SEMILINEAR EVOLUTION PDES: SPECIAL SOLUTIONS, INITIAL VALUE PROBLEM, AND STABILITY

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Warning: These notes have been prepared to serve as a guideline. They are certainly packed with mistakes!

1. Introduction

Among the many equations that arise in applied mathematics, a number of them are semilinear evolution PDEs. Of course, each of these equations has its own particularities, but general methods have been developed with a wide range of potential applications. The aim of this course is to present a selection of basic results and techniques that anyone studying a semilinear evolution EDP is likely to find useful. For the sake of simplicity, we mostly consider two model equations: the semilinear heat (NLH) and Schrödinger (NLS) equations. More precisely, we consider the nonlinear heat equation

$$u_t - \Delta u = g(u), \tag{NLH}$$

where $g \in C(\mathbb{R}, \mathbb{R})$. The unknown $u(t, x)$ is a real-valued function of $t \in [0, T]$ for some $0 < T < \infty$ and $x \in \Omega$ where Ω is a domain (i.e. a connected, open subset) of $\mathbb{R}^N$. Note that Ω can possibly be the whole space $\mathbb{R}^N$. We will also often assume some boundary conditions. One classical condition is the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ (in the sense $u(x) \to 0$ as $|x| \to \infty$ when $\Omega = \mathbb{R}^N$). Another classical condition is the Neumann boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$, where $\partial u/\partial n$ is the outward normal derivative of u. Finally, we will also often assume an initial condition of the form $u(0) = \varphi$ in Ω, where φ is a given initial value $\Omega \to \mathbb{R}$. Similarly, we consider the nonlinear Schrödinger equation

$$iu_t + \Delta u + g(u) = 0. \tag{NLS}$$

Here, the unknown $u(t, x)$ is a complex-valued function of $t \in [0, T]$ for some $0 < T < \infty$ and $x \in \mathbb{R}^N$. We will condider Dirichlet's boundary condition in the form $u(t, \cdot) \in L^2(\mathbb{R}^N))$ and often assume an initial condition of the form $u(0) = \varphi$ in $\mathbb{R}^N$, where φ is a given initial value $\mathbb{R}^N \to \mathbb{C}$. Even though we mostly study (NLH) and (NLS), the methods that we will present are useful to the analysis of other equations. We will indicate possible extensions and references. We will study the following aspects.

1. Special solutions. Certain equations possess special solutions. Stationary solutions are one example but, depending on the invariants of the equation, other solutions may exist, like standing waves, traveling waves and self-similar solutions. (NLH) or (NLS) then reduce to a semilinear elliptic equation. In one space dimension or under symmetry assumptions, the existence of solutions can be studied by elementary ODE techniques (shooting arguments). In more general situations, one

can often apply variational arguments (global minimization, constrained minimization, mountain pass theorem).

2. initial value problem. The existence of local solutions for (NLH) follows from the simplest fixed-point argument using either an explicit kernel (in the whole space) or the theory of semigroups (in a bounded domain). A certain care must be taken, though, on such issues as uniqueness and the blowup alternative. For (NLS), local existence also follows from a simple fixed-point argument, provided one makes use of the Strichartz estimates. We will also establish simple sufficient conditions for global existence and for finite time blowup.

3. Stability of special solutions. It may be useful to decide whether or not a given special solution is stable (in an appropriate sense) with respect to small perturbations of its initial value. Indeed, an unstable solution is unlikely to be seen, either experimentally or numerically. For (NLH), if the linearized operator at a given stationary solution has positive first eigenvalue, then the solution is stable. If the first eigenvalue is negative, then the stationary solution is unstable. In general, there are local stable and unstable manifolds. For (NLS), the stability of standing waves cannot be decided by those arguments because all eigenvalues of the linearized operator are purely imaginary. This corresponds to center manifolds, for which other techniques are required. We will give sufficient conditions for stability and instability, based on dynamical systems arguments and on variational (invariant sets and constrained minimization) arguments.

2. Other methods

Concerning the part on special solutions, we will not consider other important, classical methods that can be applied to study elliptic equations, among wich: The method of sub- and super-solutions. The topological degree. The implicit function theorem.

As for the existence for the evolution problem, we will apply a perturbation argument, based on the semigroup theory. There are of course other methods, which may be more appropriate, depending on the problem. In particular, there are various compactness techniques that can be used to construct solutions, among which some are based on the Galerkin approximation.

3. Special solutions

One possible way of studying the equations (NLH) and (NLS) is to look for special solutions, that is solutions of a particular form, whose properties are supposedly clear. The simplest of all such solutions are **stationary solutions** (solutions that are independent of the time variable), i.e.

$$u(t,x) \equiv v(x).$$

The problem then reduces for both (NLH) and (NLS) to the nonlinear elliptic equation

$$-\Delta v = g(v), \tag{NLE}$$

with the relevant boundary conditions. For (NLS), one can look for "standing waves, i.e. solutions of the form

$$u(t,x) \equiv e^{i\omega t} v(x), \tag{3.1}$$

where $\omega \in \mathbb{R}$. Such solutions have a **time-independent profile** $|u(t,x)|$. Suppose the nonlinearity is such that $g(e^{i\theta}z) = e^{i\theta}g(z)$ for all $\theta \in \mathbb{R}$, $z \in \mathbb{C}$, i.e.

$$g(z) = \frac{z}{|z|}g(|z|), \tag{3.2}$$

for all $z \in \mathbb{C}$. (For example, $g(z) = |z|^\alpha z$.) The problem then reduces to the elliptic equation

$$-\Delta v + \omega v = g(v),$$

which is also of the form (NLE) with $g(v)$ replaced by $g(v) - \omega v$.

Suppose that the domain Ω in which the equation is set is the whole space $\mathbb{R}^N$. Note that under the assumption (3.2) the equation (NLS) is Galilean invariant. More precisely, if $u(t,x)$ is a solution, then so is

$$z(t,x) \equiv e^{i(\frac{1}{2}\xi\cdot x - \frac{1}{4}|\xi|^2 t)}u(t, x - \xi t),$$

for every $\xi \in \mathbb{R}^N$. In particular, if u is a solution of the form (3.1), then we see that

$$z(t,x) = e^{i(\frac{1}{2}\xi\cdot x - \frac{1}{4}|\xi|^2 t + \omega t)}v(x - \xi t).$$

Thus, if $\omega = -|\xi|^2/4$, then the solution has the form $z(t,x) = e^{i\frac{\xi}{2}\cdot(x-\xi t)}v(x - \xi t)$. In other word,

$$z(t,x) = \psi(x - \xi t),$$

with $\psi(x) = e^{\frac{i}{2}\xi\cdot x}v(x)$. Such solutions are called **traveling waves**.

Still in the case $\Omega = \mathbb{R}^N$, one can also look for traveling wave solutions of (NLH). If $u(t,x) \equiv v(x - \xi t)$ with $\xi \in \mathbb{R}^N$, then v must satisfy

$$-\Delta v - \xi \cdot \nabla v = g(v).$$

On the other hand if, instead of (NLH), one starts with the perturbed heat equation

$$u_t - \Delta u + \xi \cdot \nabla u = g(u),$$

then traveling wave solutions of the form $u(t,x) \equiv v(x - \xi t)$ lead again to the equation (NLE).

Still assuming $\Omega = \mathbb{R}^N$, other possible special solutions of (NLH) are, if the nonlinearity is homogeneous, **self-similar solutions**. Suppose for example that

$$g(u) = \gamma|u|^\alpha u, \tag{3.3}$$

with $\gamma \in \mathbb{R}$ and $\alpha > 0$. If u is a solution of (NLH) on some time interval (a,b), then for every $\lambda > 0$,

$$u_\lambda(t,x) := \lambda^{\frac{2}{\alpha}}u(\lambda^2 t, \lambda x), \tag{3.4}$$

is also a solution of (NLH) on the time interval $(a/\lambda^2, b/\lambda^2)$. If $(a,b) = (0,\infty)$, then one can look for solutions which are invariant for this transformation, i.e. $u_\lambda \equiv u$ for all $\lambda > 0$, i.e. $u(t,x) = \lambda^{-\frac{2}{\alpha}}u(\lambda^{-2}t, \lambda^{-1}x)$ for all $\lambda, t > 0$, $x \in \mathbb{R}^N$. Letting $\lambda = \sqrt{t}$, this means that

$$u(t,x) \equiv t^{-\frac{1}{\alpha}}f\Big(\frac{x}{\sqrt{t}}\Big),$$

where $f = u(1,\cdot)$. Such solutions are called **forward self-similar solutions**. For this to happen, we see that f must solve the nonlinear elliptic equation

$$-\Delta f - \frac{1}{2}x\cdot\nabla f - \frac{1}{\alpha}f = \gamma|f|^\alpha f. \tag{3.5}$$

If $(a,b)=(-\infty,0)$, then one can also look for solutions which are invariant for this transformation, i.e. $u_\lambda \equiv u$ for all $\lambda>0$, i.e. $u(t,x)=\lambda^{-\frac{2}{\alpha}}u(\lambda^{-2}t,\lambda^{-1}x)$ for all $\lambda>0$, $t<0$, $x\in\mathbb{R}^N$. Letting $\lambda=\sqrt{(-t)}$, this means that

$$u(t,x)\equiv(-t)^{-\frac{1}{\alpha}}f\Big(\frac{x}{\sqrt{(-t)}}\Big),$$

where $f=u(-1,\cdot)$. Such solutions are called **backward self-similar solutions**. For this to happen, we see that f must solve the nonlinear elliptic equation

$$-\Delta f+\frac{1}{2}x\cdot\nabla f+\frac{1}{\alpha}f=\gamma|f|^\alpha f. \tag{3.6}$$

Equations (3.5) and (3.6) are similar to (NLE), the difference being the first order term $x\cdot\nabla f$.

4. ODE methods

A useful reference for this section is [8]

4.1. The one-dimensional case.

Suppose $N=1$ and $\Omega=(a,b)$ with $-\infty\le a<b\le+\infty$. In this case the equation (NLE) becomes the ODE

$$u''+g(u)=0. \tag{4.1}$$

The usual boundary conditions are Dirichlet ($u(a)=u(b)=0$) or Neumann ($u'(a)=u'(b)=0$) if the interval (a,b) is bounded. In the case of the whole line, one can also consider Dirichlet ($u(x)\to0$ as $|x|\to\infty$) or more generally (this is in particular relevant for traveling waves) $u(x)\to\ell_\pm$ as $x\to\pm\infty$.

In all this section, we suppose $g:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz.

Existence: Given any $x_0,u_0,v_0\in\mathbb{R}$, there exists $\tau>0$ and a unique function $u\in C^1([x_0-\tau,x_0+\tau])$ which is a solution of (4.1) on $[x_0-\tau,x_0+\tau]$ and satisfies the initial conditions $u(x_0)=u_0$, $u'(x_0)=v_0$. This is standard by the contraction argument. We write the problem in the form

$$u(x)=u_0+(x-x_0)v_0-\int_{x_0}^x\int_{x_0}^s g(u(\sigma))\,d\sigma=:\Phi(u)(x). \tag{4.2}$$

It is very easy to show that if $\tau>0$ is sufficiently small, then Φ is a strict contraction $E\to E$, where E is the metric space $E=\{u\in C([x_0-\tau,x_0+\tau]);\|u\|_{L^\infty}\le2|u_0|\}$ equipped with the distance $\mathrm{d}(u,v)=\|u-v\|_{L^\infty}$. Thus it has a fixed point u which is a solution of (4.2). It follows from the equation (4.2) that in fact u is C^2.

Uniqueness must be proved separately: If u,v are two solutions, then for $x_0\le x\le x_0+\tau$

$$\begin{aligned}|u(x)-v(x)|&\le C\int_{x_0}^x\int_{x_0}^s|u(\sigma)-v(\sigma)|\,d\sigma\\&\le C\int_{x_0}^x|u(\sigma)-v(\sigma)|\,d\sigma.\end{aligned}$$

The fact that $u(x)\equiv v(x)$ for $x_0\le x\le x_0+\tau$ follows from **Gronwall's lemma**. (Thomas Hakon Grönwall, Sweedish, b. 1877.) Similarly, $u(x)\equiv v(x)$ for $x_0-\tau\le x\le x_0$.

Gronwall's lemma ([18, 3])**.** *Let $T > 0$, $A \geq 0$ and let $f \in L^1(0,T)$ be a nonnegative function. If the nonnegative function $\varphi \in C([0,T])$ satisfies*

$$\varphi(t) \leq A + \int_0^t f(s)\varphi(s)\,ds,$$

for every $t \in [0,T]$, then,

$$\varphi(t) \leq A\ \exp\Big(\int_0^t f(s)\,ds\Big),$$

for every $t \in [0,T]$. In particular, if $A = 0$ then $\varphi(t) \equiv 0$.

Proof. Set $h(t) = \psi(t)\exp(-\int_0^t f(s)\,ds)$, where $\psi(t) = A + \int_0^t f(s)\varphi(s)\,ds$. $\psi, h \in W^{1,1}$ and

$$\begin{aligned} h'(t) &= (\psi'(t) - f(t)\psi(t))\exp\Big(-\int_0^t f(s)\,ds\Big) \\ &= (f(t)\varphi(t) - f(t)\psi(t))\exp\Big(-\int_0^t f(s)\,ds\Big) \leq 0. \end{aligned}$$

It follows that $h(t) \leq h(0)$, which proves the result. □

Some essential properties:

– Suppose u_0 is an equilibrium, i.e. $g(u_0) = 0$, so that $u(x) \equiv u_0$ is a solution. If there exists x_0 such that $u(x_0) = u_0$ and $u'(x_0) = 0$, then $u(x) \equiv u(x_0)$. In other words, if $u(x) \not\equiv u(x_0)$ and if $u(x_0) = u_0$ for some x_0, then $u'(x_0) \neq 0$; and if $u'(x_0) = 0$ for some x_0, then $u(x_0) \neq u_0$. This follows by uniqueness.

– Suppose $u(x) \not\equiv C$ is a solution on some interval I. It follows that $|u'(x)| + |g(u(x))| \neq 0$ for all $x \in I$ and that the zeroes of u' and the zeroes of $g(u)$ are isolated.

– Suppose u is a global solution and $u(x) \to \ell$ as $x \to \infty$. It follows that $g(\ell) = 0$. Indeed, $u''(x) \to g(\ell)$ by the equation. This is possible only if $g(\ell) = 0$.

– Energy: $\frac{1}{2}|u'|^2 + G(u) \equiv C$, where $G(u) = \displaystyle\int_0^u g(s)\,ds$.

These properties are valid for more general equations.

• **Case** $g(u) = 0$**.** Solution $u(x) = \alpha x + \beta$. For Dirichlet, only 0. For Neumann, all constants. On the line, only 0 (or all constants).

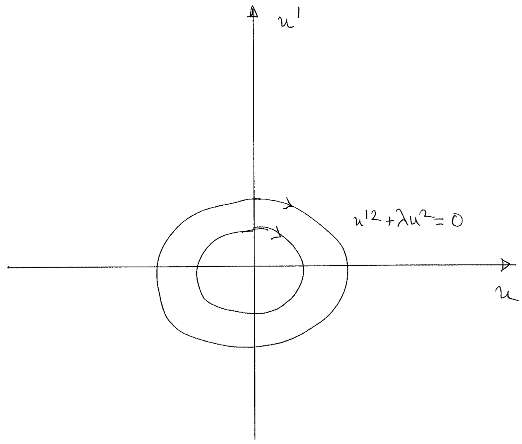
• **Case** $g(u) = \lambda u$**,** $\lambda > 0$**.** Solution

$$u(x) = \alpha \sin(x\sqrt{\lambda} + \theta).$$

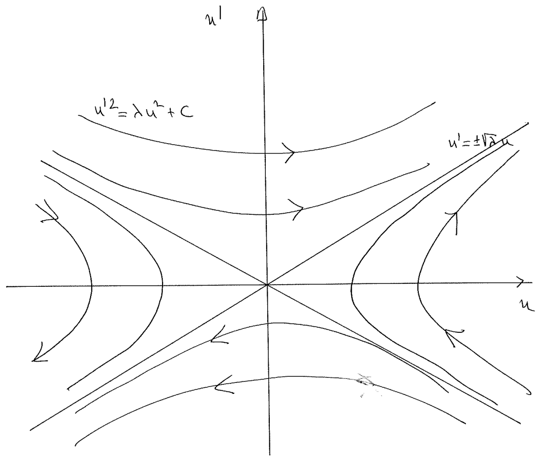
All solutions periodic **with same period**. Dirichlet and Neumann: nonzero solution if and only if

$$b - a = k\pi/\sqrt{\lambda}, \quad k \in \mathbb{N}.$$

Can be seen by **phase plane analysis**.



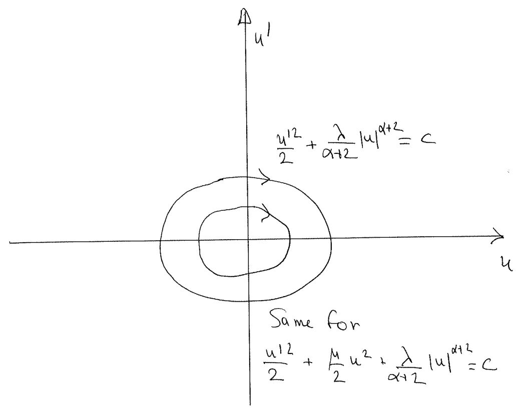
• **Case** $g(u) = -\lambda u$, $\lambda > 0$. Dirichlet, Neumann and line: only the trivial solution 0. (Immediate by phase plane analysis.)



• **Case** $g(u) = \lambda|u|^\alpha u$, $\alpha, \lambda > 0$. Note that all solutions are global by energy conservation and that if $u(x)$ is a solution, then so is (by **scaling**)

$$u_\mu(x) \equiv \mu u(\mu^{\frac{\alpha}{2}} x), \tag{4.3}$$

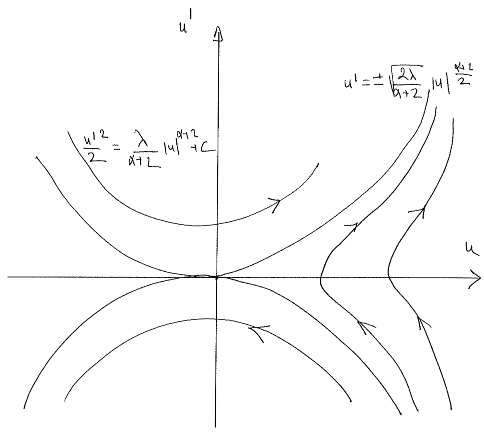
for $\mu > 0$. Phase plane + scaling $\Longrightarrow$ infinitely many nontrivial solutions for Dirichlet and Neumann. For Dirichlet, one positive solution. Only trivial solution 0 on the line. **All solutions are periodic, but with a period depending continuously on the energy!**



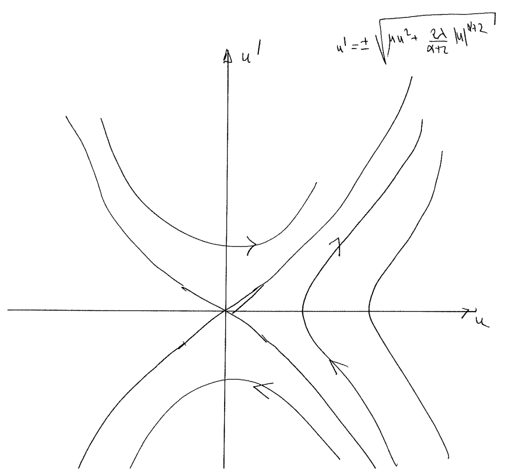
• **Case** $g(u) = \mu u + \lambda |u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** All solutions periodic. But the period $T = T(E(0))$ goes to $2\pi/\sqrt{\lambda}$ as $E(0) \to 0$, and to 0 as $E(0) \to \infty$. Always infinitely many solutions for Dirichlet and Neumann. For Dirichlet, one positive solution iff $b - a < \pi/\sqrt{\mu}$. Only trivial solution 0 on the line. **Computation of the period:**

$$\frac{1}{4}T = \int_0^{u_0} \frac{du}{\sqrt{2(G(u_0) - G(u))}}.$$

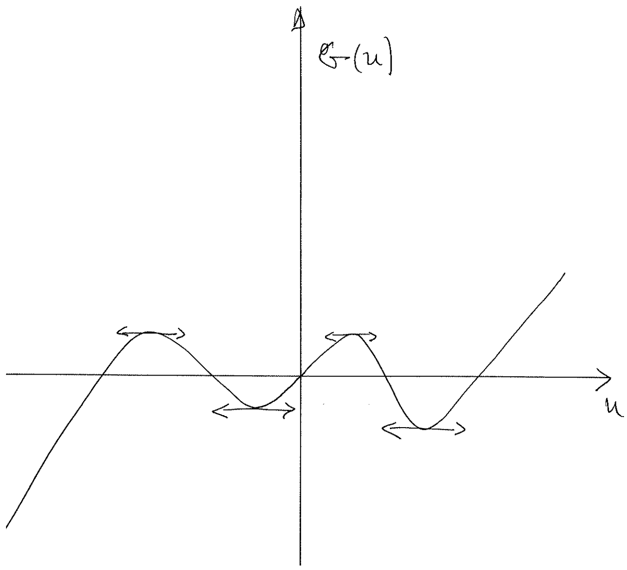
• **Case** $g(u) = -\lambda |u|^\alpha u$**,** $\alpha, \lambda > 0$**.** Dirichlet, Neumann and line: only the trivial solution 0. (Immediate by phase plane analysis.)



• **Case** $g(u) = -\mu u - \lambda |u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Same as above.

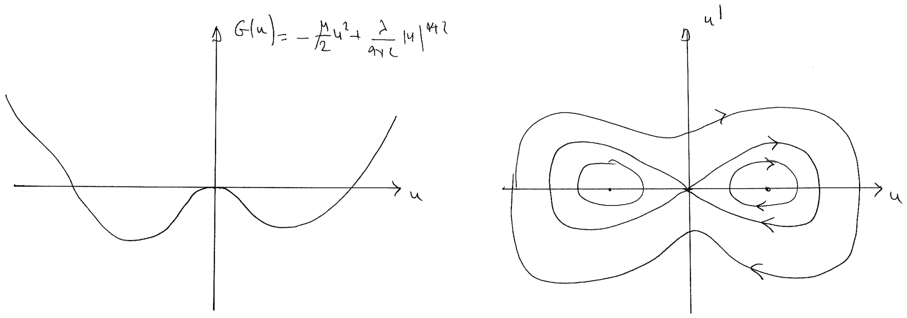


Complex cases: by phase plane or, better, by wire-ring picture.

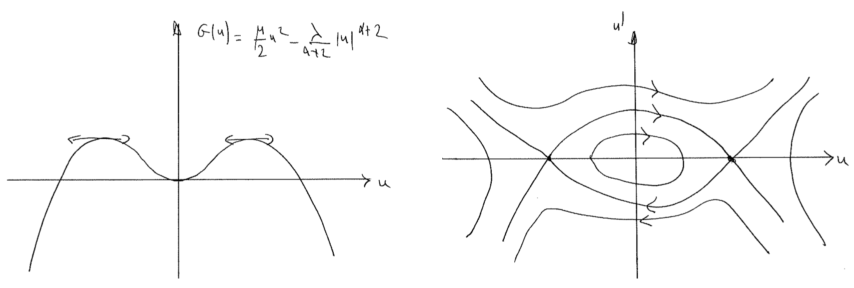


• **Case** $g(u) = -\mu u + \lambda|u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Infinitely many solutions for Dirichlet and Neumann. Always one positive solution for Dirichlet. On the line: infinitely many positive solution such that $u(x) \to 0$ as $|x| \to \infty$. All given by $u(x) = U(x - x_0)$ with $x_0 \in \mathbb{R}$ where

$$U(x) = \Big(\frac{\mu(\alpha+2)}{2\lambda}\Big)^{\frac{1}{\alpha}} \Big(\cosh\Big(\frac{\alpha\sqrt{\mu}}{2}x\Big)\Big)^{-\frac{2}{\alpha}}$$



• **Case** $g(u) = \mu u - \lambda|u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Dirichlet or Neumann: solutions iff $b - a > \pi/\sqrt{\mu}$. Under this assumption, only finitely many solutions. On the line: One odd solution such that $U(t) \to \pm(\mu/\lambda)^{\frac{1}{\alpha}}$ as $t \to \pm\infty$, the solution $-U$ and all translated of these two.



• **General** g**.** By wire-ring picture.

4.2. **The N-dimensional case.** Suppose $N \geq 2$ and $\Omega = B(0, R)$ with $0 < R \leq \infty$. If we look for **radially symmetric** solutions of (NLE), we get to the ODE

$$u'' + \frac{N-1}{r} u' + g(u) = 0. \tag{4.4}$$

The most usual boundary conditions are Dirichlet ($u(R) = 0$) or Neumann ($u'(R) = 0$) if $R < \infty$. In the case of the whole space, one considers Dirichlet ($u(r) \to 0$ as $r \to \infty$).

In order to study this problem, we consider the initial value problem (4.4) with

$$u(0) = u_0, \quad u'(0) = 0,$$

where $u_0 \in \mathbb{R}$ is given, and we look for u_0 for which the solution satisfies $u(R) = 0$ or $u'(R) = 0$, or $u(\infty) = 0$.

Existence & uniqueness: At $r_0 > 0$, the equation is **not** singular, so given any u_0, v_0 there is always (like in one dimension) a unique, local solution of (4.4) such that $u(r_0) = u_0$ and $u'(0) = v_0$. At $r_0 = 0$, given $u_0 \in \mathbb{R}$, we write the equation (4.4) with the initial conditions $u(0) = u_0$, $u'(0) = 0$ in the form

$$u(r) = u_0 - \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} g(u(\sigma))\, d\sigma. \tag{4.5}$$

Note that the integral is not singular. One solves this by the contraction mapping principle. Uniqueness is proved by Gronwall.

Wire-ring: Same as before, but **with friction**.

- **Case** $g(u) = 0$**.** Solution $u(r) \equiv u_0$. Only trivial solution for Dirichlet, all constant solutions for Neumann.
- **Case** $g(u) = -\lambda u$**,** $\lambda > 0$**.** By wire-ring picture, only trivial solution for all.
- **Case** $g(u) = \lambda u$**,** $\lambda > 0$**.** More delicate.

Lemma 4.1. *There exists an increasing sequence $(\rho_n)_{n \geq 1}$ such that $\rho_n > 0$ and $\rho_n \to \infty$ as $n \to \infty$ with the following property. Let $\lambda > 0$ and let u be the solution of the equation*

$$\begin{cases} u'' + \dfrac{N-1}{r} u' + \lambda u = 0, \\ u(0) = 1,\ u'(0) = 0. \end{cases} \tag{4.6}$$

It follows that u oscillates indefinitely and, more precisely, the zeroes of u are $\sqrt{\lambda}\rho_n$ for all $n \geq 1$.

Proof. Changing $u(r)$ to $u(\sqrt{\lambda} r)$, we are reduced to the case

$$\lambda = 1.$$

We deduce from the equation that

$$\Big(\frac{u'^2}{2} + \frac{u^2}{2} \Big)' = -\frac{N-1}{r} u'^2, \tag{4.7}$$

$$(r^{N-1} u u')' + r^{N-1} u^2 = r^{N-1} u'^2, \tag{4.8}$$

and

$$\Big(\frac{r^N}{2} u'^2 + \frac{r^N}{2} u^2 \Big)' + \frac{N-2}{2} r^{N-1} u'^2 = \frac{N}{2} r^{N-1} u^2. \tag{4.9}$$

We now show that u must have a first zero. Indeed, suppose by contradiction that $u(r) > 0$ for all $r > 0$. We note that $u''(0) < 0$, so that $u'(r) < 0$ for $r > 0$ and small. Now, if u' would vanish while u remains positive, we would obtain $u'' < 0$ from the equation, which is absurd. So $u' < 0$ while u remains positive. It follows that $u'(r) < 0$ for all $r > 0$. Thus u has a limit $\ell \ge 0$ as $r \to \infty$. Note that by (4.7), u' is bounded, so that by the equation $u''(r) \to -\ell$ as $r \to \infty$, which implies that $\ell = 0$. Observe that

$$r^{N-1}u'(r) = -\int_0^r s^{N-1}u(s); \tag{4.10}$$

and so,

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}u \ge u(r)\int_0^r s^{N-1} = \frac{r^N}{N}u(r).$$

Therefore,

$$(e^{\frac{r^2}{2N}}u(r))' \le 0,$$

which implies that

$$u(r) \le e^{-\frac{r^2}{2N}}. \tag{4.11}$$

If $N = 2$, then (4.10)-(4.11) show that $ru'(r)$ converges to a negative limit as $r \to \infty$, which is absurd. We now suppose $N \ge 3$ and we integrate (4.9) on $(0, r)$:

$$\frac{r^N}{2}u'(r)^2 + \frac{r^N}{2}u(r)^2 + \frac{N-2}{2}\int_0^r s^{N-1}u'^2 = \frac{N}{2}\int_0^r s^{N-1}u^2. \tag{4.12}$$

Letting $r \to \infty$ and applying (4.11), we deduce that

$$\int_0^\infty r^{N-1}u'(r)^2 < \infty. \tag{4.13}$$

It follows in particular from (4.11) and (4.13) that there exist $r_n \to \infty$ such that

$$r_n^N((u'(r_n)^2 + u(r_n)^2) \to 0.$$

Letting $r = r_n$ in (4.12), we deduce by letting $n \to \infty$

$$(N-2)\int_0^\infty s^{N-1}u'^2 = N\int_0^\infty s^{N-1}u^2. \tag{4.14}$$

We next integrate (4.8) on $(0, r)$:

$$r^{N-1}u(r)u'(r) + \int_0^r s^{N-1}u^2 = \int_0^r s^{N-1}u'^2. \tag{4.15}$$

We observe that $r_n^{N-1}u(r_n)u'(r_n) \to 0$. Letting $r = r_n$ in (4.15) and letting $n \to \infty$, we obtain

$$\int_0^\infty s^{N-1}u^2 = \int_0^\infty s^{N-1}u'^2.$$

Multiplying the above identity by N and adding up with (4.14), we obtain

$$0 = 2\int_0^\infty r^{N-1}u'^2 > 0,$$

which is absurd.

In fact, with the previous argument, one shows as well that if $r \ge 0$ is such that $u(r) \ne 0$ and $u'(r) = 0$, then there exists $\rho > r$ such that $u(\rho) = 0$.

To conclude, we need only show that if $\rho > 0$ is such that $u(\rho) = 0$, then there exists $r > \rho$ such that $u(r) \ne 0$ and $u'(r) = 0$. To see this, note that $u'(\rho) \ne 0$

(for otherwise $u \equiv 0$ by uniqueness), and suppose for example that $u'(\rho) > 0$. If $u'(r) > 0$ for all $r \geq \rho$, then (since u is bounded) u converges to some positive limit ℓ as $r \to \infty$; and so, by the equation, $u''(r) \to -\ell^p$ as $r \to \infty$, which is absurd. This completes the proof. □

It follows from Lemma 4.1 that there is a solution for Dirichlet iff $R = \sqrt{\lambda}\rho_n$ for some $n \geq 1$. If $(\sqrt{\lambda}\tau_n)_{n\geq 1}$ denotes the sequence of zeroes of u', so that $\rho_n < \tau_n < \rho_{n+1}$, then there is a solution for Neumann iff $R = \sqrt{\lambda}\tau_n$ for some $n \geq 1$.

Remark 4.2. It is possible to show that $u(r) \to 0$ as $r \to \infty$, so that there is a solution for Dirichlet on the whole space. However, u converges to 0 **slowly**. In particular, $u \notin L^2(\mathbb{R}^N)$. Indeed, if $u \in L^2(\mathbb{R}^N)$, we deduce from (4.9) that $\nabla u \in L^2(\mathbb{R}^N)$. Therefore, there exists $r_n \to \infty$ such that $r_n^N((u'(r_n))^2 + u(r_n)^2) \to 0$, and we conclude to a contradiction as in the proof of Lemma 4.1 above.

- **Case** $g(u) = -\lambda|u|^\alpha u$**,** $\alpha, \lambda > 0$**.** By wire-ring picture, only trivial solution.
- **Case** $g(u) = -\mu u - \lambda|u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Same as above.
- **Case** $g(u) = \lambda|u|^\alpha u$**,** $\alpha, \lambda > 0$**. Attention: delicate.**

Lemma 4.3. *Let u be the solution of*

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + |u|^\alpha u = 0, \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{4.16}$$

(i) *If $N \geq 3$ and $\alpha \geq 4/(N-2)$, then $u(r) > 0$, $u'(r) < 0$ for all $r > 0$ and $u(r) \to 0$ as $r \to \infty$. Moreover, if $\alpha = 4/(N-2)$, then*

$$u(r) = \Big(1 + \frac{r^2}{N(N-2)}\Big)^{-\frac{N-2}{2}},$$

for all $r > 0$.

(ii) *If $\alpha < 4/(N-2)$, then u oscillates indefinitely.*

Proof. (i) We note that $u''(0) < 0$, so that $u'(r) < 0$ for $r > 0$ and small. Now, if u' would vanish while u remains positive, we would obtain $u'' < 0$ from the equation (4.16), which is absurd. So $u' < 0$ while u remains positive. Next, we deduce from the equation (4.16) that

$$(r^{N-1}uu')' + r^{N-1}|u|^{\alpha+2} = r^{N-1}u'^2, \tag{4.17}$$

and

$$\Big(\frac{r^N}{2}u'^2 + \frac{r^N}{\alpha+2}|u|^{\alpha+2}\Big)' + \frac{N-2}{2}r^{N-1}u'^2 = \frac{N}{\alpha+2}r^{N-1}|u|^{\alpha+2}. \tag{4.18}$$

Assume by contradiction that u has a first zero r_0. By uniqueness, we have $u'(r_0) \neq 0$. Integrating (4.17) and (4.18) on $(0, r_0)$, we obtain

$$\int_0^{r_0} r^{N-1}u^{\alpha+2} = \int_0^{r_0} r^{N-1}u'^2,$$

and

$$\frac{r_0^N}{2}u'(r_0)^2 + \frac{N-2}{2}\int_0^{r_0} r^{N-1}u'^2 = \frac{N}{\alpha+2}\int_0^{r_0} r^{N-1}|u|^{\alpha+2};$$

and so,

$$0 < \frac{r_0^N}{2} u'(r_0)^2 = \Big(\frac{N}{\alpha+2} - \frac{N-2}{2} \Big) \int_0^{r_0} r^{N-1} u'^2 \le 0,$$

which is absurd. This shows that $u(r) > 0$ (hence $u'(r) < 0$) for all $r > 0$. In particular, $u(r)$ decreases to a limit $\ell \ge 0$ as $r \to \infty$. Since $u'(r)$ is bounded by the energy identity

$$\Big(\frac{u'^2}{2} + \frac{|u|^{\alpha+2}}{\alpha+2} \Big)' = -\frac{N-1}{r} u'^2$$

we deduce from the equation that $u''(r) \to -\ell^p$, which implies that $\ell = 0$. This proves property (i)

(ii) is proved by calculations similar to the proof of Lemma 4.1. □

Remark 4.4. Here are some comments on Lemma 4.3.

(i) By scaling, the same conclusions hold if one multiplies the nonlinearity $|u|^\alpha u$ by $\lambda > 0$ and if one considers the initial condition $u(0) = u_0 > 0$.

(ii) More generally (with the proof of Lemma 4.3 (i)) the solution of the equation

$$\begin{cases} u'' + \dfrac{N-1}{r} u' - \mu u + \lambda |u|^\alpha u = 0, \\ u(0) = u_0 > 0, \quad u'(0) = 0, \end{cases}$$

remains positive for all $r > 0$ when $\lambda, \mu > 0$ and $\alpha \ge 4/(N-2)$.

Thus we see that if $\alpha < 4/(N-2)$, there are infinitely many solutions for Dirichlet or Neumann. For the whole space, it is possible to show that every solution converges to 0 as $r \to \infty$, so also infinitely many solutions. However, these solutions decay **slowly**. Indeed, it follows from (4.17)-(4.18) that $u \notin L^{\alpha+2}(\mathbb{R}^N)$ and $\nabla u \notin L^2(\mathbb{R}^N)$. If $N \ge 3$ and $\alpha \ge 4/(N-2)$, no solution to Dirichlet or Neumann. On the whole space, infinitely many solutions. If $\alpha = 4/(N-2)$, then $\nabla u \in L^2(\mathbb{R}^N)$. Otherwise?

• **Case** $g(u) = \mu u + \lambda |u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Solutions oscillate indefinitely for all $\alpha > 0$ (contrary to the case $\mu \le 0$). Proof by **Wronskian** argument. Let u be solution and let v be the solution of

$$v'' + \frac{N-1}{r} v' + \mu v = 0,$$

with $v(0) = u_0$, $v'(0) = 0$. As long as $u, v > 0$, we have

$$(u'v - uv')' + \frac{N-1}{r}(u'v - uv') = -\lambda |u|^\alpha uv < 0.$$

Therefore $[r^{N-1}(u'v - uv')]' < 0$, so that $u'v - uv' < 0$, thus u/v is decreasing. Since v oscillates, u must vanish in finite time. Similar argument shows that u oscillates indefinitely. Note that if $u_0 \to 0$, then the equation is almost linear and the oscillation time converges to $2\pi/\sqrt{\mu}$ (period). So there are solutions for Dirichlet and Neumann if R is appropriate, or if one adjusts μ. If $\alpha < 4/(N-2)$, then infinitely many solutions for Dirichlet or Neumann for every $R > 0$ by the following lemma.

Lemma 4.5. *Fix $\rho > 0$, $\mu \in \mathbb{R}$, $\lambda > 0$ and suppose $\alpha < 4/(N-2)$. Given any $n \ge 1$, there exists $M > 0$ such that if $u_0 \ge M$ then u has at least n zeroes on $(0, \rho)$.*

Proof. Set
$$\widetilde{u}(r) = \frac{1}{u_0} u\Big(\frac{r}{u_0^{\alpha/2}}\Big).$$
It follows that
$$\widetilde{u}'' + \frac{N-1}{r}\widetilde{u}' + \frac{\mu}{u_0^\alpha}\widetilde{u} + \lambda|\widetilde{u}|^\alpha \widetilde{u} = 0,$$
with the initial condition $\widetilde{u}(0) = 1$. Fix $\tau > 0$ such that the solution v of
$$v'' + \frac{N-1}{r}v' + \lambda|v|^\alpha v = 0,$$
with the initial condition $v(0) = 1$ has n zeroes on $(0,\tau)$. It follows that there exists $M > 0$ such that if $u_0 \ge M$, then $\widetilde{u}$ has n zeroes on $(0,\tau)$. Thus u has n zeroes on $(0, \tau/M^{\alpha/2})$ and the result follows by choosing M possibly larger so that $\tau/M^{\alpha/2} \le \rho$. □

• **Case** $g(u) = -\mu u + \lambda|u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** If $\alpha < 4/(N-2)$, then the above lemma shows that there are infinitely many solutions for Dirichlet and Neumann. If $\alpha \ge 4/(N-2)$, then the proof of Lemma 4.3 (i) shows that u remains positive (and in fact bounded away from zero). So no solution for Dirichlet, Neumann or whole space. Going back to the case $\alpha < 4/(N-2)$, we study the case of the whole space. The main result is the following.

Theorem 4.6. *Let $g(u) = -\mu u + \lambda|u|^\alpha u$ with $\alpha, \lambda, \mu > 0$ and suppose $\alpha < 4/(N-2)$. There exists an increasing sequence $0 < u_0^0 < u_0^1 < \cdots$ such that the solution of (4.4) with the initial condition $u^n(0) = u_0^n$ has exactly n zeroes on $(0,\infty)$ and such that $|u(r)| + |u'(r)| \to 0$ as $r \to \infty$. Moreover, $u(r)^2 + u'(r)^2 \le Ce^{-2\sqrt{\mu}r}$ for all $r > 0$.*

Remark 4.7. Most important result. In this form, McLeod, Troy and Weissler [28]. The positive solution is **unique** (Kwong [23], see also McLeod [27]).

Proof. We first show the exponential decay. If $u \to 0$, then u has a finite number of zeroes (by wire-ring picture). Thus for r large, we have for example $u > 0$ and $u' < 0$. Set for r large
$$f(r) = (u' - \sqrt{\mu}u)^2 = u'^2 + \mu u^2 - 2\sqrt{\mu}uu' \ge u'^2 + \mu u^2.$$
It follows that
$$\begin{aligned} f' + 2\sqrt{\mu}f &= -\frac{2(N-1)}{r}(u'^2 - \sqrt{\mu}uu') + 2\lambda|u|^\alpha(\sqrt{\mu}u^2 - uu') \\ &\le 2\lambda|u|^\alpha(\sqrt{\mu}u^2 - uu') \\ &\le 2\lambda|u|^\alpha(\sqrt{\mu}u^2 - 2uu') = \frac{2\lambda}{\sqrt{\mu}}|u|^\alpha(f - u'^2) \\ &\le \frac{2\lambda}{\sqrt{\mu}}|u|^\alpha f. \end{aligned}$$
Therefore
$$\frac{f'}{f} + 2\sqrt{\mu} - \frac{2\lambda}{\sqrt{\mu}}|u|^\alpha \le 0,$$
so that
$$\frac{d}{dr}\Big(\log f + 2\sqrt{\mu}r - \frac{2\lambda}{\sqrt{\mu}}\int_{r_0}^r |u|^\alpha\Big) \le 0.$$

We first observe that if r_0 is large, then

$$\frac{2\lambda}{\sqrt{\mu}}\int_{r_0}^{r}|u|^\alpha \le \frac{2\lambda}{\sqrt{\mu}} r|u(r_0)|^\alpha \le \sqrt{\mu}r.$$

Thus we deduce that $f(r) \le Ce^{-\sqrt{\mu}r}$. This implies that

$$\int_{r_0}^{r}|u|^\alpha < \infty,$$

so we conclude that $f(r) \le Ce^{-2\sqrt{\mu}r}$, which is the desired estimate.

We now prove the existence of a positive solution. Set

$$A_0 = \{u_0 > 0;\ u > 0 \text{ on } (0,\infty)\}.$$

It follows that $u_0 = (\mu/\lambda)^{\frac{1}{\alpha}} \in A_0$ (stationary solution). Thus $A_0 \ne \emptyset$. Furthermore A_0 is bounded by Lemma 4.5, so we may define

$$u_0^0 = \sup A_0.$$

It follows by continuous dependence that $u_0^0 \in A_0$. The fact that u_0^0 has the desired properties follow from the fact that if $u_0 > u_0^0$ then u has at least one zero, continuous dependence and the wire-ring picture.

More generally, one sets

$$A_n = \{u_0 > 0;\ u \text{ has } n \text{ zeroes on } (0,\infty)\}.$$

One argues as above. The only additional thing that one has to prove is that $A_n \ne \emptyset$, which requires one more argument, see Lemma 1.3.8 in [8]. □

• **Case** $g(u) = \mu u - \lambda|u|^\alpha u$**,** $\alpha, \lambda, \mu > 0$**.** Solutions for Dirichlet and Neumann depending on R or, equivalently, on μ. See wire-ring diagram.

5. Variational methods

A very good reference for this section is the book [21] by Kavian. It is in french, but is extremely well written. Another reference is the book [35] by Struwe. It is more advanced and less friendly.

5.1. **Basic tools.**

The results of the previous section do not apply in a general domain Ω. In this section, we construct solutions to the equation (NLE) by **variational methods**. These techniques require the use of **Sobolev spaces** for the following reason. Consider the equation (NLE) in a bounded domain with, say, Dirichlet boundary conditions. Consider (formally) the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u), \tag{5.1}$$

where

$$G(t) = \int_0^t g(s)\,ds. \tag{5.2}$$

We see that

$$\begin{aligned}\frac{I(u+tv) - I(u)}{t} &= \int_\Omega \nabla u\cdot\nabla v + \frac{t}{2}\int_\Omega |\nabla v|^2 + \int_\Omega \frac{G(u+tv) - G(u)}{t}\\ &\underset{t\downarrow 0}{\longrightarrow} \int_\Omega \nabla u\cdot\nabla v - \int_\Omega g(u)v = \int_\Omega [-\Delta u - g(u)]v,\end{aligned}$$

provided u, v vanish on the boundary. Thus we see that critical points of I correspond (again, formally!) to solutions of (NLE). For example, one can try to minimize I in order to obtain a critical point.

To make this rigorous, we first need to choose an appropriate space! **The natural space $C^2(\overline{\Omega})$ is not appropriate!** For example, a minimizing sequence of I need not be bounded in C^2. The appropriate spaces are **Sobolev spaces**.

Definitions:

• $u \in L^1_{\mathrm{loc}}(\Omega)$, $\alpha \in \mathbb{N}^N$, $1 \le p \le \infty$. We say that $D^\alpha u \in L^p(\Omega)$ if there exists $f_\alpha \in L^p(\Omega)$ such that

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega f_\alpha \varphi,$$

for all $\varphi \in C^m_{\mathrm{c}}(\Omega)$. We say that $D^\alpha u = f_\alpha$. Note that f_α (if it exists- is **unique**. This definition is also **consistent** with the standard definition of derivatives, by **Green's formula**.

• $W^{m,p}(\Omega)$, $W^{m,p}_0(\Omega)$, $H^m(\Omega)$, $H^m_0(\Omega)$.

• For simplicity, we consider only Dirichlet boundary conditions, so we consider only the spaces $W^{m,p}_0(\Omega)$ $H^m_0(\Omega)$.

Elementary properties:

• Banach spaces, Hilbert spaces for the H^m

• Reflexive if $1 < p < \infty$

• Separable if $p < \infty$.

• Dual of $W^{m,p}_0(\Omega)$: $W^{-m,p'}_0(\Omega)$; of $H^m_0(\Omega)$: $H^{-m}(\Omega)$.

• Δ continuous $H^1_0(\Omega) \to H^{-1}(\Omega)$ and

$$\langle \Delta u, v \rangle_{H^{-1}, H^1_0} = -\int_\Omega \nabla u \cdot \nabla v.$$

Refined, but essential, properties:

• Compactness: $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact if $1 \le p < \infty$ and Ω bounded (Rellich-Kondrachoff). More generally, even if Ω is unbounded, if u_n is a bounded sequence, then there exists a subsequence and $u \in W^{1,p}_0(\Omega)$ such that $u_n \to u$ a.e. and $u_n \to u$ in $L^p(\Omega \cap B_R)$ for every $R < \infty$. Proof: Approximation by smooth functions and Ascoli.

• Sobolev's inequalities: If $mp < N$ and

$$\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N},$$

then

$$\|u\|_{L^{p^*}} \le C \|D^m u\|_{L^p},$$

for all $u \in W^{m,p}_0(\Omega)$. Proof: Nirenberg.

• More inequalities: Morrey (case $mp > N$), Trudinger (case $mp = N$), Gagliardo-Nirenberg.

• A particular case of Gagliardo-Nirenberg. Let $1 \le p, q, r \le \infty$ and $0 \le a \le 1$ ($0 \le a < 1$ if $N = r \ge 2$) such that

$$\frac{1}{p} = a\Big(\frac{1}{r} - \frac{1}{N}\Big) + \frac{1-a}{q}.$$

It follows that

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^r}^a \|u\|_{L^q}^{1-a},$$

for all $u \in C^1_{\mathrm{c}}(\Omega)$. In particular, if $1 \le r < \infty$ and $r < p < \frac{Nr}{(N-r)^+}$, then

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^r}^{\frac{N(p-r)}{rp}} \|u\|_{L^r}^{\frac{rp-N(p-r)}{rp}}, \tag{5.3}$$

for all $u \in W^{1,r}_0(\Omega)$.

• Sobolev's embeddings: $W^{m,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ if $p \le q \le p^*$. By duality, $L^{q'}(\Omega) \hookrightarrow W^{-m,p'}_0(\Omega)$. In particular, $L^r(\Omega) \hookrightarrow H^{-1}(\Omega)$ if $\frac{2N}{N+2} < r \le 2$ ($1 \le r \le 2$ if $N = 1$, $1 < r \le 2$ if $N = 2$).

• **Chain rule (partial):** If $1 \le p \le \infty$ and $u \in W^{1,p}_0(\Omega)$, then $|u| \in W^{1,p}_0(\Omega)$ and $|\nabla(|u|)| = |\nabla u|$ a.e.

C^1 functionals on $H^1_0(\Omega)$

• If

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2,$$

then $J \in C^1(H^1_0(\Omega), \mathbb{R})$ and $J'(u) = -\Delta u$.

• Let $g \in C(\mathbb{R}, \mathbb{R})$ and suppose $|g(s)| \le C|s|^r$ for all $s \in \mathbb{R}$ with $1 \le r < \infty$. Define

$$G(u) = \int_0^u g(s)\, ds,$$

for all $u \in \mathbb{R}$, so that $|G(u)| \le C|u|^{r+1}$. Set

$$J(u) = \int_\Omega G(u),$$

for $u \in L^{r+1}(\Omega)$. It follows that the mapping $u \mapsto g(u)$ is continuous $L^{r+1}(\Omega) \to L^{\frac{r+1}{r}}(\Omega)$. Moreover, $J \in C^1(L^{r+1}(\Omega), \mathbb{R})$ and $J'(u) = g(u)$ for all $u \in L^{r+1}(\Omega)$.

• Under the above assumptions, if $2 \le r \le \frac{2N}{N-2}$ ($2 \le r < \infty$ if $N = 1, 2$), then the mapping $u \mapsto g(u)$ is continuous $H^1_0(\Omega) \to H^{-1}(\Omega)$. Moreover, $J \in C^1(H^1_0(\Omega), \mathbb{R})$ and $J'(u) = g(u)$.

• If g is as above and

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u),$$

for all $u \in H^1_0(\Omega)$, then $J \in C^1(H^1_0(\Omega), \mathbb{R})$ and $J'(u) = -\Delta u - g(u)$.

• By decomposition, same result if $|g(s)| \le C(1 + |s|^r)$ with r as above.

Spectral decomposition of the Laplacian

• We define

$$\lambda_1 = \inf\Big\{\int_\Omega |\nabla u|^2;\ u \in H^1_0(\Omega), \|u\|_{L^2} = 1\Big\},$$

so that $\lambda_1 \geq 0$. If $|\Omega| < \infty$, then by **Poincaré's inequality**

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2},$$

(which follows immediately from (5.3) since $\|u\|_{L^r} \leq |\Omega|^{\frac{1}{r}-\frac{1}{p}}\|u\|_{L^p}$) we see that $\lambda_1 > 0$.

- Define the linear, unbounded operator A on $L^2(\Omega)$ by

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega);\ \Delta u \in L^2(\Omega)\}, \\ Au = -\Delta u, \quad u \in D(A). \end{cases}$$

It follows that A is self-adjoint. Moreover, if we suppose further that Ω is bounded, then A^{-1} is compact by Rellich. Thus the spectrum of A is a sequence of eigenvalues $(\lambda_j)_{j\geq 1}$ and λ_1, the first eigenvalue, is the same as defined above.

5.2. **Global minimization.** Recall that if X is a Banach space, $J \in C^1(X, \mathbb{R})$ and if $w \in X$ is a minimum of J, i.e.

$$J(x) = \inf\{J(x);\ x \in X\},$$

then w is a **critical point** of J, i.e. $J'(w) = 0$.

Theorem 5.1. *Suppose Ω is bounded. Let $0 < \alpha \leq 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$), $\lambda > 0$ and $\mu > \lambda_1$. It follows that there exists a solution $u \in H_0^1(\Omega)$, $u \geq 0$, $u \not\equiv 0$ of the equation*

$$-\Delta u - \mu u + \lambda |u|^\alpha u = 0, \tag{5.4}$$

in Ω. (The equation makes sense in $H^{-1}(\Omega)$.)

Proof. The equation (5.4) has the form $J'(u) = 0$ with

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\mu}{2}\int_\Omega u^2 + \frac{\lambda}{\alpha+2}\int_\Omega |u|^{\alpha+2}.$$

We know that $J \in C^1(H_0^1(\Omega), \mathbb{R})$.

We claim that $\inf J < 0$. Indeed, if $w = \varphi_1$ (first eigenvector with $\|\varphi_1\|_{L^2} = 1$) then for $t > 0$

$$J(tw) = \frac{1}{2}\lambda_1 t^2 - \frac{1}{2}\mu t^2 + \frac{\lambda}{\alpha+2}\|w\|_{L^{\alpha+2}}^{\alpha+2} t^{\alpha+2}.$$

Thus $J(tw) < 0$ for $t > 0$ and small.

Next, we claim that J has a minimum u with $u \geq 0$ (and $u \not\equiv 0$ since $\inf J < 0$). Indeed, let $(u_n)_{n\geq 1}$ be a minimizing sequence. Since $J(|u_n|) = J(u_n)$, we may replace u_n by $|u_n|$, so we may assume $u_n \geq 0$. We show that u_n is bounded in $H_0^1(\Omega)$. Let $x > 0$ be defined by

$$\frac{\mu}{2} = \frac{\lambda}{\alpha+2}x^\alpha,$$

and define

$$A(s) = \begin{cases} \frac{\mu}{2}s^2 - \frac{\lambda}{\alpha+2}t^{\alpha+2} & 0 \leq s \leq x, \\ 0 & s \geq x, \end{cases}$$

$$B(s) = \begin{cases} 0 & 0 < s \leq x, \\ -\frac{\mu}{2}s^2 + \frac{\lambda}{\alpha+2}t^{\alpha+2} & s \geq x. \end{cases}$$

We have $A, B \geq 0$ and

$$J(u_n) = \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_\Omega A(u_n) + \int_\Omega B(u_n).$$

Since A is bounded, we deduce that $\|\nabla u_n\|_{L^2}$ is bounded, so that by Poincaré u_n is bounded in $H^1_0(\Omega)$. By Rellich, there is a subsequence, whch we still denote by $(u_n)_{n\geq 1}$ and $u \in H^1_0(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ strongly, in $H^1_0(\Omega)$ weakly and a.e. Since $\|\nabla u\|_{L^2}$ is an equivalent norm on $H^1_0(\Omega)$ by Poincaré, the first term is weakly l.s.c. The second term is continuous on L^2. The third term is l.s.c by Fatou. Thus u is a minimizer, hence $J'(u) = 0$. □

Remark 5.2. Here are some comments on Theorem 5.1.

(i) In fact, the solution of Theorem 5.1 satisfies $u > 0$ by the strong maximum principle.

(ii) If $\mu \leq \lambda_1$, then the equation (5.4) has no nontrivial solution. Indeed, taking the duality product of the equation with u, we obtain

$$\begin{aligned} 0 &= \|\nabla u\|^2_{L^2} - \mu\|u\|^2_{L^2} + \lambda\|u\|^{\alpha+2}_{L^{\alpha+2}} \\ &\geq (\lambda_1 - \mu)\|u\|^2_{L^2} + \lambda\|u\|^{\alpha+2}_{L^{\alpha+2}} \\ &\geq \lambda\|u\|^{\alpha+2}_{L^{\alpha+2}} \end{aligned}$$

(iii) The assumption $\alpha \leq 4/(N-2)$ is not necessary. One can do without by working in the space $X = H^1_0(\Omega) \cap L^{\alpha+2}(\Omega)$.

(iv) Similar result if one replaces $|u|^\alpha u$ by a more general nonlinearity.

We now study the case $\Omega = \mathbb{R}^N$.

Proposition 5.3. *Suppose* $0 < \alpha < 4/(N-2)$ *(*$0 < \alpha < \infty$ *if* $N = 1, 2$*),* $\lambda > 0$ *and* $\mu \in \mathbb{R}$*. If* $u \in H^1(\mathbb{R}^N)$ *is a solution of the equation* (5.4)*, then* $u = 0$*.*

For the proof, we will use **Pohožaev's identity**, which we recall below.

Proposition 5.4. *Let* $g \in C(\mathbb{R}, \mathbb{R})$ *and* G *its primitive. Let* $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $G(u) \in L^1(\mathbb{R}^N)$ *and* $\nabla u \in L^2(\mathbb{R}^N)$*. If* $-\Delta u = g(u)$ *in* $\mathcal{D}'(\mathbb{R}^N)$*, then*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 = N\int_{\mathbb{R}^N} G(u).$$

Proof. Formally, Pohožaev's identity follows by integrating the pointwise identity

$$\begin{aligned} &\frac{N-2}{2}|\nabla u|^2 - NG(u) \\ &\quad = \nabla\cdot\Big\{\Big(\frac{1}{2}|\nabla u|^2 - G(u)\Big)x - (x\cdot\nabla u)\nabla u\Big\} - [-\triangle u - g(u)][x\cdot\nabla u] \\ &\quad = \nabla\cdot\Big\{\Big(\frac{1}{2}|\nabla u|^2 - G(u)\Big)x - (x\cdot\nabla u)\nabla u\Big\}. \end{aligned}$$

For a complete proof, see e.g. Berestycki and Lions [5], proof of Proposition 1, p. 320. □

Proof of Proposition 5.3. Recall that $\lambda_1 = 0$. If $\mu \leq 0$, then the argument of Remark 5.2 (ii) shows that $u = 0$. Suppose now $\mu > 0$. It follows by standard regularity that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, so we may use Pohožaev, which yields

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 = \frac{\mu N}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{\lambda N}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

On the other hand, taking the duality product of the equation with u, we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \mu \int_{\mathbb{R}^N} |u|^2 - \lambda \int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

Multiplying the last equation by $N/(\alpha+2)$ and making the difference, we obtain

$$0 \le \frac{4-(N-2)\alpha}{2(\alpha+2)} \int_{\mathbb{R}^N} |\nabla u|^2 = -\frac{\mu N \alpha}{2(\alpha+2)} \int_{\mathbb{R}^N} |u|^{\alpha+2} \le 0,$$

so that $u = 0$. □

5.3. **Constrained minimization.** If we want to find solutions of the equation

$$-\Delta u + \mu u - \lambda |u|^\alpha u = 0, \tag{5.5}$$

with $\lambda > 0$, we cannot use global minimization since $\inf J = -\infty$. We use instead constrained minimization, based on the following classical result. (See e.g. [21, 8] for a proof.)

Theorem 5.5 (Lagrange multipliers). *Let X be a Banach space, $F, J \in C^1(X, \mathbb{R})$ and set*

$$M = \{v \in X;\ F(v) = 0\}.$$

Let $S \subset M$, $S \neq \emptyset$, and suppose $x_0 \in S$ satisfies

$$J(u_0) = \inf_{v \in S} J(v).$$

If $F'(u_0) \neq 0$ and if $M \cap \{x \in X;\ \|x - u_0\|_X \le \eta\} \subset S$ for some $\eta > 0$, then there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $J'(u_0) = \Lambda F'(u_0)$.

Theorem 5.6. *Suppose Ω is bounded, $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$), $\mu > -\lambda_1$ and $\lambda > 0$. It follows that there exists a solution $u \in H^1_0(\Omega)$, $u \ge 0$, $u \not\equiv 0$ of the equation* (5.5).

Proof. Set

$$F(u) = \frac{\lambda}{\alpha+2} \int_\Omega |u|^{\alpha+2} - 1, \quad J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\mu}{2} \int_\Omega |u|^2, \tag{5.6}$$

so that $F, J \in C^1(H^1_0(\Omega), \mathbb{R})$. Let

$$M = S = \{u \in H^1_0(\Omega);\ F(u) = 0\}.$$

We have $F'(u) = |u|^\alpha u \neq 0$ for all $u \in S$. We construct $u \in S$ such that

$$J(u) = \inf_{v \in S} J(v) =: m. \tag{5.7}$$

Note that, since $\mu > -\lambda_1$, $\sqrt{J(u)}$ is an equivalent norm on $H^1_0(\Omega)$. In particular, J is weakly l.s.c. If $(u_n)_{n\ge1}$ is a minimizing sequence, then so is $(|u_n|)_{n\ge1}$, so that we may assume $u_n \ge 0$. Moreover, u_n is bounded so that by Rellich, there is a subsequence, which we still denote by $(u_n)_{n\ge1}$ and $u \in H^1_0(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ strongly and in $H^1_0(\Omega)$ weakly. In particular $J(u) \le m$. Moreover, by Gagliardo-Nirenberg, $u_n \to u$ in $L^{\alpha+2}(\Omega)$, so that $u \in S$. Thus there exists $\Lambda \in \mathbb{R}$ such that $J'(u) = \Lambda F'(u)$, i.e.

$$-\Delta u + \mu u = \Lambda \lambda |u|^\alpha u.$$

Multiplying by u, we obtain

$$2m = 2J(u) = \Lambda \lambda \int_\Omega |u|^{\alpha+2} = \Lambda(\alpha+2).$$

In particular, $\Lambda = 2m/(\alpha+2) > 0$. Replacing u by $\Lambda^{-\frac{1}{\alpha}}u$, we obtain a solution of the desired problem. □

Remark 5.7. The assumption $\mu > -\lambda_1$ is not necessary in order to construct a solution $u \not\equiv 0$ of the equation (5.5). It is necessary, however, for the existence of a positive (nontrivial) solution. (Multiply the equation by φ_1.)

Remark 5.8. Suppose we replace the nonlinearity $\lambda|u|^\alpha u$ by a similar, but nonhomogeneous, nonlinearity, for example $\lambda|u|^\alpha u + \gamma|u|^\beta u$ with $\lambda, \gamma > 0$, $0 < \alpha, \beta < 4/(N-2)$. One can apply the argument of Theorem (5.6) with

$$F(u) = \frac{\lambda}{\alpha+2}\int_\Omega |u|^{\alpha+2} + \frac{\gamma}{\beta+2}\int_\Omega |u|^{\beta+2} - 1.$$

One obtains a solution of the equation

$$-\Delta u + \mu u = \Lambda(\lambda|u|^\alpha u + \gamma|u|^\beta u),$$

where Λ is a Lagrange multiplier. It follows easily that $\Lambda > 0$, but there is no way to eliminate Λ. Instead, one must use the mountain pass theorem of the next section.

Remark 5.9. Solutions of minimal action $\mathcal{A}$ defined by

$$\mathcal{A}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\mu}{2}\int_\Omega |u|^2 - \frac{\lambda}{\alpha+2}\int_\Omega |u|^{\alpha+2}, \tag{5.8}$$

may be important for some applications, because they tend to be "more stable", in some appropriate sense. However, we saw that the action $\mathcal{A}$ is not bounded from below, so a solution cannot minimize the energy on the whole space $H^1_0(\Omega)$. There is still an appropriate notion of solution of minimal energy, the *ground state.* A ground state is a nontrivial solution of (5.5) which *minimizes* $\mathcal{A}$ *among all nontrivial solutions of* (5.5). It is not difficult to adapt the above constrained minimization technique in order to construct a ground state. More precisely, we have the following result.

Theorem 5.10. *Under the assumptions of Theorem* 5.6*, there exists a ground state* $u \ge 0$ *of the equation* (5.5)*.*

Proof. Let

$$F(u) = \int_\Omega |\nabla u|^2 + \mu\int_\Omega u^2 - \lambda\int_\Omega |u|^{\alpha+2},$$

set

$$M = \{u \in H^1_0(\Omega);\ F(u) = 0\}, \quad S = \{u \in M;\ u \ne 0\},$$

and consider $\mathcal{A}$ defined by (5.8). Given any $v \in H^1_0(\Omega)$, $v \ne 0$, we see that $F(tu) = 0$ for some $t > 0$. Thus $S \ne \emptyset$. We proceed in four steps.

STEP 1. $(F'(v), v)_{H^{-1},H^1_0} < 0$ and $(\mathcal{A}'(v), v)_{H^{-1},H^1_0} = 0$ for all $v \in S$. Indeed,

$$\begin{aligned}(F'(v), v)_{H^{-1},H^1_0} &= (-2\triangle v + 2\mu v - \lambda(\alpha+2)|v|^\alpha v, v)_{H^{-1},H^1_0}\\ &= 2\int_\Omega |\nabla v|^2 + 2\mu\int_\Omega v^2 - \lambda(\alpha+2)\int_\Omega |v|^{\alpha+2}\\ &= 2F(v) - \lambda\alpha\int_\Omega |v|^{\alpha+2},\end{aligned}$$

from which we deduce the first property. Since $(\mathcal{A}'(v), v)_{H^{-1},H^1_0} = F(v)$, the second property follows.

STEP 2. There exists $\delta > 0$ such that $\|v\|_{L^{\alpha+2}} \ge \delta$ for all $v \in S$. Indeed, since $F(v) = 0$ and $\lambda > -\lambda_1$, there exists a constant C such that

$$\|v\|_{H^1}^2 \le C\|v\|_{L^{\alpha+2}}^{\alpha+2},$$

for all $v \in S$. By Sobolev's inequality, we deduce that

$$\|v\|_{L^{\alpha+2}}^2 \le C\|v\|_{L^{\alpha+2}}^{\alpha+2},$$

from which the result follows.

STEP 3. There exists $u \in S$, $u \ge 0$, such that

$$\mathcal{A}(u) = \inf_{v\in S} \mathcal{A}(v) := m. \tag{5.9}$$

Indeed,

$$\begin{aligned}\mathcal{A}(v) &= \frac{1}{2}F(v) + \lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\int_\Omega |v|^{\alpha+2} \\ &= \lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\int_\Omega |v|^{\alpha+2},\end{aligned} \tag{5.10}$$

for all $v \in S$, so that $m > 0$ by Step 2. Furthermore, it follows from (5.9) and (5.10) that

$$m = \lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\inf_{v\in S}\int_\Omega |v|^{\alpha+2}. \tag{5.11}$$

Let $(u_n)_{n\ge0} \subset S$ be a minimizing sequence for (5.9), hence for for (5.11). Replacing u_n by $|u_n|$, we see that we may assume $u_n \ge 0$. Since $u_n \in S$ and $(u_n)_{n\ge0}$ is bounded in $L^{\alpha+2}(\Omega)$ (hence in $L^2(\Omega)$) by (5.11), we see that $(u_n)_{n\ge0}$ is bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence, which we still denote by $(u_n)_{n\ge0}$, and $u \in H_0^1(\Omega)$, $u \ge 0$, such that $u_n \to u$ in $L^{\alpha+2}(\Omega)$ and $\|\nabla u\|_{L^2} \le \liminf \|\nabla u_n\|_{L^2}$ as $n \to \infty$. It follows that

$$\lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\int_\Omega |u|^{\alpha+2} = m, \tag{5.12}$$

and $F(u) \le 0$. We deduce in particular that there exists $t \in (0, 1]$ such that $F(tu) = 0$, i.e. $tu \in S$. Therefore,

$$m \le \lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\int_\Omega |tu|^{\alpha+2} = t^{\alpha+2}\lambda\Big(\frac{1}{2} - \frac{1}{\alpha+2}\Big)\int_\Omega |u|^{\alpha+2} = t^{\alpha+2}m,$$

by (5.12). Since $m > 0$, this implies that $t = 1$. Therefore, $u \in S$ and thus $\mathcal{A}(u) = m$ by (5.12) and (5.10).

STEP 4. Conclusion. Let u be as in Step 3. By Step 1 we have $F'(u) \ne 0$; and so, we may apply Theorem 5.5. It follows that there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $\mathcal{A}'(u) = \Lambda F'(u)$. Since, by Step 1, $(\mathcal{A}'(u), u)_{H^{-1},H_0^1} = 0$ and $(F'(v), v)_{H^{-1},H_0^1} \ne 0$, we must have $\Lambda = 0$; and so u is a solution of the equation (5.5). It remains to show that $\mathcal{A}(v) \ge \mathcal{A}(u)$ for all solutions $v \ne 0$ of (5.5). This is clear, since any solution v of (5.5) satisfies $F(v) = 0$, i.e. $v \in S$, and u minimizes $\mathcal{A}$ on S. □

If $\Omega = \mathbb{R}^N$, then we know (by ODE techniques) that there are radially symmetric solutions of (5.5) with exponential decay at infinity. By using constrained minimization, we can show the existence of a positive ground state.

Theorem 5.11. *Suppose $\Omega = \mathbb{R}^N$, $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$) and $\lambda, \mu > 0$. It follows that there exists a ground state $u \in H^1(\mathbb{R}^N)$, $u \geq 0$, $u \not\equiv 0$ of the equation* (5.5)*, which is radially symmetric and nonincreasing.*

For the proof, we will use *symmetric-decreasing rearrangement*, or Schwarz symmetrization. Given a measurable set E of $\mathbb{R}^N$, we denote by E^* the ball of $\mathbb{R}^N$ centered at 0 and such that

$$|E^*| = |E|.$$

Accordingly, we set

$$1_E^* = 1_{E^*}.$$

Given now a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ such that $|\{|u| > t\}| < \infty$ for all $t > 0$, we set

$$u^*(x) = \int_0^\infty 1^*_{\{|u|>t\}}(x)\, dt, \tag{5.13}$$

for all $x \in \mathbb{R}^N$. It is not difficult to show that u^* is nonnegative, radially symmetric and nonincreasing. Moreover, u^* has the same distribution function as u, i.e.

$$|\{u^* \geq \lambda\}| = |\{|u| \geq \lambda\}|,$$

for all $\lambda > 0$. It follows from the above identity that if $\phi \in C(\mathbb{R})$ is continuous, nondecreasing, and $\phi(0) = 0$, then

$$\int_{\mathbb{R}^N} \phi(u^*) = \int_{\mathbb{R}^N} \phi(|u|). \tag{5.14}$$

(Integrate the function $\theta(\lambda, x) = 1_{\{\phi(u(x)) \geq \lambda\}}$ on $(0, \infty) \times \mathbb{R}^N$ and apply Fubini.) The assumption that ϕ is nondecreasing can be weakened by writing $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are nondecreasing. In particular, if $\phi \in C(\mathbb{R})$ is even, $\phi(0) = 0$, and if $\phi(u) \in L^1(\mathbb{R}^N)$, it follows that $\phi(u^*) \in L^1(\mathbb{R}^N)$ and that (5.14) holds. One can show that if $\nabla u \in L^p(\mathbb{R}^N)$ for some $1 \leq p < \infty$, then $\nabla u^* \in L^p(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla u^*|^p \leq \int_{\mathbb{R}^N} |\nabla u|^p. \tag{5.15}$$

This result, however, is more delicate. See Lieb [24] for a relatively simple proof in the case $p = 2$. See Brock and Solynin [6] for a really simple proof in the general case, via polarization.

We will also use the following lemma.

Lemma 5.12. *Let $(u_n)_{n\geq 0}$ be a bounded sequence of $H^1(\mathbb{R}^N)$, and suppose that each u_n is radially symmetric and nonincreasing. Suppose further that there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$. It follows that $u_n \to u$ strongly in $L^r(\Omega)$ for all $2 < r < 2N/(N-2)$ ($2 < r < \infty$ if $N = 1, 2$).*

Proof. It is clear that u is radially symmetric and nonincreasing. Next, note that if $w \in L^2(\mathbb{R}^N)$ is radially symmetric and nonincreasing, then for every $r > 0$

$$\|w\|_{L^2}^2 \geq \int_{\{w > w(r)\}} w^2 \geq w(r)^2 |B_r| \geq \eta r^N w(r)^2. \tag{5.16}$$

This means that

$$w(r) \leq C r^{-\frac{N}{2}} \|w\|_{L^2},$$

with C independent of w and r. Fix $\varepsilon > 0$. Given $R > 0$, we have

$$\int_{\{|x|>R\}} |u_n - u|^r \le \|u_n - u\|_{L^\infty(\{|x|>R\})}^{r-2} \|u_n - u\|_{L^2}^2 \le CR^{-\frac{N(r-2)}{2}},$$

by the above inequality. Thus if we fix R large, $\|u_n - u\|_{L^r(\{|x|>R\})} \le \varepsilon/2$ for all $n \ge 1$. By Rellich, $\|u_n - u\|_{L^r(\{|x|<R\})} \to 0$, so that $\|u_n - u\|_{L^r} \le \varepsilon$ if n is large. Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

Proof of Theorem 5.11. Let

$$F(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \mu \int_{\mathbb{R}^N} u^2 - \lambda \int_{\mathbb{R}^N} |u|^{\alpha+2},$$
$$S = \{u \in H^1(\mathbb{R}^N);\, u \ne 0 \text{ and } F(u) = 0\},$$

and

$$J(u) = \int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

Set

$$m = \inf_{v \in S} J(v).$$

If $u \in S$, then

$$\|u\|_{H^1}^2 \le C \|u\|_{L^{\alpha+2}}^{\alpha+2},$$

so that $\|u\|_{L^{\alpha+2}} \ge \eta$ for some $\eta > 0$. It follows that $m > 0$. Let $(u_n)_{n\ge1}$ be a minimizing sequence. If $v_n = |u_n|$, then $(v_n)_{n\ge1}$ is also a minimizing sequence. Let now $w_n = v_n^\star$, the symmetric-decreasing rearrangement of v_n. We have $w_n \ne 0$, $J(w_n) = J(v_n)$ and $F(w_n) \le 0$. If particular, there exists $0 < \theta_n \le 1$ such that $F(\theta_n w_n) = 0$. Setting $z_n = \theta_n w_n$, we see that $(z_n)_{n\ge1}$ is a minimizing sequence made up of radially symmetric and decreasing functions. Applying Lemma 5.12 and using the property $m > 0$, we easily conclude to the existence of a radially symmetric, nonincreasing minimizer u. We now claim that u is a solution of (5.5). Indeed, note that

$$\begin{aligned}\langle F'(u), u\rangle_{H^{-1},H^1} &= \langle -2\Delta u + 2\mu u - \lambda(\alpha+2)|u|^\alpha, u\rangle_{H^{-1},H^1}\\ &= 2\int_{\mathbb{R}^N} |\nabla u|^2 + 2\mu \int_{\mathbb{R}^N} |u|^2 - \lambda(\alpha+2)\int_{\mathbb{R}^N} |u|^{\alpha+2}\\ &= -\lambda\alpha \int_{\mathbb{R}^N} |u|^{\alpha+2} = -\lambda\alpha m.\end{aligned}$$

In particular, we see that $F'(u) \ne 0$. Moreover,

$$\langle J'(u), u\rangle_{H^{-1},H^1} = (\alpha+2)\alpha \int_{\mathbb{R}^N} |u|^{\alpha+2} = (\alpha+2)m.$$

Since there exists a Lagrange multiplier Λ such that $J'(u) = \Lambda F'(u)$, we see that

$$\Lambda = -\frac{\alpha+2}{\alpha\lambda},$$

so that u is a solution of (5.5). Finally, the fact that u is a ground state follows from the same argument as in the proof of Theorem 5.10. □

Remark 5.13. The above argument can be adapted to more general nonlinearity. There is no problem of homogeneity, one can always reduce the Lagrange multiplier by a simple scaling! See Berestycki and Lions [5].

5.4. **Mountain pass.** As observed in Remark 5.8, the constrained minimization does not apply well in bounded domains for nonhomogeneous nonlinearities. A remedy to this problem is the mountain pass theorem of Ambrosetti and Rabinowitz [2]. Before stating the theorem, we introduce the Palais-Smale condition.

Definition 5.14. *Let X be a Banach space and $J \in C^1(X,\mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the Palais-Smale condition at the level c (in brief, J satisfies $(\mathrm{PS})_c$) if the following holds. If there exists a sequence $(u_n)_{n\geq 0} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ (in $X^\star$) as $n \to \infty$, then c is a critical value (i.e. there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$). We say that J satisfies the Palais-Smale condition (in brief, J satisfies (PS)) if J satisfies $(\mathrm{PS})_c$ for all $c \in \mathbb{R}$.*

With the above definition, we have the following result.

Theorem 5.15 (The mountain pass theorem)**.** *Let X be a Banach space, and let $J \in C^1(X,\mathbb{R})$. Suppose that:*

(i) *$J(0) = 0$;*
(ii) *there exist $\varepsilon, \gamma > 0$ such that $J(u) \geq \gamma$ for $\|u\| = \varepsilon$;*
(iii) *there exists $u_0 \in X$ such that $\|u_0\| > \varepsilon$ and $J(u) < \gamma$.*

Set $\mathcal{P} = \{p \in C([0,1],X);\ p(0) = 0,\ p(1) = u_0\}$ and let

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} J(p(t)) \geq \gamma.$$

If J satisfies $(\mathrm{PS})_c$, then c is a critical value of J.

Corollary 5.16. *Let X be a Banach space and $J \in C^1(X,\mathbb{R})$. Suppose that:*

(i) *$J(0) = 0$;*
(ii) *there exist $\varepsilon, \gamma > 0$ such that $J(u) \geq \gamma$ for $\|u\| = \varepsilon$;*
(iii) *there exists $u_0 \in X$ such that $\|u_0\| > \varepsilon$ and $J(u) < \gamma$.*

If J satisfies (PS), then there exist $c \geq \gamma$ and $u \in X$ such that $J(u) = c$ and $J'(u) = 0$.

See [8] for a simple proof due to Brezis and Nirenberg.

We now give some applications of the mountain pass theorem to the resolution of the equation

$$\begin{cases} -\triangle u + \mu u = g(u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega. \end{cases} \tag{5.17}$$

Theorem 5.17. *Assume Ω is a bounded domain of $\mathbb{R}^N$, and let $\mu > -\lambda_1$. Let $g \in C(\mathbb{R},\mathbb{R})$ satisfy $g(0) = 0$, and suppose there exist $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1$ or 2), $\nu < \mu + \lambda_1$ and $\theta > 2$ such that*

$$\begin{aligned} &|g(u)| \leq C(1 + |u|^{\alpha+1}) \quad \textit{for all} \quad u \in \mathbb{R}, \\ &G(u) \leq \frac{\nu}{2}u^2 \quad \textit{for} \quad |u| \quad \textit{small}, \\ &0 < \theta G(u) \leq ug(u) \quad \textit{for} \quad |u| \quad \textit{large}, \end{aligned}$$

where $G(u) = \displaystyle\int_0^u g(s)\,ds$. It follows that there exists a solution $u \in H^1_0(\Omega)$, $u \neq 0$, of the equation (5.17).

Proof. Set

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\mu}{2}\int_\Omega u^2 - \int_\Omega G(u). \tag{5.18}$$

We will show, by applying the mountain pass theorem, that there exists a critical point $u \in H^1_0(\Omega)$ of J such that $J(u) > 0$ (and so, $u \neq 0$). We proceed in two steps.

STEP 1. J satisfies (PS). Suppose $(u_n)_{n\geq 0} \subset H^1_0(\Omega)$ satisfies $J(u_n) \to c \in \mathbb{R}$ and $J'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. Since $J'(u_n) = -\triangle u_n + \mu u_n - g(u_n)$, it follows that

$$(J'(u_n), u_n)_{H^{-1},H^1_0} = \int_\Omega |\nabla u_n|^2 + \mu \int_\Omega u_n^2 - \int_\Omega u_n g(u_n);$$

and so,

$$2J(u_n) - (J'(u_n), u_n)_{H^{-1},H^1_0} = \int_\Omega (u_n g(u_n) - 2G(u_n)).$$

Note that $ug(u) \geq \theta G(u) - C$ for all $u \in \mathbb{R}$ and some constant C. Therefore,

$$2J(u_n) - (J'(u_n), u_n)_{H^{-1},H^1_0} \geq (\theta - 2)\int_\Omega G(u_n) - C|\Omega|.$$

We deduce that

$$(\theta - 2)\int_\Omega G(u_n) \leq 2J(u_n) + \|J'(u_n)\|_{H^{-1}}\|u_n\|_{H^1} + C|\Omega|. \tag{5.19}$$

It follows that there exists a constant C such that

$$\int_\Omega G(u_n) \leq C + C\|u_n\|_{H^1}.$$

Therefore,

$$J(u_n) \geq \alpha\|u_n\|^2_{H^1} - C\|u_n\|_{H^1} - C,$$

for some $\alpha > 0$; and so $(u_n)_{n\geq 0}$ is bounded in $H^1_0(\Omega)$. We deduce that there exist a subsequence, which we still denote by $(u_n)_{n\geq 0}$ and $u \in H^1_0(\Omega)$ such that $u_n \to u$ in $L^{p+1}(\Omega)$ as $n \to \infty$ and

$$\int_\Omega \nabla u_n \cdot \nabla\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega \nabla u \cdot \nabla\varphi,$$

for all $\varphi \in H^1_0(\Omega)$. Furthermore, we may also assume that there exists $h \in L^{\alpha+2}(\Omega)$ such that $|u_n| \leq h$ a.e. in Ω. We deduce easily by dominated convergence and the growth assumption on g that $g(u_n) \to g(u)$ in $L^{\frac{\alpha+2}{\alpha+1}}(\Omega)$, hence in $H^{-1}(\Omega)$, as $n \to \infty$. It follows that

$$\int_\Omega g(u_n)\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega g(u)\varphi,$$

for all $\varphi \in H^1_0(\Omega)$. Therefore,

$$(-\triangle u_n + \mu u_n - g(u_n), \varphi)_{H^{-1},H^1_0} \underset{n\to\infty}{\longrightarrow} (-\triangle u + \mu u - g(u), \varphi)_{H^{-1},H^1_0}.$$

Since $-\triangle u_n + \lambda u_n - g(u_n) = J'(u_n) \to 0$, it follows that $J'(u) = 0$. It now remains to show that $J(u) = c$. It follows from what precedes that $-\triangle u_n + \mu u_n \to -\triangle u + \mu u$ in $H^{-1}(\Omega)$, which implies that $u_n \to u$ in $H^1_0(\Omega)$; and so, $J(u) = \lim J(u_n) = c$ as $n \to \infty$.

Step 2. Conclusion. We have $J(0) = 0$. In addition, there exists a constant C such that $G(u) \le \frac{\nu}{2} u^2 + C|u|^{\alpha+2}$ for all $u \in \mathbb{R}$; and so,

$$\int_\Omega G(u) \le \frac{\nu}{2} \int_\Omega u^2 + C\|u\|_{H^1}^{\alpha+2}.$$

Therefore,

$$J(u) \ge \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\mu - \nu}{2} \int_\Omega u^2 - C\|u\|_{H^1}^{\alpha+2}.$$

Since $\mu - \nu > -\lambda_1$, we deduce that there exists $\delta > 0$ such that

$$J(u) \ge \delta \|u\|_{H^1}^2 - C\|u\|_{H^1}^{\alpha+2}.$$

Therefore, setting $\varepsilon = (\delta/2C)^{\frac{1}{\alpha}}$, we have $J(u) \ge \delta\varepsilon^2/2 > 0$ for $\|u\|_{H^1} = \varepsilon$. We claim that there exists $u \in H_0^1(\Omega)$ such that $\|u\|_{H^1} \ge \varepsilon$ and $J(u) < 0$. Indeed, for s large, we have

$$\frac{g(s)}{G(s)} \ge \frac{\theta}{s};$$

and so, $G(s) \ge cs^\theta$ for s large. Thus $G(u) \ge cs^\theta - C$ for all $s \ge 0$. Consider now $\psi \in C_c^\infty(\Omega)$ such that $\psi \ge 0$ and $\psi \ne 0$, and $t > 0$. We have

$$J(t\psi) \le \frac{t^2}{2} \Bigl(\int_\Omega |\nabla \psi|^2 + \mu \int_\Omega \psi^2 \Bigr) + C|\Omega| - ct^\theta \int_\Omega \psi^\theta. \tag{5.20}$$

Therefore, $J(t\psi) < 0$ for t large enough, which proves the claim. Since J satisfies (PS) by Step 1, it follows from what precedes that we may apply the mountain pass theorem, from which the result follows. □

Remark 5.18. Here are some comments on Theorem 5.17.

(i) We see that Theorem 5.17 applies to more general nonlinearities than Theorem 5.6, because it does not require homogeneity. On the other hand, we do not know if the nontrivial solution that we construct is nonnegative.
(ii) Note that the assumption $\lambda > -\lambda_1$ is not essential in Theorem 5.17. However, the proof in the general case requires a slightly stronger assumption on f (namely, we need $F \ge 0$) and a more general version of the mountain pass theorem (see for example Kavian [21], Example 8.7 of Chapter 3).
(iii) Note that in Step 1 of the proof, we proved a slightly stronger property than (PS). We proved that if $(u_n)_{n\ge 0} \subset H_0^1(\Omega)$ satisfies $J'(u_n) \to 0$ and $J(u_n) \to c \in \mathbb{R}$ as $n \to \infty$, then there exist a subsequence $(u_{n_k})_{k>0}$ and $u \in H_0^1(\Omega)$ such that $u_{n_k} \to u$ in $H_0^1(\Omega)$ as $k \to \infty$ (and so, $J(u) = c$ and $J'(u) = 0$). This property is sometimes used as the definition of the Palais-Smale condition.

We saw that the energy J is not bounded from below (see (5.20)), so a solution cannot minimize the energy on the whole space $H_0^1(\Omega)$. However, there is still the notion of *ground state*, as in the preceding section. A ground state is a nontrivial solution of (5.17) which minimizes J among all nontrivial solutions of (5.17). We now show the existence of a ground state, under slightly stronger assumptions on g than in Theorem 5.17.

Theorem 5.19. *Assume Ω is a bounded domain of $\mathbb{R}^N$, and let $\mu > -\lambda_1$. Let $g \in C(\mathbb{R}, \mathbb{R})$ satisfy $g(0) = 0$, and suppose there exist* $0 < \alpha < 4/(N-2)$ *(*$0 < \alpha < \infty$

if $N = 1$ or 2), $\nu < \mu + \lambda_1$ and $\theta > 2$ such that

$$
\begin{aligned}
&|g(u)| \le C(1 + |u|^{\alpha+1}) \quad \textit{for all} \quad u \in \mathbb{R}, \\
&ug(u) \le \nu u^2 + C|u|^{\alpha+2} \quad \textit{for all} \quad u \in \mathbb{R}, \\
&0 < \theta G(u) \le ug(u) \quad \textit{for} \quad |u| \quad \textit{large},
\end{aligned}
$$

where $G(u) = \displaystyle\int_0^u g(s)\,ds$. It follows that there exists a ground state of the equation (5.17).

Proof. Since g satisfies the assumptions of Theorem 5.17, there exists a nontrivial solution of (5.17). Let $\mathcal{E} \neq \emptyset$ be the set of nontrivial solutions of (5.17), and set

$$
m = \inf_{v \in \mathcal{E}} J(v).
$$

If $v \in \mathcal{E}$, then it follows from (5.19) that

$$
\theta \int_\Omega G(v) \le 2J(v) + C|\Omega|.
$$

Since G is bounded from below, we deduce that $J(v)$ is bounded from below; and so, $m > -\infty$. Let now $(u_n)_{n\ge 0}$ be a minimizing sequence. Since $J'(u_n) = 0$ and $J(u_n) \to m \in \mathbb{R}$, it follows from Remark 5.18 (iii) that there exist a subsequence $(u_{n_k})_{k>0}$ and $u \in H^1_0(\Omega)$ such that $u_{n_k} \to u$ in $H^1_0(\Omega)$ as $k \to \infty$. In particular, $J(u) = m$ and $J'(u) = 0$. Therefore, it only remains to show that $u \neq 0$. Indeed, we have $(J'(u_n), u_n)_{H^{-1},H^1_0} = 0$, i.e.

$$
\begin{aligned}
\int_\Omega |\nabla u_n|^2 + \mu \int_\Omega u_n^2 = \int_\Omega u_n g(u_n) &\le \nu \int_\Omega u_n^2 + C \int_\Omega |u_n|^{\alpha+2} \\
&\le \nu \int_\Omega u_n^2 + C\|u_n\|_{H^1}^{\alpha+2};
\end{aligned}
$$

and so,

$$
\int_\Omega |\nabla u_n|^2 + (\mu - \nu) \int_\Omega u_n^2 \le C\|u_n\|_{H^1}^{\alpha+2}.
$$

Since $\mu - \nu > -\lambda_1$, we deduce that

$$
\|u_n\|_{H^1}^2 \le C\|u_n\|_{H^1}^{\alpha+2};
$$

and since $u_n \neq 0$, we conclude that $\|u_n\|_{H^1} \ge C^{-\frac{1}{\alpha}}$. It follows that $\|u\|_{H^1} \ge C^{-\frac{1}{\alpha}}$, so that $u \neq 0$. □

Remark 5.20. If (PS) is not satisfied, then the conclusion of Theorem 5.15 may fail. Here is such an example. Let Ω be the unit ball of $\mathbb{R}^3$ and let $X = H^1_{0,\mathrm{r}}(\Omega)$, the subset of $H^1_0(\Omega)$ made up of radially symmetric functions. Let

$$
J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{6} \int_\Omega u^6.
$$

It follows that $J \in C^1(X, \mathbb{R})$. Moreover, $J(0) = 0$, and it follows from Sobolev's inequality $\|u\|_{L^6} \le C\|\nabla u\|_{L^2}$ that if ε is small, then $J(u) \ge \eta_\varepsilon > 0$ if $\|u\|_{H^1} = \varepsilon$. Moreover, $J(tu) \to -\infty$ as $t \to \infty$ for all nontrivial $u \in X$. Thus all assumptions of the mountain pass theorem are satisfied, except possibly (PS). In fact, we claim that if $u \in X$ satisfies $J'(u) = 0$, then $u = 0$. In particular, (PS) is not satisfied and the conclusion of Theorem 5.15 does not hold. To prove the claim, we note that $J'(u) = -\Delta u - u^5$, so if $J'(u) = 0$ then

$$u'' + \frac{N-1}{r} u' + u^5 = 0,$$

for $0 < r < 1$ and $u(1) = 0$. In particular,

$$\frac{1}{2}(r^2 u u')' = -\frac{2}{2} r^2 u^6 + \frac{1}{2} r^2 u'^2,$$

$$\Big(\frac{r^3}{2} u'^2 + \frac{r^3}{6} u^6\Big)' = \frac{2}{2} r^2 u^6 - \frac{1}{2} r^2 u'^2.$$

Summing up the two inequalities and integrating on $(\varepsilon, 1)$ for $0 < \varepsilon < 1$, we obtain

$$\frac{\varepsilon^3}{2} u'(\varepsilon)^2 + \frac{\varepsilon^3}{6} u(\varepsilon)^6 + \frac{\varepsilon^2}{2} u(\varepsilon) u'(\varepsilon) = \frac{1}{2} u'(1)^2.$$

Since $u \in H^1_0(\Omega)$, $r^2 u'^2$ is integrable on $(0,1)$. Moreover, $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$, so $r^2 u^6$ is integrable on $(0,1)$. It follows that there exist $r_n \downarrow 0$ such that $r_n^{\frac{3}{2}} u'(r_n) \to 0$ and $r_n^2 u(r_n)^6 \to 0$. Choosing $\varepsilon = r_n$, we see that the left-hand side of the above inequality converges to 0 as $n \to \infty$, so that $u'(1) = 0$. Hence $u \equiv 0$.

Remark 5.21. In all the above section, we considered spaces of **real-valued** functions. For all the results that we obtained by global or constrained minimization (resulting in positive solutions), it is equivalent to consider spaces of **complex-valued** functions. Indeed, let u is a complex-valued minimizer, and set $f = |\mathrm{Re}\, u|$, $g = |\mathrm{Im}\, u|$ and $v = f + ig$. We have $|v| = |u|$ and $|\nabla v| = |\nabla u|$. Thus v is also a minimizer, hence a solution of the equation. In particular, $-\Delta f = Vf$ and $-\Delta g = Vg$ for the same potential V. Suppose, for example, that $f \not\equiv 0$. By the maximum principle, $f > 0$. Therefore, since $g \geq 0$ is a solution of the same equation, $g = \gamma f$ for some $\gamma \geq 0$. This shows that there exists a positive minimizer w such that $u = e^{i\theta} w$ for some $\theta \in \mathbb{R}$.

6. The heat equation

General reference: the book [33] by Quittner and Souplet. Also the book [9], less advanced but more elementary. Also contains a section on semigroups. In this section, we study the heat equation: IVP, global existence and blowup, stability.

We prove local existence by a perturbation technique (semilinear), so we first study the **linear** equation.

6.1. The linear equation on $\mathbb{R}^N$.

We begin with the case of the whole space, so we consider the heat equation $u_t = \Delta u$ in $\mathbb{R}^N$. The Cauchy problem is easily solved in $\mathcal{S}'(\mathbb{R}^N)$ by Fourier.

Lemma 6.1. *Given any $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, there exists a unique $u \in C([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ such that $u_t = \Delta u$ in $\mathcal{S}'(\mathbb{R}^N)$ for all $t \geq 0$ and $u(0) = \varphi$. u is given by*

$$\widehat{u}(t) = e^{-4\pi^2 t|\xi|^2} \widehat{\varphi}, \tag{6.1}$$

or, equivalently,

$$u(t) = G_t \star \varphi, \tag{6.2}$$

for all $t > 0$, where the **heat kernel** *G_t is*

$$G_t(x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}. \tag{6.3}$$

Proof. We use the Fourier transform

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\, dx.$$

Setting $v(t) = \widehat{u}(t)$ equation is equivalent to $v_t = -4\pi^2|\xi|^2 v$, whose unique solution is given by (6.1). Since $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}f \star \mathcal{F}^{-1}g$ and $\mathcal{F}^{-1}[e^{-4\pi^2 t|\xi|^2}] = G_t$, (6.2) follows. □

Definition 6.2. *Given $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, we denote by $e^{t\Delta}\varphi$ the corresponding solution of the heat equation given by Lemma* 6.1.

Corollary 6.3. *Let $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ and set $u(t) = e^{t\Delta}\varphi$ for $t \ge 0$.*

(i) *If $\varphi \in L^p(\mathbb{R}^N)$ for some $1 \le p < \infty$, then $u \in C([0,\infty), L^p(\mathbb{R}^N))$. Moreover, $u \in C^\infty((0,\infty), W^{m,q}(\mathbb{R}^N))$ for all $m \ge 0$ and $p \le q \le \infty$.*

(ii) *If $\varphi \in C_0(\mathbb{R}^N)$, then $u \in C([0,\infty), C_0(\mathbb{R}^N)) \cap C^\infty((0,\infty), W^{m,\infty}(\mathbb{R}^N))$ for all $m \ge 0$.*

(iii) *If $\varphi \in L^\infty(\mathbb{R}^N)$, then $u \in C^\infty((0,\infty), W^{m,\infty}(\mathbb{R}^N))$ for all $m \ge 0$.*

(iv) *The inequality*

$$\|e^{t\Delta}\varphi\|_{L^p} \le t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_{L^q}, \tag{6.4}$$

holds for all $t > 0$, $1 \le q \le p \le \infty$ and $\varphi \in L^q(\mathbb{R}^N)$.

6.2. **The (weak) maximum-comparison principle.** The following result describes an essential property of the heat equation. Ω is a domain of $\mathbb{R}^N$, bounded or not.

Lemma 6.4. *Fix $T > 0$, $1 < p < \infty$. Let $f \in L^1_{\rm loc}((0,T), H^{-1}(\Omega))$ and $u \in L^p((0,T), H^1(\Omega)) \cap W^{1,p'}((0,T), H^{-1}(\Omega))$ (so that $u \in C([0,T], L^2(\Omega))$) satisfy the equation*

$$u_t - \Delta u = f, \text{ for a.a. } t \in (0,T),$$

and assume that

(i) *there exists $v \in L^p((0,T), H^1_0(\Omega))$ such that $u(t) \le v(t)$ a.e. in Ω for a.a. $t \in (0,T)$;*

(ii) *$f = g + h$, with $g \in L^1_{\rm loc}((0,T), H^{-1}(\Omega))$, $g(t) \le 0$ for a.a. $t \in (0,T)$, and $h \in L^1_{\rm loc}((0,T), L^2(\Omega))$, $h(t) \le C|u(t)|$ a.e. in Ω for a.a. $t \in (0,T)$ where C is independent of t;*

(iii) *$u(0) \le 0$ a.e. in Ω.*

It follows that $u(t) \le 0$ a.e. in Ω for all $t \in (0,T)$.

Remark 6.5. Assumption (i) means that $u \le 0$ on $\partial\Omega$, and assumption (ii) means that $f \le C|u|$. Thus, formally, we have

$$\begin{cases} u_t - \Delta u \le C|u|, \\ u_{|\partial\Omega} \le 0, \\ u(0,x) \le 0. \end{cases}$$

Proof. By (i), we have $u^+(t) \in H^1_0(\Omega)$ for a.a. $t \in (0,T)$. It follows that

$$\langle u_t(t), u^+(t)\rangle_{H^{-1},H^1_0} - \langle \Delta u(t), u^+(t)\rangle_{H^{-1},H^1_0} = \langle f(t), u^+(t)\rangle_{H^{-1},H^1_0},$$

for a.a. $t \in (0,T)$. On the other hand,

$$\langle \Delta u(t), u^+(t)\rangle_{H^{-1},H^1_0} = -\int_\Omega |\nabla u^+|^2 \le 0,$$

and it follows from assumption (ii) that

$$\begin{aligned}\langle f(t), u^+(t)\rangle_{H^{-1},H_0^1} &\le \langle h(t), u^+(t)\rangle_{H^{-1},H_0^1} \\ &\le C\int_\Omega |u(t)|u^+(t)\,dx = C\int_\Omega u^+(t)^2\,dx.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega u^+(t)^2\,dx \le C\int_\Omega u^+(t)^2\,dx,$$

for a.a. $t \in (0, T)$. Integrating the above inequality and using assumption (iii), we obtain

$$\int_\Omega u^+(t)^2\,dx \le C\int_0^t\int_\Omega u^+(s)^2\,dx\,ds,$$

for all $t \in (0, T)$; and so, $u^+(t) \equiv 0$ by Gronwall's lemma. Hence the result. ☐

6.3. **The linear equation on a domain.** We now consider a **bounded domain** $\Omega \subset \mathbb{R}^N$ and we consider the heat equation with **Dirichlet** boundary condition

$$\begin{cases} u_t = \Delta u & \text{in } \Omega, t > 0, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

The construction of solutions of the initial value problem for (6.5) makes use of the **semigroup theory** (see [19, 31, 9]).

General idea: consider the linear, unbounded operator A on $L^2(\Omega)$ defined by

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega);\ \Delta u \in L^2(\Omega)\}, \\ Au = -\Delta u, \quad u \in D(A). \end{cases}$$

It follows that A is a self-adjoint, positive operator with compact inverse.

Given any $\varphi > 0$, let the **linear, continuous operator** A_λ be defined by

$$A_\lambda x = \frac{x - (I + \lambda A)^{-1}x}{\lambda} = A(I + \lambda A)^{-1}x = (I + \lambda A)^{-1}Ax,$$

where the last identity holds if $x \in D(A)$. In fact, A_φ is also self-adjoint. The exponential e^{-tA_λ} is well-defined and, given any $\varphi \in L^2(\Omega)$, $u_\lambda(t) = e^{-tA_\lambda}\varphi$ is the unique solution of the equation

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0, \quad t \ge 0,$$

with the initial condition $u_\lambda(0) = \varphi$. The semigroup theory shows that one can pass to the limit as $\lambda \downarrow 0$. The resulting object is denoted by e^{-tA}, or $e^{t\Delta}$ and the following properties hold.

(i) $e^{t\Delta}$ is a contraction on $L^2(\Omega)$ for all $t \ge 0$.
(ii) $e^{0\Delta} = I$.
(iii) $e^{t\Delta}e^{s\Delta} = e^{(t+s)\Delta}$ for all $s, t \ge 0$.
(iv) Given any $\varphi \in L^2(\Omega)$, the mapping $t \mapsto e^{t\Delta}\varphi$ belongs to $C([0,\infty), L^2(\Omega))$.
(v) Given $\varphi \in D(A)$, $u(t) = e^{t\Delta}\varphi$ satisfies $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), L^2(\Omega))$ and is the unique solution of the equation (6.5) with the initial condition $u(0) = \varphi$.
(vi) (This last property is because A is self-adjoint and ≥ 0.) Given $\varphi \in L^2(\Omega)$, $u(t) = e^{t\Delta}\varphi$ satisfies $u \in C((0,\infty), D(A)) \cap C^1((0,\infty), L^2(\Omega))$ and is the unique solution of the equation (6.5) with the initial condition $u(0) = \varphi$.

(vii) (By applying powers of A, time-derivatives.) Given $\varphi \in L^2(\Omega)$, $u(t) = e^{t\Delta}\varphi$ satisfies $u \in C^\infty((0,\infty), D(A^m))$ for all $m \geq 0$. In particular (by interior regularity) $u \in C^\infty((0,\infty) \times \Omega)$.

Proposition 6.6. *Given $\varphi \in L^2(\Omega)$, the estimate*

$$\|e^{t\Delta}\varphi\|_{L^p} \leq t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_{L^p}, \tag{6.6}$$

holds for all $t > 0$ and $1 \leq q \leq p \leq \infty$.

Proof. By comparison with the case of $\mathbb{R}^N$ (maximum principle). First, by density we may assume $\varphi \in D(A)$. Let $\psi \in L^2(\mathbb{R}^N)$ be defined by

$$\psi(x) = \begin{cases} |\varphi(x)| & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

and set $v(t) = G_t \star \psi$ and $w(t) = v(t)_{|\Omega} > 0$. If $z(t) = u(t) - w(t)$, then we see that

$$\begin{cases} z_t = \Delta z & (0,\infty) \times \Omega, \\ z_{|\partial\Omega} = -w_{|\partial\Omega} < 0 & (0,\infty) \times \partial\Omega, \\ z(0) = \varphi - |\varphi| \leq 0 & \Omega. \end{cases}$$

By comparison principle, $u \leq w$ and, similarly, $u \geq -w$. Thus $|u| \leq w$ and the result follows. □

Proposition 6.7. *Given $\varphi \in L^2(\Omega)$, the estimate*

$$\|e^{t\Delta}\varphi\|_{L^2} \leq e^{-\lambda_1 t}\|\varphi\|_{L^2}, \tag{6.7}$$

holds for all $t > 0$.

Proof. Multiply the equation by u and use Poincaré. □

Corollary 6.8. *Given any $1 \leq p \leq \infty$, there exists C such that*

$$\|e^{t\Delta}\varphi\|_{L^p} \leq Ce^{-\lambda_1 t}\|\varphi\|_{L^p}, \tag{6.8}$$

for all $t > 0$ and all $\varphi \in L^2(\Omega)$.

6.4. **The linear non-homogeneous equation (Duhamel's formula).** Suppose Ω is a bounded domain. (Not important.)

Proposition 6.9 (Duhamel's formula)**.** *Let $T > 0$, $\varphi \in D(A)$, $f \in C([0,T], L^2(\Omega))$, $u \in C([0,T], D(A)) \cap C^1([0,T], L^2(\Omega))$ and suppose*

$$\begin{cases} u_t = \Delta u + f & 0 < t < T, \\ u(0) = \varphi. \end{cases} \tag{6.9}$$

It follows that

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta} f(s)\, ds, \tag{6.10}$$

for all $0 \leq t \leq T$.

Proof. Fix $0 < t < T$. If

$$w(s) = e^{(t-s)\Delta}u(s)$$

for $0 \le s \le t$, then for all $0 \le s < s+h \le t$

$$\begin{aligned}\frac{w(s+h)-w(s)}{h} &= e^{(t-s-h)\Delta}\Big(\frac{u(s+h)-u(s)}{h} - \frac{e^{h\Delta}-I}{h}u(s)\Big)\\ &\underset{h\downarrow 0}{\longrightarrow} e^{(t-s)\Delta}(u'(s)-\Delta u(s)) = e^{(t-s)\Delta}f(s).\end{aligned}$$

□

Remark 6.10. Here are some comments.

(i) Given any $\varphi \in L^2(\Omega)$ and $f \in L^1((0,T),L^2(\Omega))$, the formula (6.10) defines a unique function $u \in C([0,T],L^2(\Omega))$. We call this u a "weak", or "mild" solution of (6.9), and we will study the nonlinear problem in this form.

(ii) Suppose $\varphi \in D(A)$ and $f \in C([0,T],L^2(\Omega))$. If $f \in L^1((0,T),D(A))$ or if $f \in W^{1,1}((0,T),L^2(\Omega))$, then the function u defined by (6.10) belongs to $C([0,T],D(A)) \cap C^1([0,T],L^2(\Omega))$ and satisfied the PDE (6.9).

6.5. **A general local existence result.** Suppose Ω is bounded and smooth and $g \in C(\mathbb{R},\mathbb{R})$ is locally Lipschitz, and consider the problem

$$\begin{cases} u_t - \Delta u = g(u) & x\in\Omega, t\in[0,T],\\ u(t,x)=0, & x\in\partial\Omega, t\in[0,T],\\ u(0,x)=\varphi(x), & x\in\Omega.\end{cases} \tag{6.11}$$

Theorem 6.11. *Given $\varphi \in L^\infty(\Omega)$, there exists a unique weak solution u of* (6.11)*, defined on a maximal time interval $[0,T_{\max})$, i.e. $u \in L^\infty((0,T)\times\Omega)$ for all $T < T_{\max}$ and*

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta} g(u(s))\,ds, \tag{6.12}$$

for all $t \in [0,T_{\max})$. Moreover, we have the **blowup** *alternative*

either $T_{\max} = +\infty$*,*

or $T_{\max} < \infty$ **and** $\lim_{t\uparrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty$.

In addition, u depends continuously on φ. More precisely, the mapping $\varphi \mapsto T_{\max}(\varphi)$ is lower semicontinuous, and for every $T < T_{\max}(\varphi)$ there exist $\varepsilon > 0$ and $C < \infty$ such that if $\|\psi-\varphi\|_{L^\infty} \le \varepsilon$, then $\|v-u\|_{L^\infty((0,T)\times\Omega)} \le C\|v_0-u_0\|_{L^\infty(\Omega)}$, where v is the solution of (6.12) *with the initial value ψ.*

Remark 6.12. If $T_{\max} = \infty$ the solution is **global**. If $T_{\max} < \infty$, the maximal existence interval of the solution is finite, and the blowup alternative shows that the solution **blows up in finite time** (in $L^\infty(\Omega)$).

Remark 6.13. Note that there is **no growth condition** on the nonlinearity. This is because we work in $L^\infty(\Omega)$.

Remark 6.14. One can show that the solution is "smooth" for positive time. More precisely, $u \in C((0,T_{\max}),W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega))\cap C^1((0,T_{\max}),L^p(\Omega))$ for every $p<\infty$.

Proof of Theorem 6.11. Step 1. Uniqueness. Suppose that u_1 and u_2 are two solutions of (6.12) on $[0,T]$. Then,

$$u_1(t)-u_2(t) = \int_0^t e^{(t-s)\Delta}(g(u_1(s))-g(u_2(s)))\,ds.$$

Taking the L^∞ norm of both sides, we find

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{L^\infty} &\le \int_0^t \|g(u_1(s))-g(u_2(s))\|_{L^\infty}\,ds\\ &\le K\int_0^t \|u_1(s)-u_2(s)\|_{L^\infty}\,ds,\end{aligned}$$

for all $t\in[0,T]$, with K the Lipschitz constant of g on $[-A,A]$,

$$A=\max\{\|u_1\|_{L^\infty((0,T)\times\Omega)},\|u_2\|_{L^\infty((0,T)\times\Omega)}\}.$$

Uniqueness now follows from Gronwall's inequality.

STEP 2. Construction of the solution. Let $M=\|\varphi\|_{L^\infty}+1$ and let $\widetilde{g}$ be defined by

$$\widetilde{g}(u)=\begin{cases} g(M) & \text{if } u>M,\\ g(u) & \text{if } |u|\le M,\\ g(-M) & \text{if } u<-M,\end{cases}$$

so that $\widetilde{g}:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz.

We obtain a weak, global solution $\widetilde{u}\in C([0,\infty),L^2(\Omega))$ of the equation

$$\widetilde{u}(t)=e^{t\Delta}\varphi+\int_0^t e^{(t-s)\Delta}\widetilde{g}(\widetilde{u}(s))\,ds. \tag{6.13}$$

This follows from the Banach fixed point theorem by working in the space $E_k=\{u\in C([0,\infty),X);\,\sup e^{-kt}\|u(t)\|<\infty\}$ with the norm $\|u\|_{E_k}=\sup e^{-kt}\|u(t)\|$. One shows indeed that, if $\Phi(u)$ is the RHS of the equation, $\|\Phi(u)\|_{E_k}\le\|\varphi\|+\frac{1}{ke}\|F(0)\|+\frac{L}{k}\|u\|_{E_k}$ and $\|\Phi(u)-\Phi(v)\|_{E_k}\le\frac{L}{k}\|u-v\|_{E_k}$, where L is the Lipschitz constant of F. It suffices to take $k>L$.

Taking the L^∞ norm of both sides, we see that

$$\|\widetilde{u}(t)\|_{L^\infty}\le\|\varphi\|_{L^\infty}+\int_0^t\|\widetilde{g}(\widetilde{u}(s))\|_{L^\infty}\,ds\le\|\varphi\|_{L^\infty}+K_Mt,$$

where $K_M=\|g\|_{L^\infty(-M,M)}$. Choose T small enough so that

$$K_MT\le 1.$$

Then $\|\widetilde{u}(t)\|_{L^\infty}\le M$ for all $t\in[0,T]$; and then $\widetilde{u}$ satisfies (6.12) on $[0,T]$.

STEP 3. Maximal interval of existence. Set

$$T_{\max}(\varphi)=\sup\{T>0;\,\exists\text{ a solution }u\in L^\infty((0,T)\times\Omega)\}$$

Step 2 implies $T_{\max}>0$ and uniqueness implies the existence of a solution defined on a maximal time interval $[0,T_{\max})$.

STEP 4. The blowup alternative. Suppose $T_{\max}<\infty$, and assume that there is a sequence $t_j\uparrow T_{\max}$ such that $\|u(t_j)\|_{L^\infty}\le A<\infty$. Fix $\delta>0$ such that

$$\delta K_{A+1}\le 1.$$

Starting with the initial value $u(t_j)$, we have a weak solution v_j of (6.11) defined on $[0,\delta]$. Gluing together u with v_j, we obtain a weak solution of (6.11) defined on $[0,t_j+\delta]$. For j large enough, $t_j+\delta>T_{\max}$, and this is impossible since u is the maximal solution.

STEP 5. Continuous dependence. Given $T < T_{\max}$, set

$$M_T = \|u\|_{L^\infty((0,T)\times\Omega)} + 1$$

Let $\widetilde{g}$ be as above, but with $M = M_T$ and let L_T be the Lipschitz constant of $\widetilde{g}$. Let $\widetilde{u}$ be the solution of (6.13) and, given $\psi \in L^\infty(\Omega)$, let $\widetilde{v}$ be the corresponding solution of (6.13). It follows that

$$\|\widetilde{u}(t) - \widetilde{v}(t)\|_{L^\infty} \le \|\varphi - \psi\|_{L^\infty} + L_T \int_0^t \|\widetilde{u}(s) - \widetilde{v}(s)\|_{L^\infty}\, ds;$$

and so, by Gronwall's inequality,

$$\|\widetilde{u} - \widetilde{v}\|_{L^\infty((0,T)\times\Omega)} \le e^{TL_T} \|\varphi - \psi\|_{L^\infty(\Omega)}.$$

In particular, if $\|\varphi - \psi\|_{L^\infty(\Omega)} \le \varepsilon$ with $\varepsilon = e^{-TL_T}$, we have $\|\widetilde{u} - \widetilde{v}\|_{L^\infty((0,T)\times\Omega)} \le M_T$, so that $\widetilde{v}$ is the solution of (6.12) on $[0, T]$ with the initial value v_0. The continuous dependence follows easily. □

Remark 6.15. All the above results hold when $g(u)$ is replaced by $g(x, u)$ which is measurable in x and locally Lipschitz in u, uniformly in x.

Remark 6.16. By the comparison principle, if $\psi \ge \varphi$ (resp. $\psi \le \varphi$) and v is the corresponding solution, then $v \ge u$ (resp. $v \le u$) as long as both u, v exist. Indeed, set $w = v - u$. It follows that $w(0) \ge 0$, $w_{|\partial\Omega} = 0$ and $w_t = \Delta w + g(v) - g(u)$. Since g is locally Lipschitz, $g(v) - g(u) \le C|w|$.

Remark 6.17. Note that by the **blowup alternative**, showing that a solution is global (i.e. $T_{\max} = \infty$) amounts in proving an **a priori estimate**, i.e. we need only show that, given any $T > 0$, there exists $M > 0$ such that $\|u(t)\|_{L^\infty} \le M$ for all $t < T_{\max}$, $t \le T$. For this property, the blowup alternative is essential.

6.6. **The condition $ug(u) \le C(1 + u^2)$ implies global existence.** Here is a condition on the nonlinearity under which **all** the solutions are global. More precisely, we assume that

$$ug(u) \le C(1 + u^2), \tag{6.14}$$

for all $u \in \mathbb{R}$.

Theorem 6.18. *For every $u_0 \in L^\infty(\Omega)$, the solution u of* (6.11) *is globally defined.*

Proof. We need only obtain an a priori estimate. Observe that the inequality (6.14) implies the existence of a constant D such that

$$g(u) \le D(u + 1), \tag{6.15}$$

for all $u \ge 0$. Let v be the solution of the linear problem

$$\begin{cases} v_t - \Delta v = D(v + 1) & \text{in } (0, \infty) \times \Omega, \\ v = 0 & \text{in } (0, \infty) \times \partial\Omega, \\ v(0, x) = |u_0(x)| & \text{in } \Omega. \end{cases}$$

By the maximum principle, it follows that $v \ge 0$ in $(0, \infty) \times \Omega$; and so, it follows from (6.15) that

$$v_t - \Delta v \ge g(x, v).$$

The comparison principle now implies that $u \le v$ in $(0, T_{\max}) \times \Omega$. More precisely, fix $0 < T < T_{\max}$ and set $w = u - v$ for $0 \le t \le T$, so that $w \in L^\infty((0, T) \times \Omega)$. It is not difficult to see that by the smoothing effect, we have $v \in L^2((0,T), H_0^1(\Omega)) \cap W^{1,2}((0,T), H^{-1}(\Omega))$. Moreover,

$$w_t - \Delta w \le g(u) - g(v) \le C|w|,$$

since g is locally Lipschitz and w is bounded. Since $w(0) \le 0$, it follows from the weak maximum principle that $w(t) \le 0$ for $0 \le t \le T$. Since $T \in (0, T_{\max})$ is arbitrary, we see that $u(t) \le v(t)$ for all $0 \le t < T_{\max}$. Similarly, one obtains a lower bound for u in $(0, T_{\max}) \times \Omega$. By the blowup alternative, we deduce that $T_{\max} = +\infty$. □

Remark 6.19. The assumption (6.14) is a one-sided condition. It means that g is sublinear in one direction only. It applies for example to the following cases:

- $g(u) = -|u|^\alpha u$, $\alpha > 0$.
- $g(u) = a|u|^\alpha u - b|u|^\beta u$, with $b > 0$, $0 \le \alpha < \beta$.

6.7. **Global existence for small initial values.** We give a condition on $g(u)$ near $u = 0$ under which which small initial values produce global solutions.

Theorem 6.20. *Assume $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(0) = 0$ and $g'(0) < \lambda_1$. There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega)$ and $\|u_0\|_{L^\infty} \le \delta$, then the solution u of* (6.11) *is globally defined. Moreover, given any $\varepsilon > 0$, there exists $C > 0$ such that*

$$\|u(t)\|_{L^\infty} \le Ce^{-(\lambda_1 - \ell - \varepsilon)t},$$

for all $t \ge 0$.

Proof. Let $\delta > 0$ to be chosen later and set $v(t) = \delta e^{\lambda_1 t} e^{t\Delta} 1_\Omega$, so that $\|v(t)\|_{L^\infty} \le M\delta$ for some finite M. Let

$$\ell = g'(0)$$

Given any $0 < \varepsilon \le \lambda_1 - \ell$, there exists $\alpha_\varepsilon > 0$ such that

$$\frac{g(t)}{t} - \ell \le \varepsilon, \text{ for } |t| \le \alpha_\varepsilon.$$

We let $\delta = \delta_\varepsilon = \alpha_\varepsilon / M$, so that $\|v(t)\|_{L^\infty} \le \alpha_\varepsilon$. Next, we set $w_\varepsilon(t) = e^{-(\lambda_1 - \ell - \varepsilon)t} v(t)$ Since $v \ge 0$, we have $0 \le w_\varepsilon \le v \le \alpha_\varepsilon$, so that $g(w_\varepsilon) \le (\ell + \varepsilon) w_\varepsilon$. Therefore,

$$\begin{cases} (w_\varepsilon)_t - \Delta w_\varepsilon = (\ell + \varepsilon) w_\varepsilon \ge g(w_\varepsilon), \\ (w_\varepsilon)_{|\partial\Omega} = 0, \\ w_\varepsilon(0) = \delta_\varepsilon. \end{cases}$$

Therefore, w_ε is a supersolution. If $\|u_0\|_{L^\infty} \le \delta_\varepsilon$, it follows that $u(t) \le w_\varepsilon(t)$ for all $t < T_{\max}$. One shows similarly that $-w_\varepsilon$ is a subsolution, so that $|u(t)| \le w_\varepsilon(t)$ for all $t < T_{\max}$. By the blowup alternative, we deduce that $T_{\max} = \infty$.

We now proceed as follows. We first fix $0 < \varepsilon_0 < \lambda_1 - \ell$. We deduce that there exists $\delta > 0$ such that if $\|u_0\|_{L^\infty} \le \delta$, then u is global and $\|u(t)\|_{L^\infty} \to 0$ as $t \to \infty$. Next, we consider any $0 < \varepsilon < \lambda_1 - \ell$. For t_0 sufficiently large, we have $\|u(t_0)\|_{L^\infty} \le \delta_\varepsilon$, so that $u(t + t_0) \le w_\varepsilon(t)$. The exponential decay follows. □

Remark 6.21. The assumption $g'(0) < \lambda_1$ is sharp. If $g(u) = \lambda u + |u|^\alpha u$ with $\lambda \ge \lambda_1$, then for every $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, $u_0 \not\equiv 0$, the corresponding solution blows up in finite time.

6.8. **Global existence near a stable equilibrium.** Suppose that g is C^1 and let $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of

$$\begin{cases} -\Delta w = g(w), \\ w_{|\partial\Omega} = 0. \end{cases} \tag{6.16}$$

and assume that w is linearly stable in the sense that

$$\lambda_1(-\Delta - g'(w)) > 0. \tag{6.17}$$

The following result is analogous to Theorem 6.20.

Theorem 6.22. *Let w be a solution of* (6.16) *and assume* (6.17). *There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega)$ and $\|u_0 - w\|_{L^\infty} \le \delta$, then the solution u of* (6.11) *is globally defined. Moreover, given any $\varepsilon > 0$, there exists $C > 0$ such that*

$$\|u(t) - w\|_{L^\infty} \le Ce^{-(\lambda_1(-\Delta - g'(w)) - \varepsilon)t},$$

for all $t \ge 0$.

Proof. If $v = u - w$, then

$$\begin{cases} v_t - \Delta v = h(x,u), \\ v_{|\partial\Omega} = 0, \\ v(0) = u_0 - w, \end{cases}$$

where

$$h(u,x) = g(u) - g(w(x)).$$

We observe that $h(x,0) = 0$ and that

$$\lambda_1(-\Delta - g'_u(x,0)) = \lambda_1(-\Delta - g'(w)) > 0.$$

Thus we can apply the method of proof of Theorem 6.20. □

6.9. **Unstable equilibria.** In this section, we still assume $g \in C^1$, we consider a solution w of (6.16) and, instead of (6.17), we assume

$$\lambda_1(-\Delta - g'(w)) < 0. \tag{6.18}$$

Theorem 6.23. *Let w be a solution of* (6.16) *and assume* (6.18). *It follows that w is unstable. More precisely, there exists $\eta > 0$ s.t. for every $\varepsilon > 0$, there exists u_0 with $\|\varphi - w\|_{L^\infty} \le \varepsilon$ but $\|u(t) - w\|_{L^\infty} \ge \eta$ for some $t \in [0, T_{\max})$.*

Proof. Let $\varphi_1 > 0$ first eigenvector with $\|\varphi_1\|_{L^\infty} = 1$, i.e.

$$-\Delta\varphi_1 - g'(w)\varphi_1 = \lambda_1\varphi_1.$$

Fix $\eta > 0$ be such that

$$|g(\varphi + s) - g(\varphi) - g'(\varphi)s| \le \frac{|\lambda_1|}{2}|s|,$$

for $|s| \le \eta$. Given any $0 < \varepsilon < \eta$, let $T > 0$ be such that

$$\varepsilon e^{-\frac{\lambda_1}{2}T} = \eta.$$

Let

$$z(t) = \varepsilon e^{-\frac{\lambda_1}{2}t}\varphi_1,$$

and

$$\overline{w}(t) = w + z(t).$$

It follows that

$$\overline{w}_t - \Delta\overline{w} - g(\overline{w}) = \Big[g(w) - g(w + z(t)) + g'(w)z(t)\Big] + \frac{\lambda_1}{2} z(t).$$

Therefore, for $0 < t < T$,

$$\overline{w}_t - \Delta\overline{w} - g(\overline{w}) \le \frac{|\lambda_1|}{2} z(t) + \frac{\lambda_1}{2} z(t) = 0.$$

Thus $\overline{w}$ is subsolution. It u is the solution with $u(0) = w + \varepsilon\varphi_1$, then $u \ge \overline{w}$ as long as $t \le T$ and u exists. Since $\|\overline{w}(T) - \varphi\|_{L^\infty} = \eta$, the result follows. □

Remark 6.24. Similarly, instability from above. In some cases, one can show blowup.

6.10. **Stable manifolds.** In view of the above results, one may wonder if an "unstable" stationary solution can be approximated by global solutions of the heat equation or if it is "isolated". In fact, it turns out that there is a finite-codimentional "local stable manifold" of initial values whose corresponding solutions converge to the stationary solution. The "unstable directions" are only finitely many. [All of this in some appropriate sense!]

We consider $0 < \alpha < \frac{4}{(N-2)}$ ($0 < \alpha < \infty$ if $N = 1, 2$) and we recall that the initial value problem for (6.11) with $g(u) = |u|^\alpha u$ is locally well-posed in $H^1_0(\Omega)$. (We use $H^1_0(\Omega)$ because it is convenient to have a Hilbert space setting.) Because of the condition on α, there are nontrivial (in fact, infinitely many) stationary solutions $w \in H^1_0(\Omega) \cap C_0(\Omega)$, i.e. $-\Delta w = |w|^\alpha w$. Fix such a solution and let L be the linearized operator $L = -\Delta - (\alpha+1)|w|^\alpha$ with domain $H^2(\Omega) \cap H^1_0(\Omega)$. Let $(\lambda_j)_{j\ge1}$ be the sequence of eigenvalues of L and $(\varphi_j)_{j\ge1}$ a corresponding orthonormal (in $L^2(\Omega)$) sequence of eigenvectors. As we have seen above, $\lambda_1 < 0$. Moreover, since L is self-adjoint with compact inverse, $\lambda_j \to \infty$ as $j \to \infty$. Thus we may set

$$2 \le \ell = \inf\{j \ge 1;\, \lambda_j > 0\}, \tag{6.19}$$

and we see that $\lambda_j \ge \lambda_\ell > 0$ for $j \ge \ell$ and $\lambda_j \le 0$ for $j \le \ell - 1$. Note that $(\varphi_j)_{j\ge1}$, which is a Hilbert basis of $L^2(\Omega)$, is also a Hilbert basis of $H^1_0(\Omega)$. Let W be the subspace of $H^1_0(\Omega)$ spanned by $(\varphi_j)_{j\ge\ell}$. It follows that W is a closed subspace of $H^1_0(\Omega)$ of codimension $\ell - 1$. Finally, let Z be the ($\ell - 1$ dimensional) subspace of $H^1_0(\Omega)$ spanned by $(\varphi_j)_{j\le\ell-1}$.

Proposition 6.25. *Under the above assumptions, there exist a neighborhood V of 0 in W and a map $g \in C^1(V, Z)$ with $g(0) = 0$, $g'(0) = 0$ so that the set*

$$\mathcal{W} = \{w + b + g(b);\, b \in W\}, \tag{6.20}$$

is tangent to W at w and satisfies the following property: given any $\varphi \in \mathcal{W}$, the corresponding solution u of (6.11) is global and $u(t) \in \mathcal{W}$ for all $t \ge 0$. Moreover, $\|u(t) - w\|_{H^1} \le Ce^{-\lambda_\ell t}$ for all $t \ge 0$. $\mathcal{W}$ is called the local stable manifold of w.

Proof. We denote by $\boldsymbol{\mathcal{U}}(t)$ the semiflow of (6.11) in $H^1_0(\Omega)$. More precisely, given any $\varphi \in H^1_0(\Omega)$, $\boldsymbol{\mathcal{U}}(t)\varphi$ is the solution of (6.11), which is defined for all $0 \le t < T_{\max}(\varphi)$. We remark that $\boldsymbol{\mathcal{U}}(w) = w$ for all $t \ge 0$. Moreover, if B is the unit ball of $H^1_0(\Omega)$, then there exists $T > 0$ such that $T_{\max}(w + \varphi) > T$ for all $\varphi \in B$. It is not difficult to show that for all $0 \le t \le T$, $\boldsymbol{\mathcal{U}}(t) \in C^1(w + B, H^1_0(\Omega))$ and that its derivative at 0 is e^{-tL}. We have $H^1_0(\Omega) = Z \oplus W$. Moreover, the eigenvalues of e^{-tL} on Z are $(e^{-\lambda_j})_{j\le\ell-1} \subset \{z \in \mathbb{C};\, |z| \ge 1\}$ and the eigenvalues of e^{-tL} on W are

$(e^{-\lambda_j})_{j\ge \ell} \subset \{z\in \mathbb{C};\ |z|\le e^{-\lambda_\ell} < 1\}$. The result now follows from Theorem C.6 in Chen, Chen and Hale [11]. In fact, Theorem C.6 in [11] concerns powers of $\boldsymbol{\mathcal{U}}(t)$ for some fixed $t>0$, but the prof in the continuous case is similar. □

Remark 6.26. It follows easily from a bootstrap argument that if $\varphi \in \mathcal{W}$, then $\|u(t)-w\|_{L^\infty} \le Ce^{-\lambda_\ell t}$ for all $t\ge 0$.

Remark 6.27. Application. $g(u)=|u|^\alpha u$, i.e. the equation $u_t-\Delta u = |u|^\alpha u$, $\alpha>0$. Stationary solutions: $-\Delta w = |w|^\alpha w$, $w_{|\partial\Omega}=0$. There is always the solution 0. If $\alpha < 4/(N-2)$ then there are (infinitely many) nontrivial solutions.

(i) The solution $w=0$ is exponentially stable.

(ii) If $w\not\equiv 0$ is **any** nontrivial stationary solution, then w is unstable in the sense of Theorem 6.23. Indeed, we need only show that $\lambda_1(-\Delta-(\alpha+1)|w|^\alpha)<0$. However (Rayleigh quotient)

$$\lambda_1(-\Delta-(\alpha+1)|w|^\alpha) = \frac{1}{2}\inf\Big\{\int_\Omega (|\nabla u|^2-(\alpha+1)|w|^\alpha u^2);\ u\in H^1_0(\Omega), \|u\|_{L^2}=1\Big\}.$$

Letting $u=w/\|w\|_{L^2}$, we see that

$$2\|w\|^2_{L^2}\lambda_1 \le \int_\Omega |\nabla w|^2 - (\alpha+1)\int_\Omega |w|^{\alpha+2} = -\alpha\int_\Omega |w|^{\alpha+2} < 0.$$

(iii) In any case, there is a finite-codimensional local stable manifold at w.

6.11. **A simple criterion for finite-time blowup.** Consider the model problem

$$\begin{cases} u_t-\Delta u = |u|^\alpha u,\\ u_{|\partial\Omega}=0,\\ u(0)=u_0,\end{cases} \tag{6.21}$$

with $\alpha>0$ in a smooth, bounded domain Ω.

Theorem 6.28. *Let λ_1,φ_1 be the first eigenvalue, eigenvector of $-\Delta$ in $H^1_0(\Omega)$ normalized by $\varphi_1>0$ and $\|\varphi_1\|_{L^1}=1$. Let $u_0\in L^\infty(\Omega)$ and let u be the corresponding solution of* (6.21) *defined on the maximal interval $[0,T_{\max})$. If $u_0\ge 0$ and*

$$\int_\Omega u_0\varphi_1 > \lambda_1^{\frac{1}{\alpha}},$$

then u blows up in finite time, i.e. $T_{\max}<\infty$.

Proof. Note that $u\ge 0$ by the maximum principle. Let

$$f(t)=\int_\Omega u(t)\varphi_1,$$

so that by Jensen's inequality

$$f'(t)\ge f(t)^{\alpha+1}-\lambda_1 f(t),$$

for all $0\le t<T_{\max}$. Let $y(t)$ be the solution of $y'=y^{\alpha+1}-\lambda_1 y$, $y(0)=f(0)$. It follows that y blows up at the time

$$T=\frac{1}{\alpha\lambda_1}\log\frac{f(0)^\alpha}{f(0)^\alpha-\lambda_1}.$$

One concludes that $T_{\max}\le T$. □

Remark 6.29. **Be careful:** this does **not** show that $\int u(t)\varphi_1 \to \infty$ as $t \uparrow T_{\max}$.

Remark 6.30. If $u_0 = \lambda\theta$ with $\theta \in L^\infty(\Omega)$, $\theta \ge 0$, $\theta \not\equiv 0$, then the assumption is satisfied if $\lambda > 0$ is large enough.

Remark 6.31. If we replace the nonlinearity $|u|^\alpha u$ by $\lambda u + |u|^\alpha u$ with $\lambda \ge \lambda_1$, then we see that there is finite-time blowup for all $u_0 \ge 0$, $u_0 \not\equiv 0$.

Remark 6.32. The techniques for proving blowup are extremely rigid. Here is a simple example to illustrate that. Given $\omega \in \mathbb{R}$, set $h_\omega(t) = 1 + \sin \omega t$. Let $\varphi \ge 0$ be such that the solution of the equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u_{|\partial\Omega} = 0, \\ u(0) = \varphi, \end{cases} \tag{6.22}$$

blows up in finite time. Given any $\omega \in \mathbb{R}$, let u_ω be the maximal solution of

$$\begin{cases} u_t = \Delta u + h_\omega(t)|u|^{p-1}u, \\ u_{|\partial\Omega} = 0, \\ u(0) = \varphi. \end{cases} \tag{6.23}$$

(Not difficult to prove.) One shows easily that if $0 < T < T_{\max}(\varphi)$, then for all sufficiently large $|\omega|$ the solution u_ω exists on $[0, T]$ and that $u_\omega \to u$ in $L^\infty((0, T)\times\Omega$. **Question: If $|\omega|$ large, does u_ω blow up?** (By the way, same Pb for ODE, if one is not willing to use explicit formula.)

7. The Schrödinger equation

General reference: the book [7]. Other reference, the book [36] by Tao (more advanced). In this section, we study the heat equation: IVP, global existence and blowup, stability.

7.1. **The linear equation.** We consider the Schrödinger equation $iu_t + \Delta u = 0$ in $\mathbb{R}^N$. The unknown u is **complex-valued**. We note that if $u(t, x)$ is a solution on $(a, b) \times \mathbb{R}^N$, then

$$v(t, x) = \overline{u}(-t, x), \tag{7.1}$$

is also a solution, on $(-b, -a) \times \mathbb{R}^N$. So solving the equation for $t \ge 0$ or for $t \le 0$ are in fact the same problem. The Cauchy problem is easily solved in $\mathcal{S}'(\mathbb{R}^N)$ by Fourier.

Lemma 7.1. *Given any $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, there exists a unique $u \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$ such that $iu_t + \Delta u = 0$ in $\mathcal{S}'(\mathbb{R}^N)$ for all $t \ge 0$ and $u(0) = \varphi$. u is given by*

$$\widehat{u}(t) = e^{-4\pi^2 it|\xi|^2}\widehat{\varphi}, \tag{7.2}$$

or, equivalently,

$$u(t) = S_t \star \varphi, \tag{7.3}$$

for all $t \ne 0$, where the **Schrödinger kernel** *S_t is*

$$S_t(x) = (4\pi it)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}}. \tag{7.4}$$

Definition 7.2. *Given $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, we denote by $e^{it\Delta}\varphi$ the corresponding solution of the Schrödinger equation given by Lemma* 7.1.

Corollary 7.3. *Let $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ and set $u(t) = e^{t\Delta}\varphi$ for $t \in \mathbb{R}$.*

(i) *If* $\varphi \in L^2(\mathbb{R}^N)$, *then* $u \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ *and* $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ *for all* $t \in \mathbb{R}$. *More generally, if* $\varphi \in H^s(\mathbb{R}^N)$ *for some* $s \in \mathbb{R}$, *then* $u \in C(\mathbb{R}, H^s(\mathbb{R}^N))$ *and* $\|u(t)\|_{H^s} = \|\varphi\|_{H^s}$ *for all* $t \in \mathbb{R}$.

(ii) *If* $\varphi \in L^{p'}(\mathbb{R}^N)$ *for some* $2 \le p \le \infty$, *then* $u \in C(\mathbb{R}^N \setminus \{0\}, L^p(\mathbb{R}^N))$. *Moreover, the inequality*

$$\|e^{t\Delta}\varphi\|_{L^p} \le |t|^{-N(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{L^{p'}}, \tag{7.5}$$

holds for all $t \neq 0$, $2 \le p \le \infty$ *and* $\varphi \in L^{p'}(\mathbb{R}^N)$.

Proof. First part is trivial. Second part: $e^{it\Delta} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ isometry. Using S_t, $e^{it\Delta} \in \mathcal{L}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))$ with norm $\le |t|^{-\frac{N}{2}}$. Then use the Riesz-Thorin interpolation theorem. □

Remark 7.4. The estimate (7.5) is optimal. It cannot be extended to other $L^p - L^q$ estimetes. Indeed, note that

$$\begin{aligned} S_t \star \varphi(x) &= (4\pi i t)^{-\frac{N}{2}} \int e^{i\frac{|x-y|^2}{4t}} \varphi(y)\, dy \\ &= (4\pi i t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}} \int e^{-i\frac{x\cdot y}{2t}} e^{i\frac{|y|^2}{4t}} \varphi(y)\, dy. \end{aligned}$$

Fix $t \neq 0$ and set $\psi(x) = e^{i\frac{|x|^2}{4t}}\varphi(x)$. We deduce that

$$\begin{aligned} S_t \star \varphi(x) &= (4\pi i t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}} \int e^{-i\frac{x\cdot y}{2t}} \psi(y)\, dy \\ &= (4\pi i t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}} \mathcal{F}\psi\Big(\frac{x}{4\pi t}\Big). \end{aligned} \tag{7.6}$$

Therefore, modulo multiplication by functions of modulus 1, $e^{it\Delta}$ is nothing else than the Fourier transform. It has exactly the same properties with respect to L^p spaces!

7.2. **Strichartz estimates.** The estimate (7.5) is the same as for the heat equation, but with a **much more restricted range** of parameters.

It could be used for the study of the nonlinear equation by a fixed point argument in the following way:

$$u \in L^{\alpha+2} \Rightarrow |u|^\alpha u \in L^{\frac{\alpha+2}{\alpha+1}} \Rightarrow e^{i(t-s)\Delta}|u|^\alpha u \in L^{\alpha+2}.$$

This is the original proof of Ginibre and Velo [13]. This is technically complicated and besides is not appropriate for more general nonlinearities like $|u|^\alpha u + |u|^\beta u$.

The relevant tool is **Strichartz estimates**. Original estimate by Strichartz [34]. Extended estimate with considerably simplified proof by Ginibre and Velo [17]. (Based on the $T - T^\star$ method used by Pecher [32] for the wave equation, itself based on Marshall [26] for Klein-Gordon equation, based on an idea of Tomas [37].) Endpoint by Keel and Tao [22].

Definition 7.5. *A pair* (q, r) *is admissible if*

$$2 \le r \le \frac{2N}{N-2} (2 \le r \le \infty \text{ if } N = 1,\ 2 \le r < \infty \text{ if } N = 2),$$

and

$$\frac{2}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

Remark 7.6. The pair $(\infty, 2)$ is always admissible. The pair $(2, \frac{2N}{N-2})$ is admissible if $N \geq 3$.

Theorem 7.7. *Let $\varphi \in L^2(\mathbb{R}^N)$ and $u(t) = e^{it\Delta}\varphi$. Given any admissible pair (q, r), it follows that $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$. Moreover, there exists a constant C independent of φ such that*

$$\|u\|_{L^q(\mathbb{R},L^r)} \leq C\|\varphi\|_{L^2}. \tag{7.7}$$

Proof. We give the proof in the non-endpoint case (i.e. $r < \frac{2N}{N-2}$ if $N \geq 3$). The proof of the endpoint estimate requires a further argument, see [22].

We first assume $\varphi \in \mathcal{S}(\mathbb{R}^N)$, by density. Next, we note that

$$\|u\|_{L^q(\mathbb{R},L^r)} = \sup\Big\{\int_{-\infty}^{\infty} (u(t), v(t))_{L^2};\ v \in C_c^\infty(\mathbb{R} \times \mathbb{R}^N), \|v\|_{L^{q'}(\mathbb{R},L^{r'})} \leq 1\Big\}.$$

So let $v \in C_c^\infty(\mathbb{R} \times \mathbb{R}^N)$ with $\|v\|_{L^{q'}(\mathbb{R},L^{r'})} \leq 1$.

$$\begin{aligned}\int_{-\infty}^{\infty} (u(t), v(t))_{L^2} &= \int_{-\infty}^{\infty} (e^{it\Delta}\varphi, v(t))_{L^2} = \int_{-\infty}^{\infty} (\varphi, e^{-it\Delta}v(t))_{L^2} \\ &= \Big(\varphi, \int_{-\infty}^{\infty} e^{-it\Delta}v(t)\Big)_{L^2},\end{aligned}$$

so that

$$\Big|\int_{-\infty}^{\infty} (u(t), v(t))_{L^2}\Big| \leq \|\varphi\|_{L^2}\Big\|\int_{-\infty}^{\infty} e^{-it\Delta}v(t)\Big\|_{L^2}.$$

Next,

$$\begin{aligned}\Big\|\int_{-\infty}^{\infty} e^{-it\Delta}v(t)\Big\|_{L^2}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{it\Delta}v(t), e^{is\Delta}v(s))_{L^2} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v(t), e^{i(s-t)\Delta}v(s))_{L^2} \\ &\leq \|v\|_{L^{q'}(\mathbb{R},L^{r'})}\Big\|\int_{-\infty}^{\infty} e^{i(s-\cdot)\Delta}v(s)\Big\|_{L^q(\mathbb{R},L^r)} \\ &\leq \Big\|\int_{-\infty}^{\infty} e^{i(s-\cdot)\Delta}v(s)\Big\|_{L^q(\mathbb{R},L^r)} \\ &\leq \Big\|\int_{-\infty}^{\infty} |\cdot - s|^{-N(\frac{1}{2}-\frac{1}{r})}\|v(s)\|_{L^{r'}}\Big\|_{L^q(\mathbb{R})} \\ &= \||\cdot|^{-N(\frac{1}{2}-\frac{1}{r})} \star \|v(\cdot)\|_{L^{r'}}\|_{L^q}.\end{aligned}$$

Finally, by Hardy-Littlewood-Sobolev

$$\||\cdot|^{-N(\frac{1}{2}-\frac{1}{r})} \star \|v(\cdot)\|_{L^{r'}}\|_{L^q} \leq C\|\|v(\cdot)\|_{L^{r'}}\|_{L^{q'}} = C\|v\|_{L^{q'}(\mathbb{R},L^{r'})} \leq C.$$

This completes the proof. □

We recall that the solution u of the nonhomogeneous equation

$$\begin{cases} iu_t + \Delta u + f = 0, \\ u(0) = 0, \end{cases}$$

is given by Duhamel's formula

$$u(t) =: \Phi_f(t) = i\int_0^t e^{i(t-s)\Delta}f(s)\,ds.$$

Strichartz estimates for the nonhomogeneous equation are as follows.

Theorem 7.8. *Given any admissible pairs* (q,r) *and* (γ,ρ)*, the map* Φ *is continuous* $L^{\gamma'}(\mathbb{R},L^{\rho'}(\mathbb{R}^N)) \to L^q(\mathbb{R},L^r(\mathbb{R}^N))$*, i.e.*

$$\|\Phi_f\|_{L^q(\mathbb{R},L^r)} \le C\|f\|_{L^{\gamma'}(\mathbb{R},L^{\rho'})},$$

with a constant C *independent of* f*.*

Proof. The proof (in the non-endpoint case) is quite similar to the proof of the homogeneous estimate.

– One shows by Hardy-Littlewood-Sobolev that, given any admissible (q,r), Φ is continuous $L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^N)) \to L^q(\mathbb{R},L^r(\mathbb{R}^N))$.

– One shows by taking the square that Φ is continuous $L^{q'}(\mathbb{R},L^{r'}(\mathbb{R}^N)) \to L^\infty(\mathbb{R},L^2(\mathbb{R}^N))$.

– One shows by duality that Φ is continuous $L^1(\mathbb{R},L^2(\mathbb{R}^N)) \to L^q(\mathbb{R},L^r(\mathbb{R}^N))$.

One concludes by using Hölder's inequality and interpolation theory. □

Remark 7.9. Here are some comments.

(i) The proof uses only the fact that $[e^{it\Delta}]^\star = e^{-it\Delta}$ and the pointwise estimate (dispersive estimate) $\|e^{it\Delta}\varphi\|_{L^p} \le |t|^{-N(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{L^{p'}}$. However, Strichartz estimates can be established for some equations for which the dispersive estimate fails.

(ii) It follows in particular that, given any $\varphi \in L^2(\mathbb{R}^N)$, $e^{it\Delta}\varphi \in L^r(\mathbb{R}^N)$ for all $2 \le r \le \frac{2N}{N-2}$ ($2 \le r \le \infty$ if $N=1$, $2 \le r < \infty$ if $N=2$) and almost all $t \in \mathbb{R}$. This is remarkable, but note that "for almost all $t \in \mathbb{R}$" **cannot** be replaced by "for all $t \neq 0$".

7.3. **Local existence.** For simplicity, we consider the pure power case

$$\begin{cases} iu_t + \Delta u + \lambda|u|^\alpha u = 0, \\ u(0) = \varphi, \end{cases} \tag{7.8}$$

where $\alpha > 0$ and $\lambda \in \mathbb{R}$. In fact, we use Duhamel's formula and we study (7.8) in the form

$$u(t) = e^{it\Delta}\varphi + i\lambda \int_0^t e^{i(t-s)\Delta}[|u|^\alpha u](s)\,ds. \tag{7.9}$$

Remark 7.10. Note that studying the equation (7.8) (or (7.9)) for $t \ge 0$ or for $t \le 0$ is the same. Indeed, if $u(t)$ is a solution of (7.8) on $(0,T)$, then $\overline{u}(-t)$ is a solution on $(-T,0)$ with initial value $\overline{\varphi}$. Thus reversing the sense of time is just changing φ to $\overline{\varphi}$.

Note that **formally** there are the two conservation laws

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \tag{7.10}$$

$$E(u(t)) = E(\varphi), \tag{7.11}$$

where the energy E is defined by

$$E(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - \frac{\lambda}{\alpha+2}\int_{\mathbb{R}^N} |w|^{\alpha+2}. \tag{7.12}$$

Remark 7.11. If $\lambda \notin \mathbb{R}$, i.e. $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, then wrong.

This suggests two appropriate spaces for solving the equation: $L^2(\mathbb{R}^N)$, and $H^1(\mathbb{R}^N)$. In the L^2 case, we have the following result.

Theorem 7.12. *Suppose $0 < \alpha < 4/N$ and set*

$$\rho = \alpha + 2, \quad \gamma = \frac{4(\alpha+2)}{N\alpha}, \tag{7.13}$$

so that (γ, ρ) is an admissible pair. Given any $\varphi \in L^2(\mathbb{R}^N)$, there exists a (global) solution of (7.9) $u \in C([0,\infty), L^2(\mathbb{R}^N)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^N))$ for all $T < \infty$. Moreover, the following properties hold.

(i) *$u \in L^q((0,T), L^r(\mathbb{R}^N))$ for all $T < \infty$ and all admissible pair (q,r).*
(ii) *Uniqueness holds in $L^\infty((0,T), L^2(\mathbb{R}^N)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^N))$.*
(iii) *$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \ge 0$.*
(iv) *If $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and u_n, u the corresponding solutions then for every $T > 0$ there exists $C(T,\varphi)$ such that $\|u_n - u\|_{L^\infty((0,T),L^2)} \le C\|\varphi_n - \varphi\|_{L^2}$ for n large.*

We first need an appropriate local existence result. We write it for a general nonlinearity.

Proposition 7.13. *Let $g \in C(\mathbb{C},\mathbb{C})$ satisfy $g(0) = 0$ and*

$$|g(u) - g(v)| \le C(1 + |u|^\alpha + |v|^\alpha)|u - v|, \tag{7.14}$$

for all $u, v \in \mathbb{C}$. Let (γ, ρ) be defined by (7.13). Given any $\varphi \in L^2(\mathbb{R}^N)$, there exist $T = T(\|\varphi\|_{L^2}) > 0$ and a solution $u \in C([0,T], L^2(\mathbb{R}^N)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^N))$ of the equation

$$u(t) = e^{it\Delta}\varphi + i\int_0^t e^{i(t-s)\Delta} g(u(s))\, ds. \tag{7.15}$$

The flow is locally Lipschitz in the following sense. If $\varphi, \psi \in L^2(\mathbb{R}^N)$,

$$T \le \min\{T(\|\varphi\|_{L^2}), T(\|\psi\|_{L^2})\}$$

and u, v the corresponding solutions, then $\|u-v\|_{L^\infty((0,T),L^2)} \le C\|\varphi-\psi\|_{L^2}$ with C independent of φ, ψ. Moreover, for every $T > 0$ uniqueness holds in $L^\infty((0,T), L^2(\mathbb{R}^N)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^N))$. In addition, u can be extended to a maximal existence interval $(0, T_{\max}(\varphi))$ and the following blowup alternative holds: either $T_{\max}(\varphi) = \infty$ or else $T_{\max}(\varphi) < \infty$ and $\|u(t)\|_{L^2} \to \infty$ as $t \uparrow T_{\max}(\varphi)$.

Proof. Let $\eta \in C^\infty([0,\infty), \mathbb{R})$ such that $\eta(r) = 1$ for $r \le 1$ and $\eta(r) = 0$ for $r \ge 2$, and set

$$g_1(u) = \eta(|u|)g(u),$$
$$g_2(u) = (1 - \eta(|u|)g(u).$$

It follows that $g_j \in C(\mathbb{C},\mathbb{C})$, $g_j(0) = 0$ and

$$|g_1(u) - g_1(v)| \le C|u - v|,$$
$$|g_1(u) - g_1(v)| \le C(|u|^\alpha + |v|^\alpha)|u - v|,$$

for all $u, v \in \mathbb{C}$. Given $T, M > 0$, let

$$E_T^M = \{u \in L^\infty((0,T), L^2(\mathbb{R}^N)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^N));$$
$$\|u\|_{L^\infty((0,T),L^2)} + \|u\|_{L^\gamma((0,T),L^\rho)} \le M\}.$$

E_T^M is a complete metric space when equipped with the distance

$$\mathrm{d}(u,v) = \|u-v\|_{L^\infty((0,T),L^2)} + \|u-v\|_{L^\gamma((0,T),L^\rho)}.$$

Given $\varphi \in L^2(\mathbb{R}^N)$ and $u \in E_T^M$, let

$$\Phi(\varphi,u)(t) = e^{it\Delta}\varphi + i\int_0^t e^{i(t-s)\Delta} g(u(s))\,ds.$$

We note that by Hölder's inequality in space and time,

$$\|g_1(u)-g_1(v)\|_{L^1(0,T),L^2)} \le CT\|u-v\|_{L^\infty((0,T),L^2)},$$

and

$$\|g_2(u)-g_2(v)\|_{L^{\gamma'}(0,T),L^{\rho'})} \le CT^{\frac{4-N\alpha}{4}}(\|u\|^\alpha_{L^\gamma((0,T),L^\rho)} + \|v\|^\alpha_{L^\gamma((0,T),L^\rho)})\|u-v\|_{L^\gamma((0,T),L^\rho)}$$

By Strichartz, $\Phi(\varphi,u) \in C([0,T],L^2(\Omega)) \cap L^q((0,T),L^r\mathbb{R}^N))$ for all admissible pair (q,r). Moreover,

$$\|\Phi(\varphi,u)-\Phi(\psi,v)\|_{L^\infty((0,T),L^2)} + \|\Phi(\varphi,u)-\Phi(\psi,v)\|_{L^\gamma((0,T),L^\rho)} \le C\|\varphi-\psi\|_{L^2} + CM^\alpha(T+T^{\frac{4-N\alpha}{4}})\mathrm{d}(u,v),$$

for all $\varphi,\psi \in L^2(\mathbb{R}^N)$ and $u,v \in E_T^M$.

Given $M>0$, we deduce that if $T=T(M)$ is sufficiently small so that

$$CM^\alpha(T+T^{\frac{4-N\alpha}{4}}) \le \frac{M}{2},$$

then $\Phi(\varphi,u) \in E_T^M$ for all $u \in E_T^M$ and all $\varphi \in L^2(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2} \le M/2$. Choosing possibly $T=T(M)$ smaller so that

$$CM^\alpha(T+T^{\frac{4-N\alpha}{4}}) \le \frac{1}{2},$$

we also see that

$$\mathrm{d}(\Phi(\varphi,u)-\Phi(\psi,v)) \le C\|\varphi-\psi\|_{L^2} + \frac{1}{2}\mathrm{d}(u,v),$$

for all $u,v \in E_T^M$ and all $\varphi,\psi \in L^2(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2}, \|\psi\|_{L^2} \le M/2$.

Thus if we fix $\varphi \in L^2(\mathbb{R}^N)$, we let $M = 2C\|\varphi\|_{L^2}$ and T sufficiently small (depending on $\|\varphi\|_{L^2}$), then $\Phi : E_T^M \to E_T^M$ is a strict contraction. Thus Φ has a fixed point, which is a solution.

If $\|\varphi\|_{L^2}, \|\psi\|_{L^2} \le M/2$ and $u,v \in E_T^M$ are the corresponding solutions, then the above estimate shows that

$$\mathrm{d}(u,v) \le 2C\|\varphi-\psi\|_{L^2}.$$

We now prove uniqueness. Suppose $T>0$ and u,v are two solutions in $L^\infty((0,T),L^2(\mathbb{R}^N)) \cap L^\gamma((0,T),L^\rho(\mathbb{R}^N))$. Suppose $u \not\equiv v$ and set

$$\tau = \sup\{t \in [0,T];\ u \equiv v \text{ on } [0,t]\},$$

so that $0 \le \tau < T$. By translating in time, we may assume $\tau = 0$, thus there exists $t_n \downarrow 0$ such that $u(t_n) \ne v(t_n)$. By the previous estimates, $u=v$ on $[0,T]$ where T is sufficiently small. Contradiction.

The maximal interval is constructed as usual and the blowup alternative follows immediately from the fact that T depends on $\|\varphi\|_{L^2}$. □

Proof of Theorem 7.12. By Proposition 7.13 and using in particular the blowup alternative, we need only show the conservation of the L^2 norm. We cannot multiply the equation by u, so we need an approximation. We use a simple argument of Ozawa [29].

Fix $T > 0$ and consider $\psi \in \mathcal{S}(\mathbb{R}^N)$ and $f \in C^1([0,T], \mathcal{S}(\mathbb{R}^N))$. Set

$$v(t) = e^{it\Delta}\psi + i\int_0^t e^{i(t-s)\Delta} f(s)\, ds.$$

It follows that $v \in C^1([0,T], \mathcal{S}(\mathbb{R}^N))$ and that $iv_t + \Delta v = f$. Multiplying by u, we obtain

$$\|v(t)\|_{L^2}^2 - \|\psi\|_{L^2}^2 = 2\int_0^t (if(s), v(s))_{L^2} ds,$$

for all $0 \le t \le T$. Let now $0 < T < T_{\max}(\varphi)$. Since $\varphi \in L^2(\mathbb{R}^N)$ and $\lambda |u|^\alpha u \in L^{\gamma'}((0,T), L^{\rho'}(\mathbb{R}^N))$, we may approximate φ and $\lambda|u|^\alpha u$ in the appropriate space by smooth functions φ_n and f_n. If we denote by u_n the corresponding function, we deduce from Strichartz that

$$\|u_n - u\|_{L^\infty((0,T),L^2)} + \|u_n - u\|_{L^\gamma((0,T),L^\rho)} \underset{n\to\infty}{\longrightarrow} 0.$$

Note also that $(if_n(s), u_n(s))_{L^2} = (if_n(s), u_n(s))_{L^{\rho'},L^\rho}$; and so for a.a. $t \in (0,T)$

$$(if_n(s), u_n(s))_{L^2} \underset{n\to\infty}{\longrightarrow} (i\lambda|u|^\alpha u(s), u(s))_{L^{\rho'},L^\rho} = 0.$$

Using dominated convergence, we see that

$$\int_0^t (if_n(s), u_n(s))_{L^2} ds \underset{n\to\infty}{\longrightarrow} 0,$$

for all $0 \le t \le T$. The result follows. □

Remark 7.14. Uniqueness without auxiliary space when $\alpha \le 2$?

Theorem 7.15. *Suppose* $0 < \alpha < 4/(N-2)$ *(*$0 < \alpha < \infty$ *if* $N = 1,2$*) and set*

$$\rho = \alpha + 2, \quad \gamma = \frac{4(\alpha+2)}{N\alpha}, \tag{7.16}$$

so that (γ, ρ) *is an admissible pair. Given any* $\varphi \in H^1(\mathbb{R}^N)$*, there exist* $T = T(\|\varphi\|_{H^1}) > 0$ *and a unique solution* $u \in C([0,T], H^1(\mathbb{R}^N))$ *of* (7.8)*.* u *can be extended to a maximal existence interval* $[0, T_{\max})$ *and the following properties hold.*

(i) *Blowup alternative: either* $T_{\max} = \infty$ *or else* $T_{\max} < \infty$ *and* $\|u(t)\|_{H^1} \to \infty$ *as* $t \uparrow T_{\max}$.
(ii) $u \in L^q((0,T), W^{1,r}(\mathbb{R}^N))$ *for all* $T < \infty$ *and all admissible pair* (q,r).
(iii) *Uniqueness holds in* $L^\infty((0,T), H^1(\mathbb{R}^N))$.
(iv) $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ *and* $E(u(t) = E(\varphi)$ *for all* $t \ge 0$.
(v) u *depends continuously on* φ *is the following way. If* $T < T_{\max}(\varphi)$ *and* $\varphi_n \to \varphi$ *in* $H^1(\mathbb{R}^N)$*, then* $T_{\max}(\varphi_n) > T$ *for* n *large and* $u_n \to u$ *in* $C([0,T], H^1(\mathbb{R}^N))$ *as* $n \to \infty$.

Proof. For convenience we set $g(u) = \lambda |u|^\alpha u$. Note that $H^1(\mathbb{R}^N) \hookrightarrow L^\rho(\mathbb{R}^N)$. In particular, given any $T > 0$,

$$\|g(u)\|_{L^{\rho'}} = |\lambda| \|u\|_{L^\rho}^{\alpha+1} \le C \|u\|_{H^1}^\alpha \|u\|_{L^\rho},$$

so that, given any $T > 0$

$$\|g(u)\|_{L^\gamma((0,T),L^{\rho'})} \le C \|u\|_{L^\infty((0,T),H^1)}^\alpha \|u\|_{L^\gamma((0,T),L^\rho)};$$

and so

$$\|g(u)\|_{L^{\gamma'}((0,T),L^{\rho'})} \le C T^{\frac{\gamma-2}{\gamma}} \|u\|_{L^\infty((0,T),H^1)}^\alpha \|u\|_{L^\gamma((0,T),L^\rho)}.$$

Similarly,

$$\|g(u) - g(v)\|_{L^{\gamma'}((0,T),L^{\rho'})} \le \\ C T^{\frac{\gamma-2}{\gamma}} (\|u\|_{L^\infty((0,T),H^1)}^\alpha + \|v\|_{L^\infty((0,T),H^1)}^\alpha) \|u - v\|_{L^\gamma((0,T),L^\rho)}. \quad (7.17)$$

Moreover, using the inequality $|\nabla(|u|^\alpha u)| \le (\alpha+1)|u|^\alpha |\nabla u|$ a.e.,

$$\|g(u)\|_{L^{\gamma'}((0,T),W^{1,\rho'})} \le C T^{\frac{\gamma-2}{\gamma}} \|u\|_{L^\infty((0,T),H^1)}^\alpha \|u\|_{L^\gamma((0,T),W^{1,\rho})}. \quad (7.18)$$

We first prove uniqueness. Note that if $u \in L^\infty((0,T), H^1(\mathbb{R}^N))$, then $g(u) \in L^\infty((0,T), L^{\rho'}(\mathbb{R}^N)) \hookrightarrow L^\infty((0,T), H^{-1}(\mathbb{R}^N))$. Hence the equation makes sense. Next, let $u, v \in L^\infty((0,T), H^1(\mathbb{R}^N))$ be two solutions and assume (see above) that $\|u - v\|_{L^\gamma((0,\tau),L^\rho)} > 0$ for all $0 < \tau \le T$. It follows from Duhamel's formula, (7.17) and Strichartz estimate that, given $0 < \tau \le T$

$$\|u - v\|_{L^\gamma((0,\tau),L^\rho)} \le C \tau^{\frac{\gamma-2}{\gamma}} (\|u\|_{L^\infty((0,T),H^1)}^\alpha + \|v\|_{L^\infty((0,T),H^1)}^\alpha) \|u - v\|_{L^\gamma((0,\tau),L^\rho)}.$$

We conclude by choosing τ small.

We next prove local existence. By a fixed point argument in the set

$$E_T^M = \{u \in L^\gamma((0,T), W^{1,\rho};\ \|u\|_{L^\gamma((0,T),W^{1,\rho})} \le M\},$$

equipped with the distance

$$\mathrm{d}(u,v) = \|u - v\|_{L^\gamma((0,T),L^\rho)}.$$

The crux here is that (E_T^M, d) **is a complete metric space**. (This follows because $W^{1,\rho}$ is reflexive.) The rest is an easy consequence of (7.17)-(7.18). The existence time can be estimated in terms of $\|\varphi\|_{H^1}$. Once there is a fixed point in E_T^M, we see that $g(u) \in L^{\gamma'}((0,T), W^{1,\rho'}(\mathbb{R}^N))$. Strichartz now implies that $u \in C([0,T], H^1(\mathbb{R}^N)) \cap L^q((0,T), W^{1,r}(\mathbb{R}^N))$ for all admissible pair (q,r).

Maximal solution and blowup alternative. Standard by uniqueness and the fact that for the local existence part $T = T(\|\varphi\|_{H^1})$.

Conservation laws. The conservation of charge can be proved directly: the equation makes sense in H^{-1} so one can "multiply" by u. Conservation of energy needs a regularization argument. The best is to use the technique of Ozawa [29]. Same as for the conservation of charge, but more technical.

Continuous dependence. It is sufficient to prove continuous dependence on a small interval depending on $\|\varphi\|_{H^1}$. Then one repeats the argument. Let $\varphi_n \to \varphi$. For $T(\|\varphi\|_{H^1})$ small and n large, we see that $T_{\max}(\varphi_n) > T$, u_n is bounded in

$L^\infty((0,T),H^1(\mathbb{R}^N))$ and $u_n \to u$ in $L^\gamma((0,T),L^\rho) \cap L^\infty((0,T),L^2)$. By Gagliardo-Nirenberg, we obtain in particular that $u_n \to u$ in $L^\infty((0,T),L^\rho(\mathbb{R}^N))$. Apply ∇ to the equation and use Strichartz.

$$\|u_n - u\|_{L^\gamma((0,T),W^{1,\rho})} + \|u_n - u\|_{L^\infty((0,T),H^1)} \\ \le C\|\varphi_n - \varphi\|_{H^1} + C\|g(u_n) - g(u)\|_{L^{\gamma'}(0,T),W^{1,\rho'})}.$$

We write $\nabla g(u) = g'(u)\nabla u$ (beware, this is matrix) so that

$$|\nabla(g(u_n) - g(u)| \le C|g'(u_n)|\,|\nabla u_n - \nabla u| + |g'(u_n) - g'(u)|\,|\nabla u|$$

The first term is estimated as above, and by possibly choosing T smaller, we obtain

$$\|u_n - u\|_{L^\gamma((0,T),W^{1,\rho})} + \|u_n - u\|_{L^\infty((0,T),H^1)} \\ \le C\|\varphi_n - \varphi\|_{H^1} + C\|\,|g'(u_n) - g'(u)|\,|\nabla u|\,\|_{L^{\gamma'}(0,T),L^{\rho'})}.$$

The last term converges to 0 by convergence in $L^\infty(L^\rho)$. □

Proposition 7.16. *If either $\lambda \le 0$ or $\lambda > 0$ and $\alpha < 4/N$, then all solutions are global. If $\lambda > 0$ and $\alpha = 4/N$, then $T_{\max}(\varphi) = \infty$ is $\|\varphi\|_{L^2}$ is sufficiently small. If $\lambda > 0$ and $\alpha \ge 4/N$, then $T_{\max}(\varphi) = \infty$ is $\|\varphi\|_{H^1}$ is sufficiently small.*

Proof. By the blowup alternative, we need only show an a priori bound on $\|u(t)\|_{H^1}$. Note also that by conservation of charge, this means a bound on $\|\nabla u(t)\|_{L^2}$.

If $\lambda \le 0$, this follows from conservation of energy.

If $\lambda > 0$, this follows from conservation of energy and Gagliardo-Nirenberg's inequality

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \le C\|\nabla u\|_{L^2}^{\frac{N\alpha}{2}}\|u\|_{L^2}^{\frac{4-(N-2)\alpha}{2}}.$$

Direct if $\alpha \le 4/N$.

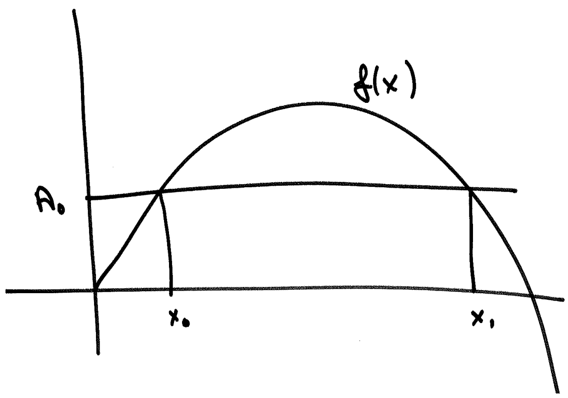
By a trapping region argument if $\alpha > 4/N$. Indeed, if $b(t) = \|u(t)\|_{H^1}^2$, then the conservation laws show that

$$b(t) \le A(\varphi) + Cb(t)^{\frac{\alpha+2}{2}},$$

where

$$A(\varphi) = 2E(\varphi) + \|\varphi\|_{L^2}^2 \le \|\varphi\|_{H^1}^2.$$

This means that $f(b(t)) \le A(\varphi)$, where $f(x) = x - Cx^{\frac{\alpha+2}{2}}$.



Choose $\|\varphi\|_{H^1}$ small enough so that $A(\varphi) \le A_0$ and $b(0) \le x_0$. Then we must have $b(t) \le x_0$ for all t. □

Proposition 7.17. *If $\lambda > 0$ and $\alpha \ge 4/N$, then some solutions blow up in finite time. More precisely, if $\varphi \in H^1(\mathbb{R}^N)$, $E(\varphi) < 0$ and $\int_{\mathbb{R}^N} |x|^2|\varphi(x)|^2 < \infty$, then $T_{\max}(\varphi) < \infty$.*

Proof. This follows from the calculation of the variance $\int_{\mathbb{R}^N} |x|^2|u(t,x)|^2$.

$$h(t) = \int_{\mathbb{R}^N} |x|^2|u(t,x)|^2\,dx,$$

for $0 \le t < T_{\max}(\varphi)$. It follows that

$$h''(t) = 8\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{4\lambda N\alpha}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}. \tag{7.19}$$

In fact, **the precise statement is as follows** : If $\varphi \in H^1(\mathbb{R}^N)$ and $|\cdot|\varphi \in L^2(\mathbb{R}^N)$, then

– The map $t \mapsto |\cdot|u(t)$ belongs to $C([0,T_{\max}), L^2(\mathbb{R}^N))$.

– $h \in C^2([0,T))$ and formula (7.19) holds.

[A rigorous proof goes as follows: approximate φ by initial values in $H^2(\mathbb{R}^N)$, so that the resulting solution is H^2. Then replace $|x|^2$ by a bounded function of x, for example $|x|^2e^{-\varepsilon|x|^2}$. Then obtain a formula like (7.19) (but more complicated). Then pass to the limit on the weight (let $\varepsilon \downarrow 0$) and show the conclusion still holds. Then pass to the limit on the initial value.]

Using conservation of energy, we deduce from (7.19) that

$$h''(t) = 16E(\varphi) - \frac{4\lambda(N\alpha-4)}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}. \tag{7.20}$$

One concludes as follows. Suppose $\alpha \ge 4/N$, $E(\varphi) < 0$ and $|\cdot|\varphi \in L^2(\mathbb{R}^N)$. It follows from (7.20) that $f''(t) \le 16E(\varphi) < 0$ for all $t \in [0,T_{\max})$. Since $h \ge 0$, $T_{\max}$ must be finite. In fact, one can be slightly more precise.

$$h(0) = \int_{\mathbb{R}^N} |x|^2|\varphi(x)|^2 =: A$$

$$h'(t) = 4\mathrm{Im}\int_{\mathbb{R}^N} \overline{u}x\cdot\nabla u,$$

so

$$h'(t) = 4\mathrm{Im}\int_{\mathbb{R}^N} \overline{\varphi}x\cdot\nabla\varphi =: B.$$

Therefore,

$$0 \le h(t) \le A + Bt + 8E(\varphi)t^2,$$

for all $0 \le t < T_{\max}$. This gives an explicit upper estimate of $T_{\max}$ in terms of A, B and $E(\varphi)$. □

Remark 7.18. As usual, the above argument **does not show** that $h(t) \to 0$ at the blowup time!

7.4. **Stability and instability of standing waves.** Suppose $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$). As we have seen, given any $\omega, \mu > 0$, there exists a ground state ψ of the equation

$$-\Delta u + \omega u = \mu |u|^\alpha u, \tag{7.21}$$

in $\mathbb{R}^N$. This means that ψ is a solution of (7.21), $\psi \not\equiv 0$ and that $E(\psi) \le E(u)$ for every nontrivial solution $u \in H^1(\mathbb{R}^N)$ of (7.21). We now study the stability of the resulting solution

$$u(t, x) = e^{i\omega t}\psi(x), \tag{7.22}$$

of (7.8). There is is simple result, which is not limited to ground states but which applies to any nontrivial solution of (7.21), when $\alpha = 4/N$.

Theorem 7.19. *Suppose $\alpha = 4/N$. If $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \not\equiv 0$ is a solution of* (7.21)*, then the resulting solution $u(t, x) = e^{i\omega t}\varphi(x)$ of* (7.8) *is unstable (by blowup) in the sense that an arbitrarily small perturbation of its initial value φ can make the perturbed solution blow up in finite time. More precisely, for every $\varepsilon > 0$ the solution of* (7.8) *with the initial value $(1 + \varepsilon)\varphi$ blows up in finite time.*

Proof. It follows from Pohožaev identity that $E(\varphi) = 0$. Therefore $E((1 + \varepsilon)\varphi) < 0$ if $\varepsilon > 0$. To apply the blowup result, we need only check that $| \cdot |\varphi \in L^2(\mathbb{R}^N)$. In fact, one shows that there exists $\eta > 0$ such that $e^{\eta|x|^2}\varphi \in L^2(\mathbb{R}^N)$. □

A similar result holds for $\alpha > 4/N$ in the case of the ground state.

Theorem 7.20. *Suppose $\alpha > 4/N$. If $\psi \in H^1(\mathbb{R}^N)$ is a ground state of* (7.21)*, then the resulting solution $u(t, x) = e^{i\omega t}\psi(x)$ of* (7.8) *is unstable (by blowup) in the sense that an arbitrarily small perturbation of its initial value ψ can make the perturbed solution blow up in finite time. More precisely, for every $\varepsilon > 0$ there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $\|\varphi - \psi\|_{H^1} \le \varepsilon$ and the solution of* (7.8) *with the initial value φ blows up in finite time.*

Proof. The proof is more involved than the proof of Theorem 7.19. We use an argument of [4], which is itself inspired from a potential well argument of Payne and Sattinger [30].

For $w \in H^1(\mathbb{R}^N)$, we let

$$Q(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{\lambda N \alpha}{2(\alpha + 2)} \int_{\mathbb{R}^N} |w|^{\alpha+2}.$$

We claim that there is a solution to the constrained minimization problem

$$\begin{cases} u \in H^1(\mathbb{R}^N), u \neq 0, Q(u) = 0, \\ \mathcal{A}(u) = \inf\{\mathcal{A}(w);\ w \in H^1(\mathbb{R}^N), w \neq 0, Q(w) = 0\}, \end{cases} \tag{7.23}$$

where the action $\mathcal{A}$ is defined as above by

$$\mathcal{A}(w) = E(w) + \frac{\omega}{2} \int_{\mathbb{R}^N} |w|^2.$$

This is proved by symmetrization, for example. Note that for $w \in S$, $\mathcal{A}(w) \approx \|w\|_{H^1}^2$. The crux is that (since $\alpha > 4/N$), S is bounded away from 0.

Next, we claim that every solution of (7.23) is a solution of (7.21). This follows from the Lagrange multiplier theorem and Pohožaev identity.

Next, we recall that the ground states of (7.23) are the solutions of (see Theorem 5.11)

$$\begin{cases} u \in H^1(\mathbb{R}^N), u \neq 0, F(u) = 0, \\ \mathcal{A}(u) = \inf\{\mathcal{A}(w);\ w \in H^1(\mathbb{R}^N), w \neq 0, F(w) = 0\}, \end{cases} \tag{7.24}$$

where

$$F(w) = \int_{\mathbb{R}^N} |\nabla w|^2 + \omega \int_{\mathbb{R}^N} |w|^2 - \lambda \int_{\mathbb{R}^N} |w|^{\alpha+2}.$$

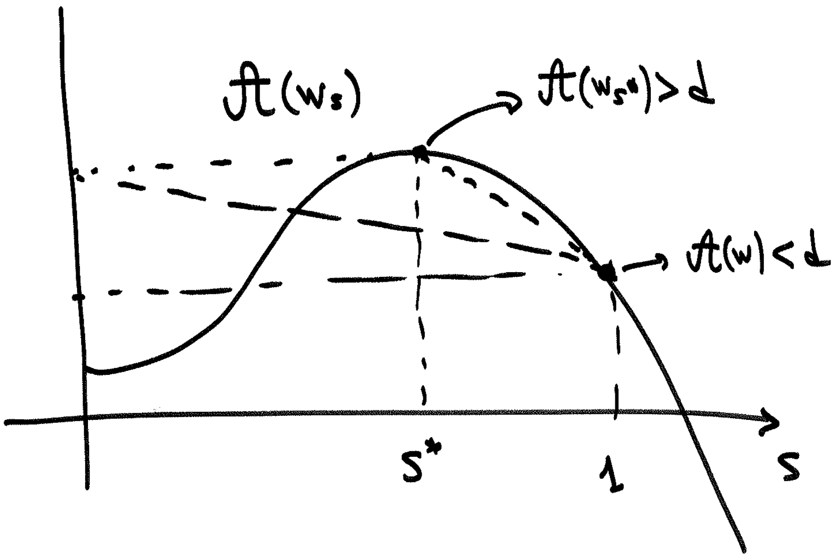
We now claim that the solutions of the problems (7.23) and (7.24) are the same. Indeed, they are all solutions of (7.21) and every solution u of (7.21) satisfies both $F(u) = 0$ and $Q(u) = 0$.

Let $d > 0$ be the infimum in (7.23) (or in (7.24)). Let $w \in H^1(\mathbb{R}^N)$ satisfy

$$\mathcal{A}(w) < d, \quad Q(w) < 0.$$

We claim that

$$Q(w) \le -[d - \mathcal{A}(w)].$$



Indeed, given $s > 0$ set $w_s(x) = s^{\frac{N}{2}} w(sx)$. It follows that

$$\mathcal{A}(w_s) = \frac{s^2}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{\omega}{2} \int_{\mathbb{R}^N} |w|^2 - \frac{\lambda s^{\frac{N\alpha}{2}}}{\alpha + 2} \int_{\mathbb{R}^N} |w|^{\alpha+2},$$

and

$$Q(w_s) = s \frac{d}{ds} \mathcal{A}(w_s). \tag{7.25}$$

Thus we see that there exists $s^\star > 0$ such that $\mathcal{A}(w_s)$ is increasing on $(0, s^\star)$ and decreasing and concave on $(s^\star, \infty)$. Since $Q(w) < 0$, we must have $s^\star < 1$. Since $\mathcal{A}(w_s)$ is concave on $(s^\star, 1)$, we see that

$$Q(w) = \frac{d}{ds} \mathcal{A}(w_s)_{|s=1} \le \frac{\mathcal{A}(w) - \mathcal{A}(w_{s^\star})}{1 - s^\star} \le \mathcal{A}(w) - \mathcal{A}(w_{s^\star}).$$

Moreover, $Q(w_{s^\star}) = 0$ by (7.25) , so that $\mathcal{A}(w_{s^\star}) \ge d$. This yields the desired estimate.

Finally, we conclude as follows. Let ψ be a ground state. In particular, $\mathcal{A}(\psi) = d$ and $Q(\psi) = d$. Thus if we let $\varphi = \psi_s$ with $s > 1$, we see that $Q(\varphi) < 0$ and

$\mathcal{A}(\varphi) < d$. In particular, if u is the corresponding solution of (7.8), then for all $0 < t < T_{\max}$ (by an obvious continuity argument)

$$Q(u(t)) \le -(d - \mathcal{A}(\varphi)) =: -\delta < 0.$$

On the other hand, as observed before, $|\cdot|\varphi \in L^2(\mathbb{R}^N)$. Since

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u(t,x)|^2\,dx = Q(u(t)),$$

we conclude as before that u blows up in finite time. The result follows, since $\psi_s \to \psi$ in $H^1(\mathbb{R}^N)$ as $s \to 1$. □

Remark 7.21. In general it is not known whether or not the standing waves corresponding to excited states are unstable. This is known only for radially symmetric standing waves, see Grillakis [14].

We now claim that if $\alpha < 4/N$ then the standing wave corresponding to the ground states are "stable". But then we must make clear the notion of stability. We cannot hope that if $\psi - \varphi$ is small, then $u(t)$ will remain close to $e^{i\omega t}\psi$ for all time. Indeed, given $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) = (1+\varepsilon)^{1/\alpha}\psi((1+\varepsilon)^{\frac{1}{2}}x),$$

and

$$u_\varepsilon(t,x) = e^{i\omega(1+\varepsilon)t}\varphi_\varepsilon(x).$$

One verifies easily that u_ε is the solution of (7.8) with initial value φ_ε. Furthermore, $\varphi_\varepsilon \to \psi$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \downarrow 0$, but one verifies easily that for every $\varepsilon > 0$,

$$\sup_{t\ge 0}\|u_\varepsilon(t,\cdot) - e^{i\omega t}\psi\|_{H^1} = \|\varphi_\varepsilon + \psi\|_{H^1} \underset{\varepsilon\to 0}{\longrightarrow} 2\|\psi\|_{H^1}.$$

Thus one must take into account a phase shift. So one might hope that if $\varphi - \psi$ is small, then

$$\inf_{\theta\in\mathbb{R}}\|u(t) - e^{i\theta}\psi\|_{H^1},$$

remains small. However, this is also impossible. Indeed, given $\varepsilon > 0$ and $y \in \mathbb{R}^N$ such that $|y| = 1$, let

$$\varphi_\varepsilon(x) = e^{i\varepsilon x\cdot y}\psi(x),$$

and

$$u_\varepsilon(t,x) = e^{i\varepsilon(x\cdot y - \varepsilon t)}e^{i\omega t}\psi(x - 2\varepsilon t y).$$

One verifies easily that u_ε is the solution of (7.8) with initial value φ_ε. Furthermore, $\varphi_\varepsilon \to \varphi$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \downarrow 0$, but one verifies easily that for every $\varepsilon > 0$,

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}\|u_\varepsilon(t) - e^{i\theta}\psi\|_{H^1} = 2\|\psi\|_{H^1}.$$

Thus we see that we must weaken further the notion of stability. We say that the solution $e^{i\omega t}\psi(x)$ is "orbitally stable" (in $H^1(\mathbb{R}^N)$) if

$$\sup_{t\ge 0}\inf_{y\in\mathbb{R}^N}\inf_{\theta\in\mathbb{R}}\|u(t) - e^{i\theta}\psi\|_{H^1},$$

is small provided $\|\varphi - \psi\|_{H^1}$ is small.

Theorem 7.22. *Suppose $\alpha < 4/N$ and let ψ be a ground state of* (7.21)*. It follows that the resulting solution $e^{i\omega t}\psi(x)$ of* (7.8) *is orbitally stable in $H^1(\mathbb{R}^N)$. In other words, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\varphi - \psi\|_{H^1} \le \delta$, then the resulting solution u of* (7.8) *is global and satisfies*

$$\sup_{t\ge 0} \inf_{y\in\mathbb{R}^N} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\psi\|_{H^1} \le \varepsilon. \tag{7.26}$$

Proof. We use the argument of [10]. Given $\eta > 0$, one first shows that the minimization problem

$$\begin{cases} u \in H^1(\mathbb{R}^N), \|u\|_{L^2} = \eta, \\ E(u) = \inf\{E(w);\, w \in H^1(\mathbb{R}^N), \|w\|_{L^2} = \eta\}, \end{cases} \tag{7.27}$$

has a solution. Estimates are obtained by Gagliardo-Nirenberg, and the infimum is negative. (let $\|w\|_{L^2} = \eta$ and compute $E(s^{\frac{N}{2}} w(s\cdot))$ for s small.)

One shows that a solution of (7.27) satisfies the equation $-\Delta u + \Lambda u = \lambda |u|^\alpha u$, for some $\Lambda = \Lambda(\eta) > 0$. This follows from The Lagrange multiplier theorem. The fact that $\Lambda < 0$ follows easily from the fact that $E(u) < 0$.

Next, it follows easily from Pohožaev's identity that all ground states have the same L^2 norm, which we call μ.

One then shows that ψ is a ground state if and only if ψ is a solution of (7.27) with $\eta = \mu$. This is a bit delicate, see [7].

Finally, one shows stability. We argue by contradiction and we assume there exist $\varphi_n \to \psi$ in $H^1(\mathbb{R}^N)$, $\varepsilon > 0$ and $(t_n)_{n\ge 1}$ such that

$$\inf_{y\in\mathbb{R}^N} \inf_{\theta\in\mathbb{R}} \|u_n(t_n) - e^{i\theta}\psi\|_{H^1} \ge \varepsilon. \tag{7.28}$$

Since $\varphi_n \to \psi$, it follows that $\|\varphi_n\|_{L^2} \to \|\psi\|_{L^2}$ and that $E(\varphi_n) \to E(\psi)$. Set now $w_n = u_n(t_n)$. By conservation of charge an energy, we deduce that

$$\|w_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \|\psi\|_{L^2}; \quad E(w_n) \underset{n\to\infty}{\longrightarrow} E(\psi).$$

Thus we see that $(w_n)_{n\ge 1}$ is a minimizing sequence for the problem (7.27). By the concentration-compactness principle of P.-L. Lions [25], there exists a sequence $(y_n)_{n\ge 1}$ and a solution v of (7.27) (i.e. a ground state) such that $w_n(\cdot - y_n) \to v$ in $H^1(\mathbb{R}^N)$. On the other hand, by uniqueness of ground states up to translation and phase shift, this means that there exist $y \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that $w_n(\cdot - y_n) \to e^{i\theta} v(\cdot - y)$ in $H^1(\mathbb{R}^N)$. This contradicts (7.28). □

Remark 7.23. If one works with radially symmetric solutions, then one can remove the space translations in the stability property. Moreover the proof can be simplified if $N \ge 2$ by using a compactness property of radially symmetric functions.

Remark 7.24. It seems that nothing is known about the stability of excited states.

Remark 7.25. There are other techniques to study the stability, the most general of which is the method of Grillakis, Shatah and Strauss [15, 16].

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