha}} (f')^{-1}\left(\frac{\xi - y_+(\xi+tf'(p_\alpha),t)}{t} + f'(p_\alpha)\right)d\xi \\
&\rightarrow p_\alpha(A_{2,\alpha} - A_{1,\alpha}) \text{ as } t \rightarrow \infty
\end{aligned}$$

this proves (3.58) and hence (5).

Let $\bar{u}_0$ is continuous in a neighbourhood of $A_{1,\alpha}$ then v_0 is differentiable in the same neighbourhood. Let $t_0 > 0$ and $x(t_0,t)$ be as in (3.65). Then from (2.13) for a.e., $r_{1,\alpha}(t_0) < x < x(t_0,t)$ and t large, $u(x,t) = u_0(y_+(x,t))$. Hence letting $t \rightarrow \infty$ and from (3.72)

$$\begin{aligned}
p_\alpha = u(r_{1,\alpha}(t_0)+, t_0) &= \lim_{t\rightarrow\infty} u_0(y_+(x,t)) \\
&= u_0(A_{1,\alpha}).
\end{aligned}$$

Similarly $p_\alpha = u_0(A_{2,\alpha})$. This proves (6).

Let $\bar{u_0}$ be mildly oscillatory at $A_{1,\alpha}$ and $A_{2,\alpha}$. Suppose they are not of same parity. Therefore assume that there exist $A_{1,\alpha} < r_1 < r_2 < A_{2,\alpha}$ such that u_0 is a non decreasing function in $(A_{1,\alpha}, r_1)$ and u_0 is non increasing function in $(r_2, A_{2,\alpha})$. Then from (3.52) choose $T > 0$ such that for all $t > T$.

$$R_-(t, r_1) = R_+(t, r_2) = R_-(t, \alpha).$$

Since $\bar{u}_0$ is continuous in $[A_{1,\alpha}, A_{2,\alpha}]$ and hence from (2.13), for a.e. $x \in (r_{1,\alpha}(t), R_-(t, \alpha))$ and $z \in (R_-(t,\alpha), r_{2,\alpha}(t))$

$$u(x,t) = \bar{u}_0(y_+(x,t)), u(z,t) = \bar{u}_0(y_+(z,t))$$

$$y_+(x,t) \in (A_{1,\alpha}, r_1), y_+(z,t) \in (r_2, A_{2,\alpha}).$$

Hence from (3.59), we have

$$p_\alpha = \bar{u}_0(A_{1,\alpha}) \leq \bar{u}_0(y_+(x,t)) = u(x,t) \tag{3.73}$$

$$p_\alpha = \bar{u}_0(A_{2,\alpha}) \leq \bar{u}_0(y_+(z,t)) = u(z,t). \tag{3.74}$$

Claim 4: There exist an $r > 0$ such that either $\bar{u}_0 > p_\alpha$ for $x \in (A_{1,\alpha}, A_{1,\alpha} + r)$ or $x \in (A_{2,\alpha} - r, A_{2,\alpha})$.

Suppose not, then by shrinking r_1 and r_2 if necessary we can assume that u_0 is constant in $(A_{1,\alpha}, r_1)$ and $(r_2, A_{2,\alpha})$ and from (3.59) for all $\theta \in (A_{1,\alpha}, r_1) \cup (r_2, A_{2,\alpha})$.

$$\bar{u}_0(\theta) = p_\alpha. \tag{3.75}$$

From (3.73) , (2.12) , (2.22), (2.23), (3.74) we have

$$\begin{aligned} f'(p_\alpha) &= \lim_{x\uparrow R_-(t,\alpha)} f'(u_0(y_+(x,t))) \\ &= \lim_{x\uparrow R_-(t,\alpha)} f'(u(x,t)) \\ &= \lim_{x\uparrow R_-(t,\alpha)} \frac{x - y_+(x,t)}{t} \\ &= \frac{R_-(t,\alpha) - y_-(t,\alpha)}{t}, \end{aligned}$$

hence $R_-(t,\alpha) = y_-(t,\alpha) + tf'(p_\alpha)$. Similarly $R_-(t,\alpha) = y_+(t,\alpha) + tf'(p_\alpha)$. This implies $y_+(t,\alpha) = y_-(t,\alpha)$ for all $t > T$, which is a contradiction since $A_{1,\alpha} < A_{2,\alpha}$. This proves the claim.

As a consequence of this claim, (3.72), (3.73) we have for $t > T$,

$$\int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x,t)dx = \int_{r_{1,\alpha}(t)}^{R_-(t,\alpha)} u(x,t)dx + \int_{R_-(t,\alpha)}^{r_{2,\alpha}(t)} u(x,t)dx > p_\alpha(A_{2,\alpha} - A_{1,\alpha}). \tag{3.76}$$

On the other hand from (3.55) , (3.58) and (3.75)

$$\begin{aligned} (A_{2,\alpha} - A_{1,\alpha})p_\alpha &= \int_{A_{1,\alpha}}^{A_{2,\alpha}} \bar{u}_0(\theta)d\theta \\ &= \int_{r_{1,\alpha}(t)}^{r_{2,\alpha}(t)} u(x,t)dx \\ &> p_\alpha(A_{2,\alpha} - A_{1,\alpha}), \end{aligned}$$

which is a contradiction.

Similar proof follows if u_0 cannot be a non increasing function in $(A_{1,\alpha}, r_1)$ and non decreasing function in $(r_2, A_{2,\alpha})$.
Suppose u_0 is a non increasing function in $(A_{1,\alpha}, r_1)$ and non increasing function in $(r_2, A_{2,\alpha})$. For $t > T$ and for a.e. x, ξ with

$$A_{1,\alpha} + tf'(p_\alpha) < x < R(t,\alpha) < \xi < A_{2,\alpha} + tf'(p_\alpha),$$

we have $y_+(x,t) \in (A_{1,\alpha}, r_1)$ and $y_-(\xi,t) \in (r_2, A_{2,\alpha})$. Therefore $f'(u_0(y_+(x,t))) \leq f'(u_0(A_{1,\alpha})) = f'(p_\alpha)$ and $f'(u_0(y_-(\xi,t))) \geq f'(u_0(A_{2,\alpha})) = f'(p_\alpha)$. Hence from

(2.13),(2.22) we have

$$\begin{aligned}\frac{R(t,\alpha)-y_-(t,\alpha)}{t} &= \lim_{x\uparrow R(t,\alpha)} \frac{x-y_+(x,t)}{t}\\ &= \lim_{x\uparrow R(t,\alpha)} f'(u(x,t))\\ &= \lim_{x\uparrow R(t,\alpha)} f'(u_0(y_+(x,t)))\\ &\le f'(p_\alpha).\end{aligned}$$

Hence

$$R(t,\alpha) \le y_-(t,\alpha) + tf'(p_\alpha).$$

Similarly by letting $z \downarrow R(t,\alpha)$ to obtian $R(t,\alpha) \ge y_+(t,\alpha)+tf'(p_\alpha)$. Hence $y_+(t,\alpha) \le y_-(t,\alpha)$ for all $t > T$, which contradicts (3.50). This proves (7).

Let $B_1 \le \alpha \le B_2$, then from (3.45) and (3.46) we have

$$f'(u_-) \le \lim_{t\to\infty} \frac{R_-(t,\alpha)-y_\pm(t,\alpha)}{t} \le f'(u_+)$$

$$f'(u_-) \le \lim_{t\to\infty} \frac{R_+(t,\alpha)-Y_\pm(t,\alpha)}{t} \le f'(u_+).$$

This implies $u_- \le p_\alpha \le u_.$ This proves (8) and hence the Lemma.

Definition 3.9 (Asymptotically single shock packet (ASSP)): *Let $B_1 \le c_1 < c_2 \le B_2$ and $u_- \le p \le u_+$ are given. Denote the parallel lines at c_1 and c_2 by*

$$\begin{aligned} r(t,c_1,p) &= c_1 + tf'(p)\\ r(t,c_2,p) &= c_2 + tf'(p)\\ D(c_1,c_2,p) &= \{(x,t); c_1+tf'(p) < x < c_2 + tf'(p)\}.\end{aligned}$$

$D(c_1,c_2,p)$ is called an Asymptotically single shock packet (ASSP) if there exist $\alpha \in (B_1,B_2)$ such that $D(c_1,c_2,p) = D_\alpha$. Denote

$$ASH = \{D(c_1,c_2,p); D(c_1,c_2,p) \textit{ is ASSP }\}$$

called the Asymptotically shock set.

Definition (3.10): *Let $\alpha \in [B_1,B_2]$ is called a singular point if α is not a left or a right regular point.*

Lemma 3.11

1. *Let $D(c_1,c_2,p)$ be ASSP. Then $D(c_1,c_2,p)$ does not contain a characteristic line and every $\alpha \in (c_1,c_2)$ is a singular point.*

2. *ASH consists of countable disjoint open sets given by* $ASH = \{D_i = D(c_{1,i}, c_{2,i}, p_i)\}_{i\in I}$ *and* $\cup_{i\in I}(c_{1,i}, c_{2,i})$ *consists of all singular points in* (B_1, B_2).

3. *Let* $\Gamma_1(t_0) \le x_0 \le \Gamma_2(t_0)$.*If* $(x_0, t_0) \notin \cup_{i\in I} D_i$, *then* (x_0, t_0) *lies on a characteristic line.*

Proof. From (3) of Lemma 3.8, D_α does not contain a characteristic line. Hence if $\alpha \in (c_1, \alpha_2)$ is a left or right regular point, then D_α contains a characteristic line, which is a contradiction. This proves (1).

From Lemma 3.8 either $D_\alpha = D_\beta$ or $D_\alpha \cap D_\beta = \phi$. Hence ASH is countable. Let $\alpha \in (B_1, B_2)$ be a singular point. Then there exist $T > 0$ such that for all $t > T$, $B_1 \le y_-(t, \alpha) < \alpha < y_+(t, \alpha) \le B_2$. Hence from Lemma 3.8, D_α is an ASSP and hence for some i, $D_\alpha = D(c_{1,i}, c_{2,i}, p_i)$. This proves (2).

Let $(x_0, t_0) \notin \cup_{i\in I} D_i, \Gamma_1(t_0) \le x_0 \le \Gamma_2(t_0)$ and $\alpha = y(x_0, t_0)$. Then $\alpha \in [B_1, B_2]$ and $R_-(t_0, \alpha) \le x_0 \le R_+(t_0, \alpha)$. If α is a singular point, then from (2), $(x_0, t_0) \in D_i$ for some i, which is a contradiction. Hence α is not a singular point. Now we have to consider three cases.

Case (1) : α is regular point.

In this case $R_-(t, \alpha)$ and $R_+(t, \alpha)$ are characteristic line denoted by

$$R_\pm(t, \alpha) = \alpha + tf'(p_\pm).$$

Then from (2.27), for $R_-(t, \alpha) < x < R_+(t, \alpha)$, $f'(u(x, t)) = \frac{x-\alpha}{t}$ and hence (x_0, t_0) lies on the characteristic line $\alpha_0 + t\left(\frac{x_0-\alpha}{t_0}\right)$.

Case (2) : α is left regular but not right regular. Then from (4) of Lemma 3.8 $D_\alpha = D(\alpha, A_{2,\alpha}, p_\alpha)$ be in ASSP. Since $(x_0, t_0) \notin D_\alpha$ and hence

$$R_-(t_0, \alpha) \le x_0 \le \alpha + t_0 f'(p_\alpha) = r(t_0) \le R_+(t_0, \alpha)$$

and $\alpha + tf'(p_\alpha)$ is a characteristic line. Since α is left regular point, hence $R_-(t, \alpha)$ is a characteristic line. Hence from (2.21), $f'(u(x_0, t_0)) = \frac{x_0-\alpha}{t}$ and $\alpha + \theta(\frac{x-\alpha}{t_0})$ is a characteristic line. Hence (x_0, t_0) lies on a characteristic line. Similar reasoning follows if α is right regular but not left regular. This proves the Lemma.

Next we decompose the region $\Omega = R \times (0, \infty)$ into several parts such that the behavior of the solution u is well understood.

Define $F_\pm$ the region of constants, $D_\pm$ the rarefaction regions, S the region of shocks, R the regular region, E the effective region where shocks and rarefaction occurs.
Let $\{D_i = D(c_{1,i}, c_{2.i}, p_i)\}$ be the collection of all ASSP. Then define

$$F_- = \{(x, t) : x \le R_-(t, A_1) = L(t)\} \tag{3.77}$$

$$F_+ = \{(x,t) : x \geq R_+(t, A_2) = R(t)\} \tag{3.78}$$

$$D_- = \{(x,t) : L(t) < x < \Gamma_1(t)\} \tag{3.79}$$

$$D_+ = \{(x,t) : \Gamma_2(t) < x < R(t)\} \tag{3.80}$$

$$E = \{(x,t); \Gamma_1(t) < x < \Gamma_2(t)\} \tag{3.81}$$

$$S = \cup_i D_i, R = E \backslash S. \tag{3.82}$$

They are mutually disjoint and have the following decomposition holds:

1. $\Omega = I\!R \times (0,\infty) = F_- \cup F_+ \cup D_- \cup D_+ \cup E.$
2. $E = S \cup R.$
3. $F_\pm, R$ are closed sets and $D_\pm, S$ are open sets.

From Lemma (3.11) if $(x,t) \in R$, then (x,t) lies on a characteristic line denoted by $r_{x,t}(\theta)$ and is given by

$$\begin{aligned} r_{x,t}(\theta) &= y_+(x,t) + \theta f'(p_{x,t}) \\ &= x + (\theta - t) f'(p_{x,t}). \end{aligned} \tag{3.83}$$

Now define the N wave by

$$f'(N(x,t)) = \begin{cases} f'(u_-) & \text{if} \quad (x,t) \in F_-, \\ f'(u_+) & \text{if} \quad (x,t) \in F_+, \\ \frac{x-B_1}{t} & \text{if} \quad (x,t) \in D_-, \\ \frac{x-B_2}{t} & \text{if} \quad (x,t) \in D_+, \\ f'(p_i) & \text{if} \quad (x,t) \in D(c_{1,i}, c_{2,i}, p_i), \\ f'(p_{x,t}) & \text{if} \quad (x,t) \in R. \end{cases} \tag{3.84}$$

For $G \subset R \times (0,\infty), t > 0$, define the t section of G by

$$G_t = \{x : (x,t) \in G\}. \tag{3.85}$$

Then we have the following main result:

Theorem 3.12 *Let u be the solution of (1.2) with initial data $\bar{u}_0$ given by (3.8). With the notations as above we have*

(1). (i). u is a single shock solution if and only if $u_- > u_+$ and there exist $T > 0$ such that u is given by

$$u(x,t) = \begin{cases} u_- & \textit{if} \quad x < S(t), \\ u_+ & \textit{if} \quad x > S(t), \end{cases} \tag{3.86}$$

where

$$S(t) = L(T) + \left(\frac{f(u_+) - f(u_-)}{u_+ - u_-}\right)(t - T).$$

(ii). Let $u_- \le u_+$ *and for* $t > 0$, $f'(p_{-,t}) = \frac{L(t)-y_+(t)}{t}, f'(p_{+,t}) = \frac{R(t)-Y_-(t)}{t}$, *then*

$$\lim_{t\to\infty}(p_{-,t}, p_{+,t}) = (u_-, u_+) \tag{3.87}$$

$$f'(p_{-,t}) + \frac{y_+(t) - B_1}{t} \le f'(N(x,t)) \le f'(u_-) \quad \text{if } (x,t) \in D_- \tag{3.88}$$

$$f'(p_{+,t}) + \frac{y_+(t) - B_2}{t} \ge f'(N(x,t) \ge f'(u_+) \quad \text{if } (x,t) \in D_+. \tag{3.89}$$

(2). Suppose there exist $s_1 \ge s, s_2 \ge 2, 2 \le j \le s_1 - 1, 2 \le l \le s_2 - 1$, *such that* $f \in C^s(I\!R)$, $s = max(s_1, s_2)$ *and satisfying for all* j *and* l,

$$f^{(j)}(u_-) = f^{(l)}(u_+) = 0,$$
$$f^{(s_1)}(u_-) \ne 0, f^{(s_2)}(u_+) \ne 0,$$

then there exist a $M_1 > 0$ *such that for* $t \ge 1$,

$$\int_{D_{-,t}} |f'(u(x,t) - f'(N(x,t))| \le \frac{M_1}{t^{1/s_1}} \tag{3.90}$$

$$\int_{D_{+,t}} |f'(u(x,t)) - f'(N(x,t))| \le \frac{M_1}{t^{1/s_2}} \tag{3.91}$$

(See also Remark 3.21 for generalization).

(3). N *is a non decreasing continuous function in* $D_{-,t} \cup D_{+,t} \cup E_t$ *and satisfying*

$$u_- \le N(x,t) \le u_+ \quad \text{if } (x,t) \in E \tag{3.92}$$

$$u(x,t) = N(x,t) \quad \text{if } (x,t) \in R \tag{3.93}$$

$$\int_{E_t} |f'(u(x,t)) - f'(N(x,t)|dx \le 2\frac{(B_2 - B_1)}{t}. \tag{3.94}$$

(4). Let $I = [-\|\bar{u}_0\|_\infty + 1, \|\bar{u}_0\|_\infty + 1]$ *and suppose there exists* $C > 0, r > 0$ *such that for all* $x, y \in I$

$$\frac{|x-y|^r}{N} \le C|f'(x) - f'(y)|, \tag{3.95}$$

then there exists $T > 0$ such that for all $t \geq T$,

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|^r dx \leq CM\left(\frac{1}{t^{1/s_1}} + \frac{1}{t^{1/s_2}}\right) + \frac{2N(B_2 - B_1)}{t}.$$

(5). Let α, β (may be different from k_1, k_2 as defined in (3.2) and (3.3)) be such that for $t > 0$

$$A_1 + tf'(\alpha) \leq L(t) \leq R(t) \leq A_2 + tf'(\beta)$$

then

$$\alpha \leq u_- \leq u_+ \leq \beta. \tag{3.96}$$

If $\alpha = \beta$, then for all $(x,t) \in I\!R \times I\!R_+$

$$N(x,t) = u_- = u_+. \tag{3.97}$$

Proof.

(1). (i). First deal with the single shock case. That is there exists $T > 0$ such that $L(T) = R(T)$ and therefore from Lemma 3.1 $u_- > u_+$ and u is given by (3.86). Conversely, let $u_- > u_+$. Suppose for all $t > 0, L(t) < R(t)$ then from (3.34), $\Gamma_1(t) \leq \Gamma_2(t)$ for all $t > 0$ and hence $f'(u_-) \leq f'(u_+)$ which contradicts $u_- > u_+$. Therefore there exist $T > 0$ such that $L(T) = R(T)$ and (3.86) follows from Lemma 3.1.

(ii). Let $u_- \leq u_+$, (3.87) follow from (3.30) and (3.31). Let $(x,t) \in D_-$, then $f'(N(x,t)) = \frac{x - B_1}{t}$ and hence

$$\begin{aligned} f'(p_{-,t}) + \frac{y_+(t) - B_1}{t} &= \frac{L(t) - B_1}{t} \\ &\leq \frac{x - B_1}{t} = f'(N(x,t)) \\ &\leq \frac{\Gamma_1(t) - B_1}{t} = f'(u_-) \end{aligned}$$

this proves (3.88) and (3.89) follows similarly.

(2). From (3.32) we have

$$L(t) \geq y_+(t) + tf'(u_-) - \frac{Mt}{t^{1/s_1}}$$

and for a.e $x \in D_{-,t}, f'(u(x,t)) = \frac{x-y_+(x,t)}{t}$. Since $L(t) < x < \Gamma_1(t)$ and $y_\pm(\Gamma_1(t),t) = B_1$, hence from monotonicity, $A_1 \le y_+(x,t) \le B_1$. Therefore

$$\int_{L(t)}^{\Gamma_1(t)} |f'(u(x,t)) - f'(N(x,t))| \quad dx$$

$$\begin{aligned}
&\le \int_{y_+(t)+tf'(u_-)-\frac{Mt}{t^{1/s}},}^{B_{1+}tf'(u_-)} \left| \frac{x-y_+(x,t)}{t} - \frac{x-B_1}{t} \right| dx \\
&\le \frac{B_1-A_1}{t}[B_1 - y_+(t) + \frac{Mt}{t^{\frac{1}{s}}}].
\end{aligned}$$

This proves (3.90) and (3.91) follows similarly.

(3). Since N is a rarefaction wave in $D_\pm$ and hence it is continuous and non decreasing function in $\bar{D}_{\pm,t}$. Let $y \in E_t$, then from (3) of Lemma (3.9) either $y \in D_{i,t}$ for some i or $y \in R_t$. Since N is constant on $D_{i,t}$ hence to prove continuity or monotonicity of N in E_t, it is enough to prove this in R_t. Let $y_k \le y_1 < y_2$ are in R_t with $\lim\limits_{k\to\infty} y_k = y_1$. Let r_k, r_1, r_2 be the characteristic lines with reciprocal slopes $f'(p_k), f'(q_1), f'(q_2)$ respectively. Hence $y_k = r_k(t), y_1 = r_1(t), y_2 = r_2(t)$ and from monotonicity of x in $y_\pm(x,t)$, and from Lemma (3.2), $r_k(0) \le r_1(0) \le r_2(0)$. This implies $r_k(s) \le r_1(s) \le r_2(s)$ for all s, $p_k \le q_1 \le q_2$ and $\lim\limits_{k\to\infty} p_k = q_1$. Therefore from Lemma 3.2,

$$u(y_1,t) = q_1 = N(y_1,t), N(y_1,t) = q_1 \le q_2 = N(y_2,t)$$

$$\begin{aligned}
N(y_1-,t) &= \lim_{k\to\infty} N(y_k,t) \\
&= \lim_{k\to\infty} p_k \\
&= q_1 \\
&= N(y_1,t).
\end{aligned}$$

Similarly for the right limit. Since $\partial E = \Gamma_1(t) \cup \Gamma_2(t)$ and from monotonicity of N, it follows that $u_- \le N(x,t) \le u_+$ and

$$\begin{aligned}
\int_{E_t} |f'(u(x,t)) - f'(N(x,t)|dx &= \sum_{i\in I} \int_{D_{i,t}} |f'(u(x,t)) - f'(p_i)| \\
&= \sum_i \int_{c_{1,i}+tf'(p_i)}^{c_{2,i}+tf'(p)} \left| \frac{x-y_+(x,t)}{t} - f'(p_i) \right| \\
&= \sum_i \int_{c_{1,i}+tf'(p_i)}^{c_{2,i}+tf'(p_i)} \left| \frac{x-tf'(p_i)}{t} - \frac{y_+(x,t)}{t} \right| dx
\end{aligned}$$

$$\leq \frac{2}{t}\sum |c_{2,i} - c_{1,i}| \leq \frac{2}{t}|B_2 - B_1|$$

since $c_{1,i} \leq y_+(x,t) \leq c_{2,i}$. This proves (3).

(4). From (3.87) to (3.95) there exist a $T > 0$ such that for all $t \geq T, N(x,t) \in I$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|^r dx &= \int_{D_-,t} |u(x,t) - N(x,t)|^r dx + \\
\int_{E_t} |u(x,t) - N(x,t)|^r dx &+ \int_{D_+,t} |u(x,t) - N(x,t)|^r dx
\end{aligned}$$

$$\begin{aligned}
&\leq C[\int_{D_-,t} |f'(u(x,t) - f'(N(x,t)|dx + C\int_{E_t} |f'(u(x,t)) - f'(N(x,t))|dx \\
&+ C\int_{D_+,t} |f'(u(x,t)) - f'(C(x,t))|dx \\
&\leq CM(\frac{1}{t^{1/s_2}} + \frac{1}{t^{1/s_2}}) + \frac{2C(B_2 - B_1)}{t}.
\end{aligned}$$

This proves (4).

(5). Since $L(t) \leq \Gamma_1(t) \leq \Gamma_2(t) \leq R(t)$, hence dividing by t and letting $t \to \infty$ to obtain, $f'(\alpha) \leq f'(u_-) \leq f'(u_+) \leq f'(\beta)$. If $\alpha = \beta$, then $u_- = u_+$ and hence from (3.92), $N = u_-$. This proves (5) and hence the theorem.

Next we give some criterion for non existence of ASSP.

Theorem 3.13 *Let $\bar{u}_0$ be a continuous function with compact support. Let*

$$G = \{\alpha \in I\!R : \bar{u}_0(\alpha) = 0\} \tag{3.98}$$

and assume that for any $\alpha < \beta, \alpha, \beta \in G$

$$\int_{\alpha}^{\beta} \bar{u}_0(\theta)d\theta \neq 0, \tag{3.99}$$

then ASSP does not exist.

Proof. Let $D_i = D(c_{1,i}, c_{2,i}, p_i)$ be a ASSP. Since $u_- = u_+ = 0$ and hence $p_i = 0$ and from (3.59) $\bar{u}_0(c_{1,i}) = \bar{u}_0(c_{2,i}) = 0$. Hence $c_{1,i}, c_{2,i} \in G$ and from (3.99) and (3.58) we have

$$\int_{c_{1,\alpha}}^{c_{2,\alpha}} \bar{u}_0(\theta)d\theta = 0,$$

which is a contradiction. This proves the theorem.

Example 3.14. (Existence of $\mathbf{u_0} \in \mathbf{C_c^\infty}$ for which the solution of (1.2) admits infinitely many ASSP): Let $u_0 \in C_c^\infty(I\!R)$ and u be the solution of (1.2) with initial data u_0. First observe that if an ASSP exist, then u_0 necessarily satisfy the compatibility conditions (3.58), (3.59) and having the same parity at the end points. Hence we look for data satisfying these conditions.

First we construct a solution having single ASSP. From this we construct a data having infinitely ASSP.

Let $A_1 < \alpha_1 < \alpha_2 < A_2$ and $u_0 \in C_c^\infty(I\!R)$ such that

1. Support of $u_0 \subset [A_1, A_2]$.
2. $\int\limits_{A_1}^{A_2} u_0(x)dx = 0$.
3. $0 < u(\alpha_1) = \max u, \ 0 > u(\alpha_2) = \min u$,

and u_0 is strictly increasing in $(A_1, \alpha_1) \cup (\alpha_2, A_2)$ and strictly decreasing in (α_1, α_2). Then

Lemma 3.15 *Assume that $f(0) = f'(0) = 0$ and u be the solution of (1.2) with initial data u_0. Then u admits a single ASSP in $D = (A_1, A_2) \times (0, \infty)$ and satisfies for all $x \notin (A_1, A_2), t > 0$*

$$u(x,t) = u(A_1+, t) = u(A_2-, t) = 0. \tag{3.100}$$

Proof. Let $\alpha_1 < \beta < \alpha_2$ be such that $u_0(\beta) = 0$. Then $u_0 > 0$ in (A_1, β) and $u_0 < 0$ in (β, A_2). Let $R_1(t) = R_-(t, \alpha_1, u_0), R_2(t) = R_+(t, \alpha_2, u_0)$.

$$y_\pm(t) = y_\pm(R_1(t), t, u_0), Y_\pm(t) = y_\pm(R_2(t), t, u_0).$$
$$y_\pm(x,t) = y_\pm(x, t, u_0).$$

Claim 1: Let $t_0 > 0$ such that for $0 < t < t_0, A_1 < R_1(t)$. Then for all $z < A_1 < x < R_1(t)$

$$A_1 \leq y_-(x,t) \leq y_+(x,t) \leq \alpha_1 \tag{3.101}$$
$$u(A_1+, t) = u(z,t) = 0. \tag{3.102}$$

Suppose for some $x_1 \in (A_1, R_1(t)), y_-(x_1, t) < A_1$, then for all $x < x_1, y_-(x,t) < A_1$. Since u_0 is differentiable and hence from (2.12) and (2.13) for a.e. $x \in (A_1, x_1)$

$$0 = f'(0) = f'(u_0(y_-(x,t)) = f'(u(x,t)) = \frac{x - y_-(x,t)}{t}$$

and hence $y_-(x,t) < A_1 < x = y_-(x,t)$ which is a contradiction. This proves (3.101).

Let $z_0 < A_1$ be such that $y_+(z_0,t) > A_1$, then for all $z \in (z_0, A_1), y_+(z,t) > A_1$. Then from NIP, it follows that $y_+(z,t) \leq y_-(t) \leq \alpha_1$ and hence $u_0(y_+(z,t)) > 0$. Hence from (2.12) and (2.13)

$$0 < f'(u_0(y_+(z,t))) = f'(u(z,t)) = \frac{z - y_+(z,t)}{t}$$

and hence $A_1 > z > y_+(z,t) \geq A_1$ which is a contradiction. Hence $y_+(z,t) \leq A_1$ for all $z < A_1$. Suppose for some $z_0 < A_1, y_+(z_0,t) = A_1$, this implies that for all $z \in (z_0, A_1), y_\pm(z,t) = A_1$. For $z < R_-(A_1,t), y_+(z,t) < A_1$ and hence from (2.13),

$$\frac{z - y_+(z,t)}{t} = f'(u(z,t)) = f'(u_0(y_+(z,t))) = f'(0) = 0.$$

Hence $z = y_+(z,t)$ and hence from (2.22),$y_-(R_-(A_1,t),t) = R_-(A_1,t)$. Let $v_0 = \int\limits_{A_1}^{x} u_0(\theta)d\theta$, and $y_\pm = y_\pm(R_-(A_1,t),t)$. Since $y_- = R_-(A_1,t), y_+ = A_1$, hence from (2.3)

$$\begin{aligned} v_0(y_-) + tf^*\left(\frac{R_-(y_-,t) - y_-}{t}\right) &= v_0(y_+) + tf^*\left(\frac{R_-(A_1,t) - y_+}{t}\right) \\ 0 = f^*(f'(0)) = f^*(0) &= f^*\left(\frac{R_-(A_1,t) - y_+}{t}\right). \end{aligned}$$

Hence from the strict convexity, $A_1 > R_-(A_1,t) = y_+ = A_1$ which is a contradiction. Therefore $y_+(z,t) < A_1$ for $z < A_1$ and from (2.13), $0 = f'(u_0(y_+(z,t))) = f'(u(z,t)) = \frac{z - y_+(z,t)}{t}$. Hence $z = y_+(z,t)$ and $u(z,t) = 0$. Therefore $u(A_1-,t) = \lim\limits_{z \uparrow A_1} u(z,t) = 0$. Hence from RH condition across $x = A_1, 0 < t < t_0$ gives $u(A_1+,t) = 0$. This proves (3.102).

Claim 2: Let $t_1 > 0$ such that for $0 < t < t_1, R_2(t) < A_2$, then for $R_2(t) < x < A_2 < z$

$$\alpha_2 \leq y_-(x,t) \leq y_+(x,t) \leq A_2 \tag{3.103}$$

$$u(A_2-,t) = u(z,t) = 0. \tag{3.104}$$

Proof follows exactly as in claim 1.

Claim 3: Let $t_0 > 0$ such that for all $0 < t < t_0$,

$$A_1 < R_1(t) \leq R_2(t) < A_2 \tag{3.105}$$

then

$$A_1 < R_1(t_0) < A_2, A_1 < R_2(t_0) < A_2. \tag{3.106}$$

Suppose $R_1(t_0) = A_1$. Then $A_1 = R_1(t_0) \leq R_2(t_0) \leq A_2$. At $R_1(t_0), y_+(t_0) \geq \alpha_1$, hence for $R_1(t_0) < x < \alpha_1, y_+(x,t_0) \geq \alpha_1$. Hence from (2.12) and (2.13) for a.e. $x \in (R_1(t_0), \alpha_1)$,

$$f'(u_0(y_+(x,t)) = f'(u(x,t)) = \frac{x - y_+(x,t_0)}{t_0} < \frac{\alpha_1 - \alpha_1}{t_0} = 0.$$

Therefore $u_0(y_+(x,t)) < 0$ and hence $y_+(x,t) > \beta$. Hence from monotonicity, for all $x > R_1(t_0) = A_1, y_+(x,t) > \beta$. Hence from (3.103), for all $x \in (A_1, A_2)$.

$$\beta \leq y_+(x,t) < A_2$$

. Therefore from the hypothesis on u_0 and from (2.13) we have for a.e. $x \in (A_1, A_2)$

$$u(x,t_0) = u_0(y_+(x,t_0)) < 0. \tag{3.107}$$

From (3.103) , (3.103) and (3.104) $u(A_1+,t) = u(A_2-,t) = 0$. Hence from (1.107) we have

$$\begin{aligned} 0 = \int_{A_1}^{A_2}\int_0^{t_0} (u_t + f(u)_x) dx dt &= \int_{A_1}^{A_2} u(x,t_0)dx - \int_{A_1}^{A_2} u_0(x)dx \\ &= \int_{A_1}^{A_2} u(x,t_0)dx < 0, \end{aligned}$$

which is a cotnradiction. Similarly $R_2(t_0) \in (A_1, A_2)$ and this proves the claim.

Since for t small, $A_1 < R_1(t) < R_2(t) < A_2$ and hence from claim (3), $A_1 < R_1(t) \leq R_2(t) < A_2$ for all $t > 0$. Since $A_1 \leq y_\pm(t) \leq A_2$ and $A_1 < R_1(t) < A_2$ hence from Lemma 3.8 there exist ASSP $D = (\tilde{A}_1, \tilde{A}_2, p)$ such that

$$\begin{aligned} (\tilde{A}_1, \tilde{A}_2) &= \lim_{t \to \infty} (y_-(t), y_+(t)) \\ f'(p) &= f'(0) = 0 \\ u_0(\tilde{A}_1) &= u_0(\tilde{A}_2) = 0, \end{aligned}$$

and $\tilde{A}_1$ and $\tilde{A}_2$ have the same parity. Since u_0 has only three zeros $\{A_1, \beta, A_2\}$ and A_1 and A_2 are the only pair of points having same parity. Hence $\tilde{A}_1 = A_1, \tilde{A}_2 = A_2$ and this proves the Lemma.

Theorem 3.16 *Let $f(0) = f'(0) = 0$. Then there exist a $\bar{u}_0 \in C_c^\infty(I\!R)$ such that the corresponding solution u of (1.2) with initial data has infinitely ASSP.*

Proof. Let u_0 be as in Lemma (3.15) with $A_1 = -1, A_2 = 1$. For $n \geq 2$. define $\{\psi_{0,n}, u_n\}$ as follows

$$\psi_{0,n}(x) = e^{-n} u_0(2n^2(x - \frac{1}{n})).$$

Then $\psi_{0,n} \in C_c^\infty(I\!R)$ and support of $\psi_{0,n}$is contained in $(\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2})$ and for each k, there exists $c(k) > 0$ such that

$$\left\| \frac{d^k \psi_{0,n}}{dx^k} \right\|_\infty \leq c(k) n^2 e^{-n} \to 0 \text{ as } n \to \infty. \tag{3.108}$$

$\psi_{0,n}$ satisfies all the hypothesis of Lemma 3.14 and let u_n be the corresponding solution. Define $\bar{u}_0$ and u by

$$\bar{u}_0(x) = \begin{cases} \psi_{0,n}(x) & \text{if } x \in (\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}), \\ 0 & \text{otherwise.} \end{cases} \tag{3.109}$$

$$u(x,t) = \begin{cases} u_n(x,t) & \text{if } x \in (\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}), \\ 0 & \text{otherwise,} \end{cases} \tag{3.110}$$

then from (3.102),(3.104) , u is a solution of (1.2) with initial condition $\bar{u}_0$. From (3.108), $\bar{u}_0 \in C_c^\infty(R)$ and u is ASSP in $\left(\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}\right)$ for all $n \geq 2$. This proves the theorem.

Example 3.17: (See Figure 13) Next we construct a ASSP which contains infinitely many shocks. Let $f(u) = \frac{u^2}{2}$ and $0 = x_0 < x_1 < x_2 < \ldots < 1$ and $0 = t_0 < t_1 <$

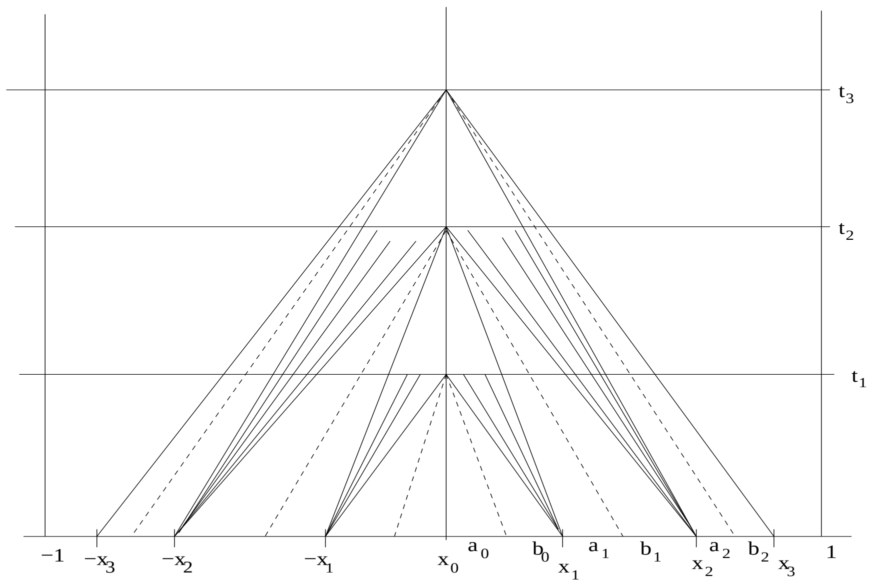

Figure 13:

$t_2 \ldots < \infty$ be such that

$$\lim_{n\to\infty} x_n = 1, \lim_{n\to\infty} \frac{x_n}{t_n} = 0.$$

Define for $n \geq 1, a_n, b_n, l_n, m_n, \tilde{a_n}, \tilde{b_n}, \tilde{l_n}, \tilde{m}_n$ as follows:

$$\begin{aligned}
f'(a_n) &= -f'(\tilde{a_n}) = \frac{-x_{n-1}}{t_n} \\
f'(b_n) &= -f'(\tilde{b}_n) = \frac{-x_n}{t_n} \\
l_n(t) &= -\tilde{l}_n(t) = x_{n-1} + tf'(a_n) \\
m_n(t) &= -\tilde{m}_n(t) = x_n + tf'(b_n). \\
s_n(t) &= (t - t_n)\frac{f(a_n) - f(b_n)}{a_n - b_i} \\
\tilde{s}_n(t) &= -s_n(t) \\
-\tilde{\alpha}_n &= \alpha_n = s_n(0).
\end{aligned}$$

Since $x_{n-1} < x_n$ and hence $f'(a_n) = -\frac{x_{n-1}}{t_n} > -\frac{x_n}{t_n} = f'(b_n)$, therefore $b_n < a_n$ and

$$\begin{aligned}
f'(b_n) &< \frac{f(a_n) - f(b_n)}{a_n - b_n} < f'(a_n), \\
x_{n-1} = -t_n f'(a_n) &< -t_n \frac{f(a_n) - f(b_n)}{a_n - b_n} < -t_n f'(b_n) = x_n, \\
l_n(0) &< s_n(0) < m_n(0).
\end{aligned}$$

Define $u_+(x,t)$ for $x > 0, t > 0$ by

$$u_+(x,t) = \begin{cases} a_n & \text{if } \ 0 < t < t_n, l_n(t) < x < s_n(t), \\ b_n & \text{if } \ 0 < t < t_n, s_n(t) < x < m_n(t), \\ (f')^{-1}\left(\frac{x - x_{n+1}}{t}\right) & \text{if } \ 0 < t < t_n, m_n(t) < x < l_{n+1}(t), \\ 0 & \quad \text{otherwise}, \end{cases}$$

and u and u_0 by

$$u(x,t) = \begin{cases} u_+(x,t) & \text{if } \ x > 0, t > 0, \\ -u_+(-x,t) & \text{if } \ x < 0, t > 0, \end{cases}$$

$$u_0(x) = u(x,0).$$

On the line $x = 0, u(0+,t) = -u(0-t)$ and hence $\frac{1}{2}(u(0+,t) + u(0-,t)) = 0$. this implies u satisfies RH condition across $x = 0$ and hence u is the solution of (1.2) with initial data u_0. Furthermore $u = 0$ for $|x| \geq 1$ and $D = (-1,1) + (0,\infty)$ is an ASSP and each line $s_n(t)$ and $\tilde{s}_n(t)$ for $0 < t < t_n$ is a shock curve.

Remark 3.18 *In example (3.14), one can relax the conditions on u_0 as follows Let u_0 be a bounded measurable function with support of u_0 is in $[A_1, A_2]$. Let*

$$z(u_0) = \{x \in [A_1, A_2] : u_0(x) = 0\}$$

be its zero set and satisfies the following conditions

(1). u_0 is monotone in a neighbourhood of $z(u_0)$.

(2). $\int_{A_1}^{A_2} u_0(x)dx = 0$ *and whenever* $\alpha < \beta, \alpha, \beta \in z(u_0), \int_{\alpha}^{\beta} u_0(x)dx = 0$, *then* $\alpha = A_1, \beta = A_2$.

(3). u_0 is a non decreasing in a neighbourhood of A_1, A_2, then

Lemma 3.19 *Let u_0 satisfies the above conditions and u be the solution of (1.2) with initial data u_0. Then $D = (A_1, A_2) \times (0, \infty)$ is the only ASSP for u and $u(x,t) = 0$ for $x \notin (A_1, A_2)$.*

Proof follows exactly as in example (3.14).

Remark 3.20 *First observe that if u is a solution of (1.2) such that $u(x,t)$ is non decreasing function in (α, β). Then $u(.,t)$ is continuous in (α, β). This follows easily from entropy condition. For $\alpha < x_1 < x < x_2 < \beta$, then $u(x,t) \leq u(x,t) \leq u(x_2,t)$ and hence $u(x-,t) \leq u(x+,t)$. From the entropy condition $u(x-,t) \geq u(x+,t)$ and hence $u(x-,t) = u(x+,t)$ and this proves the continuity.*

Remark 3.21 *In Theorem 3.21, the decay estimates (3.90),(3.91) are obtained under the assumption that f is smooth . If $f(u) = |u|^p$ for $1 < p < 2$, then f is strictly convex and C^1 but not C^2. This case had been dealt by Liu-Pierre [20] and generalised by Kim [17] to obtain decay estimates when $u_- = u_+ = 0$.*

Since decay estimates of Theorem 3.21 depends on (3.40). Hence here also one can relax the condition on f so is to give the proper decay estimates and is as follows: Assume that f satisfies

(1). f is C^1 and strictly convex. That is for $a, b \in \mathbb{R}$

$$f(a) - f(b) - (a-b)f'(b) = 0 \text{ if and only if } a = b. \tag{3.111}$$

(2). $f^* \in C^1$ and both f andf^* are of super linear growth.

(3). At $u_\pm$, there exist a $r > 0, \delta > 0, C > 0$ such that for $|h| \leq \delta$

$$f(u_\pm) - f(u_\pm + h) - hf'(u_\pm + h) \geq C|h|^\gamma. \tag{3.112}$$

(4). For any compact interval I, there exist $C(I) > 0$ such that

$$|f'(a) - f'(b)| \leq C(I)|a-b|^{\gamma-1}. \tag{3.113}$$

Then we have the following

Decay estimates at the boundary: Let f satisfies (1) to (4). Let $L(t), R(t), y_+(t)$,

$Y_-(t)$ be above and define by

$$f'(P_{+,t}) = \frac{L(t) - y_+(t)}{t} \qquad (3.114)$$

$$f'(P_{-,t}) = \frac{R(t) - Y_-(t)}{t}. \qquad (3.115)$$

Then from superlinearity of f^* and from Lemma 3.2, 3.40, $\{P_{\pm,t}\}$ are bounded and satisfies

$$f(u_\pm) - f(P_{\pm,t}) - (u_\pm - P_{\pm,t})f'(P_{\pm,t}) = O(\frac{1}{t}). \qquad (3.116)$$

Let for a sequence $t_k \to \infty, P_{\pm,t_k} \to P_\pm$. Then letting $t_k \to \infty$ in (3.111), to obtain

$$f(u_\pm) - f(P_\pm) - (u_\pm - P_\pm)f'(P_\pm) = 0.$$

Hence from (3.11), $u_\pm = P_\pm$ and therfore

$$\lim_{t\to\infty}(P_{+,t}, P_{-,t}) = (u_+, u_-). \qquad (3.117)$$

From (3.12) and (3.117), for large t, it follows that

$$|u_\pm - P_{\pm,t}| = O(\frac{1}{t^{1/\gamma}}), \qquad (3.118)$$

and from (3.114)

$$|f'(u_\pm) - f'(P_{\pm,t})| = O(\frac{1}{t^{(\gamma-1)/\gamma)}}). \qquad (3.119)$$

This implies that

$$\int_{-\infty}^{\infty} |f'(u(x,t)) - f'(N(x,t))| = O(\frac{1}{t^{(\gamma-1)/\gamma)}}). \qquad (3.120)$$

Furthermore, if f satisfies (3.95) then

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|^r = O(\frac{1}{t^{(\gamma-1)/\gamma)}}). \qquad (3.121)$$

Lemma 3.21 *Let u and u_0 be as in Lemma 3.19. Assume that u_0 is continuous in a neighbourhood of A_1 and A_2. Then there exists $T > 0$ such that for all $t > T, u(x,t)$ has just one discontinuity in $x \in (A_1, A_2)$.*

Proof. Let $A_1 < r_1 < r_2 < A_2$ be such that u_0 is continuous non decreasing function in $(A_1, r_1) \cup (r_2, A_2)$. Let T be the first point at which $R_0 = R_-(T, r_1, u_0) = R_+(T, r_2, u_0) \in D$. Then from (2.27), for all $t > T, R(t) = R_-(t, r_1, u_0) = R_+(t, r_2, u_0)$. Let $A_1 < x_1 < x_2 < R(t)$, then $A_1 \leq y_\pm(x_1, t, u_0) \leq y_\pm(x_2, t, u_0) \leq r_2$. Since u_0 is non decreasing function, hence $u_0(y_\pm(x_1, t, u_0)) \leq u_0(y_\pm(x_2, t, u_0))$ and hence from Remark 2.20 $u_0(., t)$ is continuous in $(A_1, R(t))$. Similarly $u(., t)$ is continuous in $(R(t), A_2)$. This proves the Lemma.

4 Exact Controllability:

In this section we give the proof of theorems (1.1) to (1.3). Basically following two main ideas are used to prove these results

(a) Free regions: Using the proper choice of the parameters for initial value problems,we create a region (called free region) on which the solution will remain constant. This is achieved in Lemmas 2.9 and 2.10. For example from (2.94), the region
$$\{(x,t) : x < L_1(t)\} \tag{4.1}$$
is a free region since u^λ is constant.

(b) Identify the regions where we would like to construct a solution with given target functions as free regions and can be achieved from Lemma 2.9 and 2.10. Then construct the entropy solutions in these free region and using RH condition, glue all the solutions so that it turns out to be the required solution.

We state the following Lemmas which deal with this construction. First we complete the proof of the theorems(1.1) to (1.3) and then give the proof of these Lemmas.

Let $u_0 \in L^\infty(I\!R), 0 \le \delta < T, A, C \in I\!R$. Let $l(., \delta, A, C)$ be the line joining between (C, T) and (A, δ) with slope $1/f^{'}(a(\delta, A, C))$, intersecting $t = 0$ axis at $D(\delta, A, C)$ and is given by

$$\begin{aligned} f^{'}(a(\delta, A, C)) &= \frac{C-A}{T-\delta}, && (4.1)\\ l(t, \delta, A, C) &= A + f^{'}(a(\delta, A, C))(t-\delta), && (4.2)\\ D(\delta, A, C) &= A - \delta f^{'}(a(\delta, A, C)), && (4.3)\\ &= A - \frac{\delta(C-A)}{T-\delta}. && (4.4)\end{aligned}$$

LEMMA 4.1 *1. Let $\wedge > 0,\ A < C$ and $\rho \in LA((A, C))$ satisfying*

$$\delta \le \rho(x) \le T, \tag{4.5}$$

$$\left|\frac{x-A}{T-\rho(x)}\right| \le \wedge. \tag{4.6}$$

Let $\Omega = (A, \infty) \times (\delta, T)$. Then there exists a $\tilde{b}_1(t, \delta, A, C) \in L^\infty((\delta, T))$ and a solution $\tilde{u}_1(x, t, \delta, A, C)$ of (1.2) satisfying

$$f^{'}(\tilde{u}_1(x, T, \delta, A, c)) = \frac{x-A}{T-\rho(x)}, \quad x \in (A, C), \tag{4.7}$$

with initial and boundary conditions

$$\begin{aligned}\tilde{u}_1(A,t,\delta,a,C) &= \tilde{b}_1(t,\delta,A,C), \quad t \in (\delta,T), & (4.8)\\ \tilde{u}_1(x,t,\delta,a,C) &= a(\delta,A,C), \quad x > l(t,\delta,A,C), & (4.9)\\ \tilde{u}_1(l(t,\delta,A,C)-,t,\delta,a,C) &= a(\delta,A,C). & (4.10)\end{aligned}$$

2. *Let $C < A$ and $\rho \in RA((C,A))$ satisfying (4.5) and (4.6). Let $\Omega = (-\infty, A) \times (\delta,T)$. Then there exist $\tilde{b}_2(t,\delta,A,C) \in L^\infty((\delta,T))$ and a solution $\tilde{u}_2(x,t,\delta,a,C)$ of (1.2) satisfying*

$$f'(\tilde{u}_2(x,T,\delta,A,C)) = \frac{x-A}{T-\rho(x)}, \quad x \in (C,A), \tag{4.11}$$

with initial and boundary conditions

$$\begin{aligned}\tilde{u}_2(A,t,\delta,A,C) &= \tilde{b}_2(t,\delta,A,C), \quad t \in (\delta,T), & (4.12)\\ \tilde{u}_2(x,t,\delta,A,C) &= a(\delta,a,C), \quad x < l(t,\delta,A,C), & (4.13)\\ \tilde{u}_2(l(t,\delta,A,C)+,t,\delta,A,C) &= a(\delta,A,C), \quad t \in (\delta,T). & (4.14)\end{aligned}$$

LEMMA 4.2 *Let $A_1 < A_2$, $C_1 < C_2$, $\rho \in IA((C_1,C_2))$ such that for all $x \in (C_1,C_2)$,*

$$A_1 \le \rho(x) \le A_2. \tag{4.15}$$

Let $\Omega = I\!R \times I\!R_+$, for $i = 1,2$, $l_i(t) = l(t,\delta,A_i,C_i)$, $a_i = a_i(\delta,A_i,C_i)$, then there exist $\tilde{u}_0 \in L^\infty((A_1,A_2))$ and a solution $\tilde{u}$ of (1.2) such that for $\delta < t < T$,

$$\begin{aligned}f'(\tilde{u}(x,T)) &= \frac{x-\rho(x)}{T-\delta}, \quad \text{for } x \in (C_1,C_2), & (4.16)\\ \tilde{u}(l_1(t)+,t) &= a_1, & (4.17)\\ \tilde{u}(l_2(t)-,t) &= a_2, & (4.18)\end{aligned}$$

with initial conditions

$$\tilde{u}(x,\delta) = \begin{cases} a_1 & \text{if} \quad x < A_1,\\ \tilde{u}_0(x) & \text{if} \quad A_1 < x < A_2,\\ a_2 & \text{if} \quad x > A_2.\end{cases} \tag{4.19}$$

Proof of theorem 1.1: Let $\Omega = I\!R \times (0,T), A_i, B_i, C_i, g$ and s be as in theorem (1.1). Let

$$f'(a_1) = \frac{C_1 - A_1}{T},\ f'(a_2) = \frac{C_2 - A_2}{T},\ l_1(t) = A_1 + tf'(a_1),\ l_2(t) = A_2 + tf'(a_2).$$

Then from Lemmas 2.9 and 2.10 we can choose λ and μ and solutions u_1 and u_2 in Ω with initial data's u_0^1 and u_0^2 respectively and is given by

$$u_0^2(x) = \begin{cases} a_2 & \text{if} \quad x < A_2, \\ \lambda & \text{if} \quad A_2 < x < B_2, \\ u_0(x) & \text{if} \quad x > B_2. \end{cases} \tag{4.19}$$

$$u_2(x,t) = a_2 \text{ if } x < l_2(t), \tag{4.20}$$
$$u_2(l_2,(t)+,t) = a_2. \tag{4.21}$$

$$u_0^1(x) = \begin{cases} a_1 & \text{if} \quad x > A_1, \\ \mu & \text{if} \quad B_1 < x < A_1, \\ u_0(x) & \text{if} \quad x < B_1. \end{cases} \tag{4.22}$$

$$u_1(x,t) = a_1 \text{ if } x > l_1(t). \tag{4.23}$$
$$u_1(l_1(t)-,t) = a_1. \tag{4.24}$$

From (1.35) and Lemma 4.2 there exist a solution u_3 of (1.2) and $\bar{u}_0 \in L^\infty(A_1, A_2)$ satisfying

$$u_3(x,T) = g(x), \text{ if } x \in (C_1, C_2) \tag{4.25}$$

$$u_3(x,0) = \begin{cases} a_1 & \text{if} \quad x < A_1, \\ \bar{u}_0(x) & \text{if} \quad A_1 < x < A_2, \\ a_2 & \text{if} \quad x > A_2, \end{cases} \tag{4.26}$$

and

$$u_3(x,t) = \begin{cases} a_1 & \text{if} \quad x < l_1(t), \\ a_2 & \text{if} \quad x > l_2(t). \end{cases} \tag{4.27}$$

$$u_3(l_1(t)+,t) = a_1, u_3(l_2(t)-,t) = a_2. \tag{4.28}$$

From (4.21), (4.24) , (4.28) and RH condition, glue u_1, u_2, u_3 to form a single solution u of (1.2) for $0 < t < T$ by

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if} \quad x < l_1(t), \\ u_3(x,t) & \text{if} \quad l_1(t) < x < l_2(t), \\ u_2(x,t) & \text{if} \quad l_2(t) < x. \end{cases} \tag{4.29}$$

Then from (4.25) and (4.26), $(u, \bar{u}_0)$ is the required solution. This proves the theorem.

Proof of Theorem 1.2: Let $f'(a) = \frac{C}{T-\delta}$ and $l(t)$ be the line joining (C,T) *and* $(0,\delta)$ given by $l(t) = (t-\delta)f'(a)$. Let $A = l(0) = -\delta f'(a) < 0$. From Lemma (2.9) by choosing λ large, we can find a solution u_1 of (1.2) in $\Omega = I\!R \times (0,T)$ satisfying

$$u_1(x,0) = \begin{cases} a & \text{if} \quad x < A, \\ \lambda & \text{if} \quad A < x < 0, \\ u_0(x) & \text{if} \quad x > 0. \end{cases} \tag{4.30}$$

$$u_1(x,t) = a \text{ if } x < l(t), \tag{4.31}$$

$$u_1(l(t)+,t) = a. \tag{4.32}$$

From (1.36) , (1.37) and (1) of Lemma 4.1, choose a solution u_2 of (1.2) and $b_1 \in L^\infty(\delta, T)$ such that

$$\begin{aligned} u_2(x,T) &= g(x) && (4.33)\\ u_2(0,t) &= b_1(t) \text{ if } \delta < t < T, && (4.34)\\ u_2(x,t) &= a \quad \text{if } x > l(t), t > \delta, && (4.35)\\ u_2(l(t)-,t) &= a \quad \text{if } t > \delta. && (4.36) \end{aligned}$$

From (4.30), (4.32), (4.36) and RH conditions we glue the solutions u_1 and u_2 to obtain a solution u of (1.2) by

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if} \quad x > l(t), 0 < t < T, \\ u_2(x,t) & \text{if} \quad 0 < x < l(t), \delta < t < T. \end{cases} \tag{4.37}$$

Define $b \in L^\infty(0,T)$ by

$$b(t) = \begin{cases} u_1(0,t) & \text{if} \quad 0 < t < \delta, \\ b_1(t) & \text{if} \quad \delta < t < T. \end{cases} \tag{4.38}$$

Then from (4.30) , (4.33), (u,b) is the require solution. This proves the theorem.

Proof of theorem (1.3): Let $f'(a_1) = \frac{C-A_1}{T}, f'(a_2) = \frac{C-A_2}{T}, l_1(t) = A_1 + tf'(a_1) = A_2 + tf'(a_2)$ be the respective lines joining $(C,T),(A_1,0)$ and $(C,T),(A_2,0)$.

From (2.135), choose (λ, μ) and a solution u_3 of (1.2) in $I\!R \times (0,T)$ satisfying

$$u_3(x,t) = \begin{cases} a_1 & \text{if} \quad x < l_1(t), \\ a_2 & \text{if} \quad x > l_2(t), \end{cases} \tag{4.39}$$

with initial condition

$$u_3(x,0) = \begin{cases} a_1 & \text{if} \quad x < A_1, \\ \lambda & \text{if} \quad A_1 < x < B_1, \\ u_0(x) & \text{if} \quad B_1 < x < B_2, \\ \mu & \text{if} \quad B_2 < x < A_2, \\ a_2 & \text{if} \quad x > A_2. \end{cases} \tag{4.40}$$

(a). Since g_i is a non decreasing function for $i = 1,2$ satisfying (1.38) , (1.39) and hence

$$D_1 = g_1(B_1) \le A_1, A_2 \le g_2(B_2) = D_2.$$

Let η_i be the line joining (B_i, T) and $(D_i, 0)$ with $f'(m_i) = \frac{B_i - D_i}{T}$ for $i = 1, 2$. Then from Lemma 4.2, there exist solutions u_i of (1.2) in $I\!R \times (0, T)$ with initial condition $u_0^i \in L^\infty(D_i, A_i)$ for $i = 1, 2$ such that

$$\begin{aligned} u_1(x,T) &= g_1(x,T) \quad \text{if } x \in (B_1, C), & (4.41)\\ u_2(x,T) &= g_2(x,T) \quad \text{if } x \in (C, B_2), & (4.42)\\ u_1(x,t) &= m_1 \quad \text{if } x < \eta_1(t), & (4.43)\\ u_1(l_1(t)-, t) &= u_1(l_1(t)+, t) = a_1, & (4.44)\\ u_2(x,t) &= m_2 \quad \text{if } x > \eta_2(t), & (4.45)\\ u_2(l_2(t)-, t) &= u_2(l_2(t)+, t) = a_2, & (4.46) \end{aligned}$$

and

$$u_1(x,0) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ a_1 & \text{if } x > A_1. \end{cases} \tag{4.47}$$

$$u_2(x,0) = \begin{cases} m_2 & \text{if } x > D_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ a_2 & \text{if } x < A_2. \end{cases} \tag{4.48}$$

From (4.39) , (4.40) ,(4.44), (4.46) and from RH conditions, we can glue u_1, u_2, u_3 to a solution u of (1.2) with initial data $\bar{u}_0$ given by

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if } x < l_1(t), \\ u_3(x,t) & \text{if } l_1(t) < x < l_2(t), \\ u_2(x,t) & \text{if } x > l_2(t), \end{cases} \tag{4.49}$$

and initial condition $\bar{u}_0$ is given by

$$\bar{u}_0(x) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ m_2 & \text{if } x > D_2. \end{cases} \tag{4.50}$$

From (4.41) , (4.42) u satisfies

$$u(x,T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2, \end{cases} \tag{4.51}$$

and $(u, \bar{u}_0)$ is the required solution. This proves (a).

(b). Given $\delta > 0$ choose $A_1 < B_1 < B_2 < A_2$ such that $\max(l_1(B_1), l_2(B_2)) = \delta$ and u_3 be the solution of (1.2) as in (4.39). From (1.40),(1.41) and from Lemma 4.1, there exist solution u_1 of (1.2) in $(B_1, \infty) \times (\delta, T)$ and boundary data $\tilde{b}_1$, u_2 of (1.2) in $(-\infty, B_2) \times (\delta, T)$ and boundary data $\tilde{b}_2$ such that

$$\begin{aligned} u_1(x,T) &= g_1(x) \quad \text{if } x \in (B_1, C), \\ u_2(x,T) &= g_2(x) \quad \text{if } x \in (C, B_2), \end{aligned}$$

and for $\delta < t < T$,

$$\begin{aligned} u_1(B_1,t) &= \tilde{b}_1(t), \quad u_1(l_1(t)-,t) = a_1, \\ u_2(B_2,t) &= \tilde{b}_2(t), \quad u_2(l_2(t)+,t) = a_2. \end{aligned}$$

Then from RH condition glue u_1, u_2, u_3 in $\Omega = (B_1, B_2) \times (0,T)$ by

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if} \quad 0 < t < \delta, B_1 < l_1(x) < t, \\ u_2(x,t) & \text{if} \quad 0 < t < \delta, t < l_1(x) < B_2, \\ u_3(x,t) & \text{otherwise.} \end{cases}$$

Then u is a solution of (1.2) satisfying the boundary conditions (b_1, b_2) given by

$$b_1(t) = \begin{cases} \tilde{b}_1(t) & \text{if} \quad \delta < t < T, \\ u_3(B_1,t) & \text{if} \quad 0 < t < \delta, \end{cases}$$

$$b_2(t) = \begin{cases} \tilde{b}_2(t) & \text{if} \quad \delta < t < T, \\ u_3(B_2,t) & \text{if} \quad 0 < t < \delta. \end{cases}$$

Then (u, b_1, b_2) is the solution for problem (1.3). This proves the Lemma.

Proof of Lemma (4.1) and (4.2) :

Initial Value problem partitions: (See Figure 14) Let $0 \le \delta < T,\ I = (A_1, A_2)$, $J = (C_1, C_2)$. Let $P = \{y_0, y_1 \ldots y_n, x_0, x_1 \ldots x_n\}$ is called a partition of (I, J) if

$$A_1 = y_0 < y_1 < \ldots < y_n = A_2,\ C_1 = x_0 \le x_1 \le \ldots \le x_n = C_2.$$

Let $P(I,J) = \{P : P \quad \text{is a partition of } (I,J)\}$. For a partition P denote $a_i(P), s_i(P)$, $b_i(P), a_i(t,P),\ s_i(t,P),\ b_i(t,P)$ by

$$\begin{aligned} f^{'}(a_i(p)) &= \frac{x_i - y_i}{T - \delta}, \\ f^{'}(b_i(P)) &= \frac{x_i - y_{i+1}}{T - \delta}, \\ s_i(P) &= \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)}, \\ a_i(t,P) &= x_i + f^{'}(a_i(P))(t - T), \\ b_i(t,P) &= x_i + f^{'}(b_i(P))(t - T), \\ s_i(t,P) &= x_i + s_i(P)(t - T). \end{aligned}$$

Clearly $a_i(\delta, P) = y_i,\ b_i(\delta, P) = y_{i+1}$.

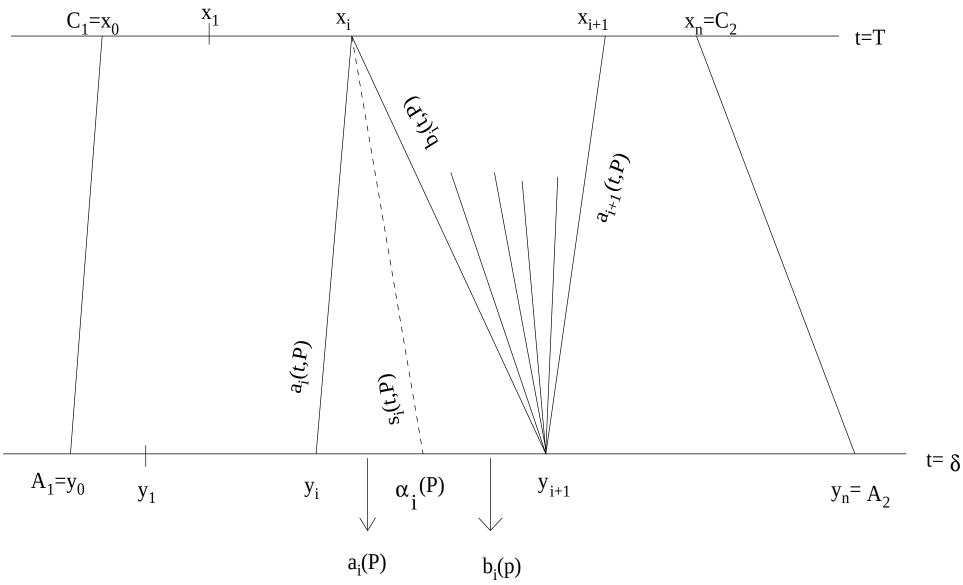

Figure 14:

LEMMA 4.3 *Let* $\alpha_i(P) = s_i(\delta, P)$*, then for* $\delta \le t \le T$,

$$b_i(P) < a_i(p),\ b_i(P) \le a_{i+1}(P), \tag{4.59}$$

$$y_i < \alpha_i(P) < y_{i+1}, \tag{4.60}$$

$$a_i(t,P) < s_i(t,P) < b_i(t,P) \quad \textit{for} \quad \delta < t < T. \tag{4.61}$$

Proof. Since $y_i < y_{i+1}$, hence

$$f'(a_i(P)) = \frac{x_i - y_i}{T - \delta} > \frac{x_i - y_{i+1}}{T - \delta} = f'(b_i(P)),$$

therefore $a_i(P) > b_i(P)$. By convexity of f

$$f'(a_i(p)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f'(b_i(P)),$$

$$\frac{x_i - y_i}{T - \delta} > \frac{x_i - \alpha_i(P)}{T - \delta} > \frac{x_i - y_{i+1}}{t - \delta},$$

and hence $y_i < \alpha_i(P) < y_{i+1}$. Since $x_i \le x_{i+1}$ and hence $f'(b_i(P)) = \frac{x_i - y_{i+1}}{T-\delta} \le \frac{x_{i+1} - y_{i+1}}{T-\delta} = f'(a_{i+1}(P))$. This implies $b_i(P) \le a_{i+1}(P)$. This proves (4.59) to (4.61) and hence the Lemma.

Let $\Omega_i(P) = \{(x,t) : \delta < t < T, a_i(t,P) < x < a_{i+1}(t,P)\}$. In view of Lemma 4.5, let $u_i(x,t,P)$ be a solution of (1.2) in $\Omega_i(P)$ defined by

$$u_i(x,t,P) = \begin{cases} a_i(P) & \text{if} \quad a_i(t,P) < x < s_i(t,P), \\ b_i(P) & \text{if} \quad s_i(t,P) < x < b_i(t,P), \\ (f')^{-1}\left(\frac{x-y_{i+1}}{t-\delta}\right) & \text{if} \quad b_i(t,P) \le x < a_{i+1}(t,P). \end{cases} \tag{4.62}$$

Hence $u_i(a_{i+1}(t,P)-,t,P) = a_{i+1}(P) = u_{i+1}(a_{i+1}(t,P)+,t,P)$. Therefore define the solution $u(x,t,P)$ of (1.2) in $I\!R \times (\delta, T)$ by

$$u(x,t,P) = \begin{cases} u_i(x,t,P) & \text{if} \quad (x,t) \in \Omega_i(P), \\ a_0(P) & \text{if} \quad x < a_0(t,P), \\ a_n(P) & \text{if} \quad x > a_n(t,P), \end{cases} \tag{4.63}$$

satisfying the initial condition

$$u(x,\delta,P) = \begin{cases} u_0(x,P) & \text{if} \quad x \in (A_1, A_2), \\ a_0(P) & \text{if} \quad x < A_1, \\ a_n(P) & \text{if} \quad x > A_2, \end{cases} \tag{4.64}$$

where u_0 is given by

$$u_0(x,P) = \begin{cases} a_i(P) & \text{if} \quad y_i < x < \alpha_i(P), \\ b_i(P) & \text{if} \quad \alpha_i(P) < x < y_{i+1}. \end{cases} \tag{4.65}$$

Furthermore at $t = T, x \in (C_1, C_2), u$ satisfies

$$f'(u(x,T,P)) = \sum_{i=0}^{n-1} \chi_{[x_i,x_{i+1})}(x)\left(\frac{x-y_{i+1}}{T-\delta}\right). \tag{4.66}$$

Next we calculate the L^∞ and TV bounds of u_0. First observe that $f'(a_0(P)) = \frac{C_1-A_1}{T-\delta}$ and $f'(a_n(P)) = \frac{C_2-A_2}{T-\delta}$, hence $a_0(P),\ a_n(P)$, are independent of P and denote

$$a_0 = a_0(P), a_n = a_n(P),$$

$$a_0(t) = a_0(t,P), a_n(t) = a_n(t,P).$$

Let $M = \frac{\max(C_2,A_2)-\min(C_1,A_1)}{T-\delta}$, then $|f'(a_i(P)| \le \left|\frac{x_i-y_i}{T-\delta}\right| \le M,\ |f'(b_i(P))| \le M$ and hence

$$|f'(u_0(x,p))| \le M \tag{4.67}$$

$$\begin{aligned} TV(f'(u_0(\cdot,P)) &= \sum_{i=0}^{n} |f'(a_i(P)) - f'(b_i(P))| \\ &= +\sum_{i=0}^{n-1} \left|f'(b_i(P)) - f'(a_{i+1}(P))\right| \\ &= \sum_{i=0}^{n} \left|\frac{x_i-y_i}{T-\delta} - \frac{x_i-y_{i+1}}{T-\delta}\right| + \sum_{i=0}^{n-1} \left|\frac{x_i-y_{i+1}}{T-\delta} - \frac{x_{i+1}-y_{i+1}}{T-\delta}\right| \\ &= \frac{1}{T-\delta}(A_2-A_1) + \frac{1}{T-\delta}(C_2-C_1). \end{aligned} \tag{4.68}$$

Since $|f'(\theta)| \to \infty$ as $|\theta| \to \infty$ and hence we have

LEMMA 4.4 *There exist a constant M_1 independent of P such that*

$$\|u_0(\cdot, P)\|_\infty + TV(f'(u_0(\cdot, P)) \leq M_1. \tag{4.69}$$

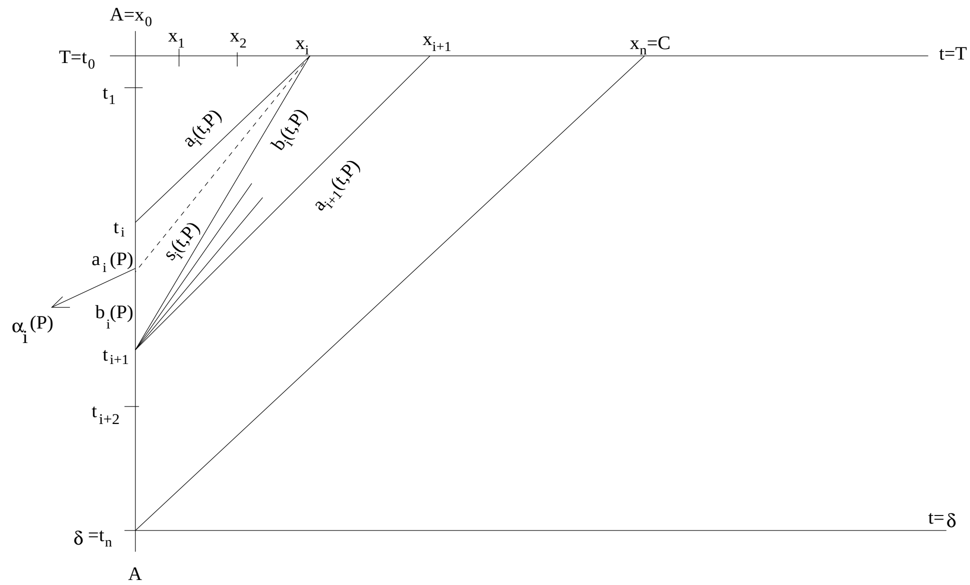

Figure 15:

Boundary value partition: (See Figure 15) Let $0 \leq \delta < T$, $A < C$, $I = (A, C)$, $J = (\delta, T)$. Let $P = \{t_0, t_1 \ldots t_n, x_0, x_1 \ldots x_n\}$ is called a boundary value partition if

$$T = t_0 > t_1 > t_2 \ldots > t_n = \delta, \ A = x_0 \leq x_1 \leq x_2 < \ldots \leq x_n = C.$$

$$\text{Let } P(I, J) = \{P : P \quad \text{is a boundary value partition of I , J}\}. \tag{4.70}$$

For a $P \in P(I, J)$ denote $a_i(P), s_i(P), b_i(P), a_i(t, P), s_i(t, P), b_i(t, P)$ by

$$\begin{aligned}
f^{'}(a_i(P)) &= \frac{x_i - A}{T - t_i}, \\
f^{'}(b_i(P)) &= \frac{x_i - A}{T - t_{i+1}}, \\
s_i(P) &= \frac{f(a_i(P) - f(b_i(P))}{a_i(P) - b_i(P)}, \\
a_i(t, P) &= x_i + f^{'}(a_i(P))(t - T), \\
b_i(t, P) &= x_i + f^{'}(b_i(P))(t - T), \\
s_i(t, P) &= x_i + s_i(P)(t - T),
\end{aligned}$$

Clearly $a_i(t_i, P) = t_i$, $b_i(t_{i+1}, P) = t_{i+1}$.

LEMMA 4.5 *Define $\alpha_i(P)$such that $s_i(\alpha_i(P), P) = A$. Then for $t \leq T$*

$$a_i(P) > b_i(P),\ \ a_{i+1}(P) \geq b_i(p), \tag{4.71}$$

$$t_i > \alpha_i(P) > t_{i+1}, a_i(t, P) \leq s_i(t, P) \leq b_i(t, P), \tag{4.72}$$

Proof. Since $t_i > t_{i+1}, x_i \leq x_{i+1}$, hence

$$\frac{x_i - A}{T - t_i} > \frac{x_i - A}{T - t_{i+1}},\ \frac{x_i - A}{T - t_{i+1}} \leq \frac{x_{i+1} - A}{T - t_{i+1}}.$$

This implies (4.71). From strict convexity of f and (4.71), we have

$$f'(a_i(P)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f^{'}(b_i(P)),$$

hence $t_i > \alpha_i(P) > t_{i+1}$ and for all $t < T$, $a_i(t, P) \leq s_i(t, P) \leq b_i(t, P)$. This proves the Lemma.

Let $\Omega_i(P) = \{(x, t) : a_i(t, P) < x < a_{i+1}(t, P), t_{i+1} < t < T\}$. In view of Lemma 4.5, let $u_i(x, t, P)$ be a solution of (1.2) in $\Omega_i(P)$ defined by

$$u_i(x, t, P) = \begin{cases} a_i(P) & \text{if } \ a_i(t, P) < x < s_i(t, P), \\ b_i(P) & \text{if } \ s_i(t, P) < x < b_i(t, P), \\ (f^{'})^{-1}\left(\frac{x-A}{T-t_{i+1}}\right) & \text{if } \ b_i(t, P) < x < a_{i+1}(t, P). \end{cases} \tag{4.73}$$

Then

$$u_{i+1}(a_{i+1}(t, P)+, t, P) = a_{i+1}(P) = u_i(a_{i+1}(t, P)-, t, P). \tag{4.74}$$

Also $a_n(P)$ and $a_n(t, P)$ are independent of P and denote by $a_n, a_n(t)$. Then from (4.74) it follows that $u_{n-1}(a_n(t)-, t, P) = a_n$. Therefore define the solution $u(x, t, P)$ of (1.2) in $\Omega = (A, \infty) \times (\delta, T)$ by

$$u(x, t, P) = \begin{cases} u_i(x, t, P) & \text{if } \ (x, t) \in \Omega_i(P),\ 0 < i \leq n-1, \\ a_n & \text{if } \ x > a_n(t),\ \delta < t < T, \end{cases} \tag{4.75}$$

and $u(x, t, P)$ takes the boundary value $b(t, P)$ and initial value a_n given by

$$u(A, t, P) = b(t, P) = \begin{cases} \theta_f & \text{if } \ t_1 < t < T, \\ a_i(P) & \text{if } \ \alpha_i(P) < t < t_i, \\ b_i(P) & \text{if } \ t_{i+1} < t < \alpha_i(P). \end{cases} \tag{4.76}$$

$$u(x, \delta, P) = u_0(x, P) = a_n = (f)^{-1}\left(\frac{C - A}{T - \delta}\right) \quad \text{if} \quad x \in (A, \infty). \tag{4.77}$$

Further more at $t = T$, and $x \in (A, C)$, u satisfies

$$f^{'}(u(x, T, P)) = \sum_{i=1}^{n} \chi_{[x_i, x_{i+1})}(x) \left(\frac{x - A}{T - t_i}\right). \tag{4.78}$$

Next we calculate the L^∞ and TV bounds of the boundary value $b(\cdot, P)$.

$$\begin{aligned}
|f^{'}(b(t,P))| &= \max_{1\le i\le n}(|f^{'}(a_i(p))|, |f^{'}(b_i(P))|) \\
&= \max_{1\le i\le n}\left(\left|\tfrac{x_i-A}{T-t_i}\right|, \left|\tfrac{x_i-A}{T-t_{i+1}}\right|\right) \\
&= \max_{1\le i\le n}\left(\left|\tfrac{x_i-A}{T-t_i}\right|\right).
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
TV(f^{'}(b(\cdot,P)) &= \sum_{i=1}^{n-1}\left|f^{'}(a_i(P)) - f^{'}(b_i(P))\right| + \sum_{i=0}^{n-1}\left|f^{'}(b_i(P)) - f^{'}(a_{i+1}(P))\right| \\
&= \sum_{i=1}^{n-1}\left|\tfrac{x_i-A}{T-t_i} - \tfrac{x_i-A}{T-t_{i+1}}\right| + \sum_{i=1}^{n-1}\left|\tfrac{x_i-A}{T-t_{i+1}} - \tfrac{x_{i+1}-A}{T-t_{i+1}}\right| + \left|\tfrac{x_1-A}{T-t_1}\right| \\
&= \sum_{i=1}^{n-1}\tfrac{(x_i-A)(t_i-t_{i+1})}{(T-t_i)(T-t_{i+1})} + \sum_{i=1}^{n-1}\tfrac{(x_{i+1}-x_i)}{(T-t_{i+1})} + \left|\tfrac{x_1-A}{T-t_1}\right| \\
&\le \left(\tfrac{T-\delta}{T-t_1}\right)\max_{1\le i\le n}\left|\tfrac{x_i-A}{T-t_i}\right| + \left(\tfrac{C-A}{T-t_1}\right).
\end{aligned} \tag{4.80}$$

Analysis of Discretization and Convergence: Let $\rho : [A, C] \to [\delta, T]$ be a non increasing right continuous function. Then it follows that $\{x : \rho(x) \le t\}$ is a closed interval for any t. Let $0 < \epsilon < C - A$, define

$$\rho_\epsilon(x) = \min\{\rho(x), \rho(A+\epsilon)\}.$$

Then ρ_ϵ is a non-increasing right continuous function. Let m, n be non negative integers and let $T = t_0 > t_1 = \rho(A+\epsilon) > t_2 > \ldots > t_n = \delta$ be such that $|t_i - t_{i+1}| \le \frac{1}{m}$ for all $i \ge 1$. Let $k \le n-1$ such that $\{x : \rho_\epsilon(x) \le t_{k+1}\} = \phi, \{x : \rho(x) \le t_k\} \ne \phi$ and define $\{x_i\}$ by $x_i = C$ if $i \ge k+1$ and for $1 \le i \le k$,

$$\{x : \rho_\varepsilon(x) > t_1\} = (x_i, C).$$

Denote $P_{n,m,\epsilon}$ by $P_{m,n,\epsilon} = \{t_0, t_1, \ldots t_n, x_0, x_1 \ldots, x_n\}$ the partition depending on n, m and ϵ. Associate to $P_{m,n,\epsilon}$ define

$$\rho(x, P_{m,n,\epsilon}) = \sum_{i=1}^{n-1} t_i \chi_{[x_{i-1},x_i)}(x) + t_n \chi_{[x_{n-1},x_n]}(x). \tag{4.81}$$

Then it follows from definition,

$$\sup_n |\rho_\epsilon(x) - \rho(x)| \le \sup_{A<x<\varepsilon} |\rho(x) - \rho(A+\varepsilon)| \tag{4.82}$$

$$\sup_n |\rho_\epsilon(x) - \rho(x, P_{m,n,\epsilon})| \le \frac{1}{m}. \tag{4.83}$$

Definition: Let $\epsilon_2 < \epsilon_1, n_2 \ge n_1$. For $i = 1, 2$, let $P_{m,n_i,\epsilon_i} = \{t_0, t_{1,i}, \ldots t_{n_i,i}, x_0, x_{1,i}, \ldots x_{n_i,i}\}$ be the partitions. Then we say P_{m,n_2,ϵ_2} dominates P_{m,n_1,ϵ_1} and is denoted by $P_{m,n_2,\epsilon_2} \ge P_{m,n_1,\epsilon_1}$ if for $1 \le j \le n_1$

$$\begin{aligned}
t_{j,1} &= t_{n_2-n_1+j,2}, \\
x_{j,1} &= x_{n_2-n_1+j,2}.
\end{aligned} \tag{4.84}$$

For a partition $P_{m,n,\epsilon,}$ define $\Omega(P_{m,n,\epsilon})$ by

$$\Omega(P_{m,n,\epsilon}) = \{(x,t) : a_1(t, P_{m,n,\epsilon}) < x,\ \delta < t < T\}. \tag{4.85}$$

Properties of the domination: Let $\epsilon_2 < \epsilon_1, n_2 \geq n_1$ and let for $i = 1, 2, u_i(x,t) = u(x, t, P_{m,n_i,\epsilon_i}), b_i(t) = b(t, P_{m,n_i,\epsilon_i})$ as in (4.75) and (4.76) respectively. Let $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$ then from the construction it follows

$$\rho_{\epsilon_1}(x) = \rho_{\epsilon_2}(x) \quad \text{if} \quad x \geq \ \varepsilon_1 + A, \tag{4.86}$$

$$u_1(x,t) = u_2(x,t) \quad \text{if} \quad (x,t) \in \Omega(P_{m,n,\epsilon_1}), \tag{4.87}$$

$$b_1(t) = b_2(t) \quad \text{if} \quad \delta < t \leq \rho(A + \epsilon_1), \tag{4.88}$$

$$f^{'}(u_i(x,T)) = \frac{x - A}{T - \rho(x, p_{m,n_i,\epsilon_i})}, \quad i = 1, 2. \tag{4.89}$$

$$\begin{aligned}
\int_{\delta}^{T} |f^{'}(b_1(t)) - f^{'}(b_2(t))|dt &= \int_{\rho(\epsilon_1+A)}^{\rho(\epsilon_2+A)} |f^{'}(b_2(t)) - f^{'}(b_1(t)))|dt \\
&\leq |\rho(A+\epsilon_1) - \rho(A+\epsilon_2)| \max_j \left\{ \left| \tfrac{x_{j,2}-A}{T-t_{j,2}} \right| \right\} \\
&= |\rho(A+\epsilon_1) - \rho(A+\epsilon_2)| \max_j \left\{ \left| \tfrac{x_{j,2}-A}{T-\rho_{\epsilon_2}(x_{j,2})} \right| \right\}.
\end{aligned} \tag{4.90}$$

Construction of dominations: Let $\epsilon_2 < \epsilon_1$ and $P_{m,n_1,\epsilon_1} = \{t_0, t_{1,1}, \ldots t_{n_1,1}, x_0, x_{1,1}, \ldots x_{n_1,1}\}$. Now choose $\rho(\epsilon_2 + A) = t_{1,2} > t_{2,2} > \ldots t_{r_2,2} = t_{11} = \rho(\epsilon_1 + A)$ such that $|t_{i,2} - t_{i+1,2}| \leq \frac{1}{m}$ for $1 \leq i \leq r_2 - 1$. Let $n_2 = n_1 + n_2$ and define $t_{i,2}$ for $i \geq r_2$ by

$$t_{i,2} = t_{i-r_2+1,1},$$

and $\{x_{i,2}\}$ be associated to $\{t_{i,2}\}$. Let $n_2 = r_2 + n_1 - 1$ and $P_{m,n_2,\epsilon_2} = \{t_0, t_{1,2} \ldots t_{n_2,2}, x_0, x_{1,2}, \ldots x_{n_2,2}\}$, then $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$.

Let $0 < \epsilon_{i+1} < \epsilon_i < C - A$, $\lim\limits_{i\to\infty} \epsilon_i = 0$. Let $m \geq 1$ and $\{P_{m,n_1,\epsilon_1}\}_m$ be a partition corresponding to ρ_{ϵ_1}. From the above construction, extend this partition to $\{P_{m,n_2,\epsilon_2}\}_m$ to ρ_{ϵ_2} such that $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$. By induction there exist partitions $\{P_{m,n_j,\epsilon_j}\}_m$ of ρ_{ϵ_j} such that

$$P_{m,n_j,\epsilon_j} \geq P_{m,n_{j-1},\epsilon_{j-1}}. \tag{4.91}$$

Denote $P_{m,n_j,\epsilon_j} = \{t_0, t_{1,m,j}, \ldots t_{n_j,m,j}, x_0, x_{1,m,j}, \ldots x_{n_j,m,j}\}$. Since $\rho_{\epsilon_j} \leq \rho$ and hence

$$\left| \frac{x - A}{T - \rho_{\epsilon_j}(x)} \right| \leq \left| \frac{x - A}{T - \rho(x)} \right|,$$

and

$$\left| \frac{x_{k,m,j} - A}{T - t_{k,m,j}} \right| = \left| \frac{x_{k,m,j} - A}{T - \rho_{\epsilon_j}(x_{k,m,j})} \right| \leq \max_x \left| \frac{x - A}{T - \rho(x)} \right|. \tag{4.92}$$

Assume that ρ satisfies (4.6). Then from (4.92)

$$\max_{k\le n_j}\left\{\left|\frac{x_{k,m,j}-A}{T-t_{j,m,j}}\right|\right\}\le\wedge. \tag{4.93}$$

For each m,j, let

$$u_{m,j}(x,t)=u(x,t,P_{m,n_j,\epsilon_j}),$$
$$b_{m,j}(t)=b(t,P_{m,n_j,\epsilon_j}),$$

where u and b are given in (4.75) and (4.76) respectively. From (4.79), (4.80) and (4.93) we have for all m,j

$$\left|f^{'}(b_{m,j}(t))\right|\le\wedge. \tag{4.94}$$

$$TV(f^{'}(b_{m,j}))\le\left(\frac{T-\delta}{T-\rho(\epsilon_j)}\right)\wedge+\frac{C-A}{T-\rho(\epsilon_j+A)}. \tag{4.95}$$

Let $j>k$, then from (4.90)

$$\int_{\delta}^{T}\left|f^{'}(b_{m,j}(t)-f^{'}(b_{m,k}(t)\right|dt\le\wedge\left|\rho(\epsilon_j+A)-\rho(\epsilon_k+A)\right|. \tag{4.96}$$

Under the above notations we have

Proof of Lemma 4.1: Let ρ satisfies (4.6), then for $\rho(\epsilon_j+A)<T$ and from (4.94), (4.95), for each j, $\{f'(b_{m,j})\}_{m\in N}$ is bounded in total variation norm. Therefore from super linearity of f, $\{b_{m,j}\}_{m\in N}$ is uniformly bounded in L^∞ for all j,m. Hence from Helly's theorem and Cantors diagonalization, we can extract a subsequence still denoted by $\{b_{m,j}\}$ such that for every $j, f'(b_{m,j})\to f'(b_j)$ as $m\to\infty$ in L^1 and for *a.e.* t. Since $(f')^{-1}$ exist and hence $b_{m,j}\to b_j$ a.e. t and by dominated convergence theorem, $b_{m,j}\to b_j$ in L^1. Let $\rho_{m,j}(x)=\rho(x,P_{m,n_j,\epsilon_j})$, then from (4.83) $\rho_{m,j}(x)\to\rho_{\varepsilon_j}(x)$ uniformly. Since $f^{'}(u_{m,j}(x,\delta))=\frac{C-A}{T-\delta}$, hence by L^1_{loc} contraction, $u_{m,j}$ converges in L^1_{loc} and for *a.e.* (x,t) to a solution u_j of (1.2) with initial boundary condition

$$u_j(A,t)=b_j(t) \tag{4.97}$$

$$f^{'}(u_j(x,\delta))=\frac{C-A}{T-\delta}. \tag{4.98}$$

From (4.78) , (4.81) and (4.83), for a.e. $x\in(A,C)$

$$f^{'}(u_j(x,T))=\frac{x-A}{T-\rho_{\epsilon_j}(x)}. \tag{4.99}$$

Letting $m\to\infty$ in (4.96) to obtain

$$\int_{\delta}^{T}\left|f^{'}(b_j(t))-f^{'}(b_k(t))\right|\le\wedge\left|\rho(A+\epsilon_j)-\rho(A+\epsilon_k)\right|. \tag{4.100}$$

Since ρ is right continuous and hence $|\rho(A+\epsilon_j)-\rho(A+\epsilon_k)| \to 0$ as $j,k \to \infty$. Therefore from L^1_{loc} contractivity, there exist a subsequence still denoted by j such that $u_j \to \tilde{u}_1$, a solution of (1.2), $b_j \to \tilde{b}_1$ in L^1_{loc} and *a.e.* Letting $j \to \infty$ in (4.97) to (4.99), then $(\tilde{u}_1, \tilde{b}_1)$ satisfies (4.6) to (4.9). From Rankine-Hugoniot condition across $a_n(t,\delta), \tilde{u}_1$ satisfies (4.9). This proves (1). Similarly (2) follows and hence the Lemma.

Proof of Lemma 4.2 is much simpler than that of Lemma 4.1 because of (4.67) and (4.68), TV bounds exist for discretization of ρ. Let $\rho : [C_1, C_2] \to [A_1, A_2]$ be in $IA([C_1, C_2])$. First assume that ρ is a strictly increasing continuous function. Let $n \geq 1$ and $A_1 = y_0 < y_1 < \ldots < y_n = A_2$ such that $|y_i - y_{i+1}| \leq \frac{1}{n}$. Let k be such that $\{x : \rho(x) < y_{k-1}\} = \phi$ and $\{x : \rho(x) < y_k\} \neq \phi$. Define $x_i = A_1$ if $i \leq k-1$ and for $i \geq k$,

$$\{x : \rho(x) < y_i\} = [A_1, x_i).$$

Let $P_n = \{y_0, y_1 \ldots y_n, x_0, x_1 \ldots x_n\}$ be the corresponding partition and define

$$\rho(x, P_n) = \sum_{i=0}^{n-2} y_i \chi_{[x_i, x_{i+1})}(x) + y_{n-1}\chi_{[x_{n-1}, C_2]}(x).$$

Clearly

$$|\rho(x) - \rho(x, P_n)| \leq \frac{1}{n},$$

and hence $\rho(x, P_n) \to \rho(x)$ uniformly as $n \to \infty$.

Proof of Lemma 4.2: First assume that ρ is a strictly increasing continuous function. For $n \geq 1$, let P_n and $\rho_n(x) = \rho(x, P_n)$ be constructed as above. Let $u_n(x,t) = u(x,t,P_n)$ as in (4.63) a solution of (1.2) with initial data $u_{0,n}(x) = u_0(x, \delta, P_n)$ as in (4.64). From (4.67), $\{f'(u_{0,n})\}$ is bounded in $BV_{loc}(I\!R)$ and hence by Helly's theorem, there exist a subsequence still denoted by $\{f'(u_{0,n})\}$ converges to $f'(u_0)$ in L^1_{loc} and *a.e.* Since $u_{0,n}$ is uniformly bounded and f' is strictly increasing function, therefore $u_{0,n} \to u_0$ in L^1_{loc}. Hence from L^1_{loc} contractivity, u_n converges to $\tilde{u}$ a solution of (1.2) *a.e.* $(x,t) \in I\!R \times (\delta, T)$ with initial data $f'(u_0)$. Since $f^{'}(a_0(P)) = \frac{C_1 - A_1}{T-\delta}, f^{'}(a_n(P)) = \frac{C_2 - A_2}{T-\delta}$, hence if $\tilde{u}_0 = \tilde{u}_0|_{[A_1, A_2]}$, then from Rankine-Hugoniot condition across $a_0(t), a_n(t), (\tilde{u}, \tilde{u}_0)$ satisfies (4.17) to (4.19). At $t = T$,

$$f^{'}(u_n(x,T)) = \frac{x - \rho_n(x)}{T - \delta}, \quad \text{if } x \in [C_1, C_2],$$

and hence letting $n \to \infty$, for *a.e.* $x \in (C_1, C_2)$,

$$f^{'}(u(x,T)) = \frac{x - \rho(x)}{T - \delta}.$$

Then $(\tilde{u}, \tilde{u}_0)$ is the required solution satisfying (4.16). Let $\rho \in IA((C_1, C_2))$ and ρ_n be a strictly increasing continuous function with values in (A_1, A_2) and converging to ρ in L^1 and *a.e.* Let $(\tilde{u}_n, \tilde{u}_{n,0})$ be the corresponding solutions satisfying (4.16) to

(4.19). Hence from Helly's theorem, there exist a subsequence still denoted by $\tilde{u}_{n,0}$ converging to $\tilde{u}_0$ in L^1_{loc} and *a.e.* Therefore from L^1_{loc} contractivity, for a subsequence still denoted by $\tilde{u}_n$ converging to $\tilde{u}$ a.e to a solution of (1.2) satisfying (4.17) to (4.19). For *a.e.* $x \in (C_1, C_2)$ we have

$$f^{'}(\tilde{u}(x,t)) = \lim_{n\to\infty} f^{'}(\tilde{u}_n(x,T)) = \lim_{n\to\infty} \frac{x-\rho_n(x)}{T-\delta} = \frac{x-\rho(x)}{T-\delta}.$$

This proves the Lemma.

5 Extensions:

PROPOSITION 5.1 (Controllability of constant states):

1. *In theorem (1.1),* $g(x) = m$ *a constant if and only if* m *satisfies*
$$\frac{C_2 - A_2}{T} \leq f'(m) \leq \frac{C_1 - A_1}{T}. \tag{5.1}$$
2. *In theorem (1.2),* $g(x_0) = m$ *a constant if and only* m *satisfies*
$$f'(m) \geq \frac{C}{T-\delta}. \tag{5.2}$$
3. *In theorem (1.3),* $g_1(x) = m_1, g_2(x) = m_2$ *are constants. then* g_1, g_2 *is controllable if and only if* m_1, m_2 *satisfies*
$$f'(m_1) \geq \frac{C - A_1}{T-\delta}, f'(m_2) \leq \frac{A_2 - C}{T-\delta}. \tag{5.3}$$

Proof. (1). $g(x) = m$ if and only if $\rho(x) = x - Tf'(m)$ for all $x \in (C_1, C_2)$. Hence from (1.35) we have $A_1 \leq \rho(x) \leq A_2$ implies that $\frac{x-A_2}{T} \leq f'(m) \leq \frac{x-A_1}{T}$ and hence (5.1) holds.

(2). From (1.36) , $g(x) = m$ if and only if $\delta \leq \rho(x) \leq T$ and hence $\delta \leq x - Tf'(m) \leq T$. This implies (5.2). Similarly (5.3) follows from (1.38) and (1.39). This proves the theorem.

(3). Follows similarly.

5.1 Controllability on the boundary

As mentioned in the introduction problems (I) and (III) deal with the controllability at time $t = T$. What about the controllability at $x = A_2$. More precisely

Problem (IV): Let $T > 0$ and $A_1 < A_2$. Given $u_0 \in L^\infty(I\!\!R)$, $g \in L^\infty(0,T)$ find $\bar{u}_0 \in L^\infty((A_1, A_2))$ and u a solution of (1.2) in $\Omega = (-\infty, A_2) \times (0,T)$ such that

$$f'(u(A_2,t)) = g(t) \text{if} \quad 0 < t \leq T, \tag{5.4}$$

and

$$u(x,0) = \begin{cases} u_0(x) & \text{if} \quad x < A_1 \\ \bar{u}_0(x) & \text{if} \quad A_1 < x < A_2. \end{cases} \tag{5.5}$$

Then we have the following

Theorem 5.2 *Let $A_1 < B < A_2, \wedge > 0$ and $\rho : [0,T] \to [B, A_2]$ be a non increasing left continuous function such that for all $t \in [0,T]$,*

$$\left| \frac{A_2 - \rho(t)}{t} \right| \leq \wedge, \tag{5.6}$$

and $f'(g(t)) = \frac{A_2 - \rho(t)}{t}$. Then there exist $(u, \bar{u}_0)$ satisfying (5.4) and (5.5).

Proof. Proof follows on the same lines as in theorem (1.2) and hence only sketch the main idea of the proof.

Step 1. This step is analogous to Lemma 4.1. Frist assume that ρ is discrete. That is there exist a partition $0 = t_n \leq t_{n-1} \leq \ldots \leq t_0 = T$ and $B = x_0 < x_1 < \ldots < x_n = A_2$. Define a_i and b_i by

$$\begin{aligned} f'(a_i) &= \frac{A_2 - x_i}{t_i}, f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} \\ s_i(t) &= A_2 + (t - t_i)\frac{f(a_i) - f(b_i)}{a_i - b_i} \\ f'(a_i) &= \frac{A_2 - x_i}{t_i} > \frac{A_2 - x_{i-1}}{t_i} = f'(b_i). \end{aligned}$$

Then

$$f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} < \frac{A_2 - x_{i-1}}{t_{i-1}} = f'(a_{i-1}).$$

Hence $a_i > b_i, a_{i+1} > b_i$ and from convexity. $f'(a_i) > \frac{f(a_i)-f(b_i)}{a_i - b_i} > f'(b_i)$. Therefore

$$\begin{aligned} x_i = A_2 - tf'(a_i) &< A_2 - t\tfrac{f(a_i)-f(b_i)}{a_i-b_i} \\ &= s_i(0) < A_2 - tf'(b_i) \\ &= x_{i+1}. \end{aligned}$$

Hence for $0 \leq t \leq T$,

$$l_i(t) \leq s_i(t) \leq m_i(t),$$

where $l_i(t) = x_i + f'(a_i)t, m_i(t) = x_{i-1} + f'(b_i)t$. Define ρ_n and g_n by

$$\begin{aligned} \rho_n(t) &= x_0\chi_{[T,t_1]} + \sum_{i=1}^{n} x_i \chi_{(t_i, t_{i+1}]}(t) \\ f'(g_n(t)) &= \tfrac{A_2 - \rho_n(t)}{t}. \end{aligned}$$

Define u_n in $\Omega = (-\infty, A_2) \times (0,T)$ by

$$f'(u_n(x,t)) = \begin{cases} a_n & \text{if } \ x \le l_n(t), \\ a_i & \text{if } \ l_i(t) \le x < s_i(t), \\ b_i & \text{if } \ s_i(t) < x \le m_i(t), \\ (f')^{-1}(\frac{A_2 - x_i}{t}) & \text{if } \ m_i(t) \le x \le l_{i-1}(t), \end{cases}$$

then u_n is a solution of (1.2) in Ω satisfying

$$\begin{aligned} f'(u_n(x,t)) &= a_n \ \text{ if } x \le l_n(t) = B + t(\tfrac{A_2 - B}{T}) \\ f'(u_n(A_2,t)) &= \tfrac{x - \rho_n(t)}{t}. \end{aligned}$$

Let $\bar{u}_{n,0}(x) = u_n(x,0)$ for $B \le x \le A_2$, then as in the proof of Lemma (4.1) and from (5.6), for a subsequence $u_n \to \tilde{u}$ in $L^1_{ioc}(\Omega),\ u_n(\cdot,0) \to \tilde{u}_0$ in $L^1((B,A_2)), \rho_n \to \rho$ a. e. such that u satisfies (1.2) and for a.e. t,

$$f'(\tilde{u}(A_2,t)) = \frac{A_2 - \rho(t)}{t} \quad \text{if} \quad t \in (0,T), \tag{5.7}$$

$$\tilde{u}(x,0) = \tilde{u}_0(x) \quad \text{if} \quad x \in (B,A_2), \tag{5.8}$$

$$f'(\tilde{u}(x,t)) = \frac{A_2 - B}{T} \quad \text{if} \quad x \le l_0(t). \tag{5.9}$$

Step 2. From Lemma (2.10) there exist a μ and a solution u_1 of (1.2) in Ω satisfying

$$f'(u_1(x,t)) = a_0 \quad \text{if} \quad x > l_0(t),\ 0 \le t < T \tag{5.10}$$

$$u_1(x,0) = \begin{cases} a_0 & \text{if } \ x > B, \\ \mu & \text{if } \ A_1 < x < B, \\ u_0(x) & \text{if } \ x < A_1. \end{cases}$$

Now define $(u, \bar{u}_0)$ in Ω by

$$u(x,t) = \begin{cases} \tilde{u}(x,t) & \text{if } \ x > l_1(t), \\ u_1(x,t) & \text{if } \ x < l_0(t), \end{cases}$$

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } \ x < A_1, \\ \mu & \text{if } \ A_1 < x < B, \\ \tilde{u}_0(x) & \text{if } \ x \in (B,A_2). \end{cases}$$

Then $(u, \bar{u}_0)$ is the required solution to problem (IV).

5.2 Controllability of initial and boundary values:

All three problems deals with finding either initial data or purely boundary data. In fact one can combine both and is as follows.

Problem V: Let $u_0 \in L^\infty, T > 0, 0 < C_1 < C_2, 0 < A$, Let $\rho_1 : [0, C_1] \to [0,T], \rho_2 : [C_1, C_2] \to [0, A]$ be such that

(i) ρ_1 is a non increasing right continuous function.

(ii) ρ_2 is a non decreasing function.

Define g_1 and g_2 by

$$f'(g_1(x)) = \frac{x}{T-\rho_1(x)} \quad \text{if } x \in [0, C_1]$$

$$f'(g_2(x)) = \frac{x-\rho_2(x)}{T} \quad \text{if } x \in [C_1, C_2].$$

Then the problem is to find $b \in L^\infty(0,T)$ and $\bar{u}_0 \in L^\infty(0,A)$ such that a solution u of (1.2) in $I\!R \times (0,T)$ satisfying the following initial boundary data

$$u(0,t) = b(t) \quad \text{if } 0 < t < T. \tag{5.11}$$

$$u(x,0) = \begin{cases} \bar{u}_0(x) & \text{if } x \in (0,A), \\ u_0(x) & \text{if } x \in (A,0), \end{cases} \tag{5.12}$$

and

$$f'(u(x,t)) = \begin{cases} g_1(x) & \text{if } x \in (0,C_1), \\ g_2(x) & \text{if } x \in (C_1,C_2). \end{cases} \tag{5.13}$$

Theorem 5.3 *Let $\lambda > 0, 0 < A_1 < A$ be given. Let ρ_1 and ρ_2, g_1 and g_2 be as above and satisfying*

$$0 \le \rho_2(x) \le A_1, \quad \left|\frac{x}{T-\rho_1(x)}\right| \le \wedge \tag{5.14}$$

then problem (V) admits a solution.

Idea of the proof. First get a free region by choosing λ large such that the solution u_λ of (1.2) in $I\!R \times (0,\infty)$ satisfying for $0 < t < T$,

$$u_\lambda(x,t) = a_1 = \frac{C_2 - A_1}{T}, \quad \text{if } x < A_1 + tf'(a_1) = l_1(t),$$

$$u_\lambda(x,0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < A, \\ u_0(x) & \text{if } x > A. \end{cases}$$

Existence of u_λ is guaranteed from Lemma 2.9. Let $f'(a_0) = \frac{C_1}{T}$ and for $0 < t < T$ define the free region F_1 and F_2 by

$$F_1 = \{(x,t) : 0 < x < l_0(t) = tf'(a_0)\}, F_2 = \{(x,t) : l_0(t) < x < l_1(t) = A_1 + tf'(a_1)\}.$$

Since $0 \le \rho_1(x) \le T$ for $x \in (0, C_1)$ and satisfying (5.14), therefore from Lemma (4.1), there exist a solution u_1 of (1.2) in F_1 and $b \in L^\infty(0,T)$ such that

$$\begin{aligned} u_1(0,t) &= b(t) \\ u_1(x,T) &= g_1(x) \quad \text{if } x \in (0,C_1) \\ u_1(l_0(t)-,t) &= a_0. \end{aligned}$$

From (5.14) and Lemma (4.2), there exist a solution u_2 of (1.2) in F_2 and $\tilde{u}_0 \in L^\infty(0, A_1)$ such that

$$\begin{array}{lcl} u_2(x,T) & = & g_2(x) \quad \text{if} \quad x \in (C_1, C_2) \\ u_2(x,0) & = & \tilde{u}_0(x) \quad \text{if} \quad x \in (0, A_1) \\ u_2(l_0(t)+, t) & = & a_0, \ \ u_2(l_1(t)-, t) = a_1. \end{array}$$

From RH conditions, glue u_1, u_2, u_λ to a single solution u of (1.2) in $0, \infty) \times (0, T)$ by

$$u(x,t) = \begin{cases} u_1(x,t) & \text{if} \quad (x,t) \in F_1, \\ u_2(x,t) & \text{if} \quad (x,t) \in F_2, \\ u_\lambda(x,t) & \text{if} \quad x > l_1(t), \end{cases}$$

and

$$u(x,0) = \begin{cases} \tilde{u}_0(x) & \text{if} \quad x \in (0, A_1), \\ \lambda & \text{if} \quad x \in (A_1, A), \\ u_0(x) & \text{if} \quad x \in (A, \infty). \end{cases}$$

Then $(u, u(x,0), b)$ is the required solution to problem (V) The same method allows to generalize problem III also.

6 Optimal Control:

Let $T > 0, k$ a measurable function such that $f'(k) \in L^2(I\!R)$ and u be a solution of (1.2) with initial data u_0. Consider the optimal control problem

$$m = \inf_{u_0} J(u_0) = \inf_{u_0} \int\limits_{I\!R} |f'(u(x,T)) - f'(k(x))|^2 dx. \tag{6.1}$$

For the Burger's equation ($f(u) = \frac{u^2}{2}, f'(u) = u$), this problem was studied by Castro-Palcious-Zuazua [6]. Under the assumption that k and u_0 have compact support. They prove an existence of a minimizer and derived an algorithm to capture this minimizer. The main difficulty is that the functional is not differentiable .

In this paper we reduce the problem to a simple projection problem on a convex set and then use the standard numerical method to capture the unique minimizer. Then from Lemma (4.2) we can construct a minimizer to (6.1).

Reduction to a simpler problem: Let $k \in L^2_{loc}(I\!R)$ and $I = (C_1, C_2)$ such that

$$k(x) = \theta_f \quad \text{if} \quad x \notin I. \tag{6.2}$$

Define the admissible class A of functions by

$$A = \{u_0 \in L^\infty(I\!R) : u_0(x) = \theta_f \quad \text{outside a compact set }\}.$$

Since $f'(k(x)) = f'(\theta_f) = 0$ for $x \notin I$, hence J is well defined on A and the optimal control problem is to find $\tilde{u}_0$ such that

$$J(\tilde{u}_0) = \inf_{u_0 \in A} J(u_0). \tag{6.3}$$

For a given $u_0 \in A$, let u be the solution of (1.2) with initial data u_0. Then from Lax-Olienik explicit representation (2.12), there exist a non decreasing function ρ in x such that for $a.e.x$

$$f'(u(x,t)) = \frac{x - \rho(x)}{T}. \tag{6.4}$$

From the finite domain of dependence, $u(x,T) = \theta_f$ outside a compact set and hence $f'(u(x,t)) = 0$ for x outside a compact set. Therefore for outside a compact set $\rho(x) = x$. Let

$$B = \left\{ \rho \in L^\infty_{loc}(I\!R) : \begin{array}{l} (i)\rho \quad \text{is a non decreasing function} \\ (ii)\rho(x) = x \quad \text{outside a compact set} \end{array} \right\} \tag{6.5}$$

then the problem (6.3) reduces to find a $\tilde{\rho} \in B$ such that

$$J(\bar{\rho}) = m = \inf_{\rho \in B} \int_{I\!R} |\frac{x - \rho(x)}{T} - f'(k(x))|^2 dx. \tag{6.6}$$

From the definition of k and B, J is well defined. Let $N > 0$ and define

$$B_N = \{\rho \in B; \rho(x) = x \quad \text{for} \quad |x| > N\}. \tag{6.7}$$

Then $B = \cup_{N=1}^\infty B_N$. For $\rho \in B_N$, define $\tilde{\rho} \in B$ as follows:

$$\tilde{\rho}(x) = \begin{cases} x & \text{if} \quad x \leq \min(C_1, \rho(C_1)) \text{ or } x > \max(C_2, \rho(C_2)), \\ \rho(C_1) & \text{if} \quad \rho(C_1) < x \leq C_1, \\ \rho(x) & \text{if} \quad C_1 < x \leq C_2, \\ \rho(C_2) & \text{if} \quad C_2 < x \leq \rho(C_2), \end{cases} \tag{6.8}$$

and $0 < x - \tilde{\rho}(C_1) \leq x - \rho(x)$ if $\rho(C_1) < x \leq C_1$ and $0 \leq \rho(C_2) - x \leq \rho(x) - x$ if $C_2 < x < \rho(C_2)$. Since $f'(k(x)) = 0$ if $x \notin (\min(C_1, J(C_1)), \max(C_2, J(C_2))$ and hence

$$\begin{aligned} J(\tilde{\rho}) &= \int_{\min(C_1,J(C_1))}^{\mathrm{Max}(C_2,J(C_2))} |\tfrac{x-\tilde{\rho}(x)}{T} - f'(k(x))|^2 dx \leq \int_{I\!R} |\tfrac{x-\rho(x)}{T} - f'(k(x))|^2 dx \\ &= J(\rho). \end{aligned} \tag{6.9}$$

Therefore from (6.9), if

$$\tilde{B} = \{\rho \in B; \rho = \tilde{\rho}\}$$

then

$$m = \inf_{\rho \in \tilde{B}} J(\rho). \tag{6.10}$$

LEMMA 6.1 *Let $\rho_n \in \tilde{B}$ be a minimizing sequence of (6.10). Then $\{\rho_n(C_1), \rho_n(C_2)\}$ are bounded.*

Proof. Let $\rho_0(x) = x \in \tilde{B}$ and suppose $\rho_k(C_1) \to -\infty$ as $k \to \infty$, then

$$\begin{aligned} \|f'(k)\|_2^2 = \int_{C_1}^{C_2} |f'(k)|^2 dx = J(\rho_0) &\geq \int_{\rho_k(C_1)}^{C_1} |\tfrac{x-\rho_k(x)}{T} - f'(k(x))|^2 dx \\ &= \int_{\rho_k(C_1)}^{C_1} |\tfrac{x-\rho_k(C_1)}{T}|^2 dx = \tfrac{1}{2T^2}|C_1 - \rho_k(C_1)|^3 \\ &\to \infty \text{ as } k \to \infty, \end{aligned} \tag{6.11}$$

which gives a contradiction. Similarly $\rho_n(C_2)$ is bounded from above. This proves the Lemma.

Theorem 6.2 There exist a unique minimizer ρ to (6.10) satisfying

$$C_1 - (2T^2\|f(k)\|_2^2)^{1/3} \leq \rho(C_1) \leq \rho(C_2) \leq C_2 + (2T^2\|f(k)\|_2^2)^{1/3}. \tag{6.12}$$

Proof. In view of Lemma 6.1, there exist $N > 0$ such that for $\tilde{B}_N = \tilde{B} \cap B_N$

$$m = \inf_{\rho \in \tilde{B}_N} J(\rho).$$

Now $\tilde{B}_N$ is a closed convex set in $L^2((-N, N))$ and hence there exist a unique minimizer $\bar{\rho} \in \tilde{B}_N$. Now (6.12) follows from (6.11). This proves the theorem.

Construction of Optimal control:
Let

$$A_1 = C_1 - (2T^2\|f(k)\|_2^2)^{1/3},\ A_2 = C_2 + (2T^2\|f(k)\|_2^2)^{1/3},$$

and $J = (A_1, A_2)$. then $\bar{\rho} \in IA(J)$ (see 1.32) with values in J. Hence from Lemma (4.2), there exist $\tilde{u}_0 \in L^\infty(J)$ and a solution $u_1(x,t)$ of (1.2) in $\Omega = J \times (0,T)$ satisfying

$$\begin{aligned} f'(u_1(x,T) &= \frac{x - \bar{\rho}(x)}{T} \\ u_1(0,T) &= \tilde{u}_0 \\ f'(u_1(A_1,t)) &= f'(u_1(A_2,t)) = 0. \end{aligned}$$

Now define $\bar{u}_0$ and $\bar{u}$ by

$$\bar{u}_0(x) = \begin{cases} \tilde{u}_0(x) & \text{if } x \in J, \\ \theta_f & \text{if } x \notin J. \end{cases}$$

$$\bar{u}(x,t) = \begin{cases} u_1(x,t) & \text{if } (x,t) \in \Omega, \\ \theta_f & \text{if } (x,t) \notin \Omega, \end{cases}$$

then $\bar{u}_0$ is a solution to optimal control problem (6.1)

Numerically it is fairly easy to obtain $\bar{\rho}$ since the minimization is a simple convex minimization problem. Once $\bar{\rho}$ is obtained, then using Lemma (4.2), approximate $\bar{u}_0$ can be constructed.

Remark 6.3 *In the optimal Control Problem: condition on k can be relaxed and is as follows. We can assume that for $x \notin (C_1, C_2)$*

$$k(x) = \begin{cases} \alpha_1 & \text{if} \quad x < C_1, \\ \alpha_2 & \text{if} \quad x > C_2. \end{cases}$$

Then the class B is defined as follows. Let $\rho \in L^\infty_{loc}(I\!R)$ be such that

(i). ρ is non decreasing function

(ii). There exist $A_1 < A_2$ such that

$$\rho(x) = \begin{cases} x - \alpha_1 T & \text{if} \quad x < A_1, \\ x - \alpha_2 T & \text{if} \quad x > A_2. \end{cases}$$

Let

$$B = \{\rho; \rho \text{ satisfying (i) and (ii) }\}.$$

Then by the similar arguments one can show that there exist a unique $\tilde{\rho} \in B$ such that

$$J(\tilde{\rho}) = \inf_{\rho \in B} J(\rho). \tag{6.13}$$

In view of the controllability of initial and boundary value problems, we can extend the optimal controllability for the boundary value problem. To illustrate this, let us consider one sided initial boundary value problem. Let $0 < T, 0 < C$, and $k \in L^2_{loc}(I\!R)$ such that $k(x) = \theta_f$ for x large. Let $u_0 \in L^\infty, b \in L^\infty(0,T)$ and u be the solution of (1.2) in $\Omega = (0, \infty) \times (0, T)$ with

$$u(t, 0) = b(t) \quad 0 < t < T,$$

$$u(x, 0) = u_0(x) \quad x > 0,$$

$$J(u_0, b) = \int_{I\!R} |f'(u(x,T) - f'(k(x))|^2 dx.$$

In order to make integral finite assume $u_0(x) = \theta_f$ for x large. Hence define

$$A = \{(u_0, b) \in L^\infty(I\!R) \times L^\infty(0,T); u_0(x) = \theta_f, \text{ for large } x\}.$$

Then optimal control problem is to find $(\tilde{u}_0, \tilde{b})$ such that

$$J(\tilde{u}_0, \tilde{b}) = \inf_{(u_0, b) \in A} J(u_0, b). \tag{6.14}$$

From Joseph-Gowda [16] and Lax - Olienik [10] formulas for any $(u_0, b) \in A$. there exist $0 \le C_1 \le C$ and $\rho_1 : [0, C_1] \to [0, T]$ a non increasing function and $\rho_2 : [C_1, C] \to I\!R$ a non decreasing function such that

(i). $\rho_2(x) = x$ for x large and

$$f'(u(x,T)) = \frac{x - \rho_2(x)}{T} \quad \text{if} \quad x \in (C_1, C).$$

(ii). $f'(u(x,t)) = \frac{x}{T - \rho_1(x)}, \quad 0 < x < C_1$ and

$$\left| \frac{x}{T - \rho(x)} \right| \le \wedge,$$

where $\wedge$ is a constant depending on the Lipschitz constant of f on $[-\|b\|_\infty, \|b\|_\infty]$.

Therefore

$$\begin{aligned} J(u_0, b) &= \int_0^{C_1} \left| \frac{x}{T - \rho_1(x)} - f'(k(x)) \right|^2 dx + \int_{C_1}^{C} \left| \frac{x - \rho_2(x)}{T} - f'(k(x)) \right|^2 dx \\ &= J_1(\rho_1, C_1) + J_2(\rho_2, C_1) \\ &= J(\rho_1, \rho_2, C_1) \end{aligned}$$

$$m = \inf_{u_0, b} J(u_0, b) = \inf_{\rho_1, \rho_2, C_1} \{J_1(\rho_1, C_1) + J_2(\rho_2, C_1)\}.$$

Let $(\rho_{1,k}, \rho_{2,k}, C_{1,k})$ be a minimizing sequence. As in the previous case, it follows that there exists $A > 0$, such that $\rho_{2,k}(x) = x$ for all k and $x \ge A$. Hence $\{\rho_{i_k}\}$ are uniformly bounded monotone functions, therefore from Helly's theorem there exist a subsequence still denoted by $\{\rho_{1,k}, \rho_{2,k} C_k\}$ converges $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ for all $x \in [0, C]$. Hence by Fatau's Lemma $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ is an optimal solution and from Lemma 4.1 , 4.2 we can construct the solution $(\tilde{u}_0, \tilde{b})$.

If $\tilde{C}$ is known, then $\tilde{\rho}_1$ and $\tilde{\rho}_2$ can be obtained from the L^2 - projection as follows: Let $\eta_1(x) = \frac{x}{T - \rho_1(x)}$ then $\frac{x}{\eta_1} \le T$, $\frac{\eta_1(x)}{x}$ is a non increasing right continuous function. Let $\eta : [0, C] \to I\!R$ be function such that

1. $\frac{\eta(x)}{x}$ is non increasing right, continuous function in $[0, \tilde{C}]$ and $\frac{x}{\eta(x)} \le T$ in $[0, \tilde{C}]$.
2. $\eta|_{(\tilde{C}, C)}$ is non decreasing function.

Let

$$K_N = \{\eta; \eta \text{ satisfying } (1)\ ,\ (2) \text{ and } \ 0 \le \eta \le N\}.$$

Then K_N is a closed convex set in $L^2((0,C))$ and let $\tilde{\eta}_k \in K_N$ such that

$$J(\tilde{\eta}) = \inf_{\eta \in K_N} J(\eta) = \inf\{\int_0^{\tilde{C}} |\eta(x) - f'(k(x))|^2 dx + \int_{\tilde{C}}^{C} |\frac{x-\eta(x)}{T} - f'(k(x))|^2 dx\}.$$

Then for large η, if we define

$$\begin{aligned} \tilde{\rho}_1(x) &= T - \frac{x}{\tilde{\eta}(x)} \quad \text{for} \quad x \in (0, \tilde{C}) \\ \tilde{\rho}_2,(x) &= \tilde{\eta}(x) \qquad \text{for} \quad x \in (\tilde{C}_1, C) \end{aligned}$$

then $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{C})$ is the optimal solution.

References

[1] Adimurthi, Shyam Sundar Ghoshal, G.D.Veerappa Gowda, Finer analysis of characteristic curves, and its applications to shock profile, exact and optimal controllability of conservation law with discontinuous strict convex fluxes - preprint , 2011.

[2] Adimurthi, G.D. Veerappa Gowda,, Conservation Law with discontinuous flux, J.Math, Kyoto Univ, 43, no.J , 2003 , 27-70.

[3] F. Ancona, A,Marson, On the attainability set for scalar non linear conservation laws with boundary control, SIAM J.Control optim., vol 36, No.1 , 1998, 290-312.

[4] F.Ancona, A.Marson, Scalar non linear conservation laws with integrable boundary data, Nonlinear Analysis , 35, 1999, 687-710.

[5] Bardos, C., Leroux, A. Y., and Nedelec, J. C., First-order quasilinear equations with boundary conditions, Comm. in PDE 4, 1979, pp. 1017-1037.

[6] C.Castro, F.Palacios, E.Zuazua, Optimal control and vanishing viscosity for the Burgers equations, Integral methods in science and engineering, vol 2, 65-90, Birkhouser Boston Inc, Boston MA 2010.

[7] J.M.Coron, Global asymptotic stabilization for controllable systems without drift, Mth. Control signals systems, 5, 1992, 295-312.

[8] C.M.Dafermos, Characteristics in hyperbolic conservations laws, A study of the structure and the asymptotic behavious of solutions, Research notes in maths., Pitman Vol 1, (1977), 1 - 56.

[9] Dafermos, Constantine,Shearer, Michael, Finite time emergence of a shock wave for scalar conservation laws. J. Hyperbolic Differ. Equ. 7 (2010), no. 1, 107-116.

[10] L.C. Evans, Partial differential equations, Graduate studies in Mathematics, vol 19, AMS 1998.

[11] O. Glass, S.Guerrero, On the uniform controllability of the Burgers equation, SIAM J. Control optim. 46 mo.4, 2007 , 1211-1238.

[12] Hopf, Eberhard The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3, (1950). 201-230.

[13] S.Guerrero, O . Yu. Immunauvilov, Remarks on global controllability for the Burgers equation with two control forces, Preprint 2009.

[14] E.Godleweski, P.A.Raviant, Hyperbolic systems of conservation laws, Mathematiques and Applications 3/4, ellipses, Paris, 1991.

[15] T. Horsin, On the controllability of the Burger equation, ESIAM, Control optimization and Calculus of variations, 3, 1998, 83-95.

[16] Joseph.K.T, Veerappa Gowda, G.D, Explicit formula for the solution of Convex conservation laws with boundary condition, Duke Math.J..62 , No.2, 1991, 401-416.

[17] Yong Jung Kim, Asymptotic behavior of solutions to scalar conservation laws and optimal convergence orders to N - waves, J,Diff cans, 192, no 1, (2..3) , 202-224.

[18] P.Lax , Hyperbolic systems of conservation Laws II, comm Pure Appl. Math, 10 (1957), 535- 566.

[19] T.P.Liu, Invariants and Asymptotic behavior of solutions of a conservation law, Proc AMS, vol 71, No.2 (1978) 227- 231.

[20] T.P.Liu an D.M.Pierre, Source solutions and asymptotic behavious in conservation laws, J.Diff. equns , 51, (1984), 419 - 441.

[21] David G.Schaffer, A regularity theorem for conservation laws, Advances in Mathematics, 11 (1973) 368 - 380.

[22] Kazumi Tanuma, Asymptotic beahvious of the shock curve and the Entropy solution to the scalar conservation law with periodic initial data, Funk cialaj Ekvacioj, 35 (1992), 467- 484.