What is a dynamical system?
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

**Popular answer:** a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?

Popular answer: a math subject that produces beautiful pictures!
What is a dynamical system?
What is a dynamical system?

Informal answer: a system that evolves with time
What is a dynamical system?

Informal answer: a system that evolves with time

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.
What is a dynamical system?

Informal answer: a system that evolves with time

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.
What is a dynamical system?

**Informal answer: a system that evolves with time**

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.
What is a dynamical system?

Informal answer: a system that evolves with time

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.
What is a dynamical system?

**Informal answer:** a system that evolves with time

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.

---

**finance**

**fluids**

**weather prediction**

**material science**
What is a dynamical system?
Informal answer: a system that evolves with time

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.
The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space.

finance
fluids
weather prediction

material science
population dynamics
chemical reactions
What is a dynamical system?
What is a dynamical system?

Formal answer: a math definition
What is a dynamical system?

**Formal answer: a math definition**

A dynamical system is a tuple $(T, M, \Phi)$

- $T$: monoid (time)
- $M$: set (state space)
- $\Phi$: map (evolution function)

satisfying the two following properties

\[
\begin{cases}
\Phi(0, x) = x \\
\Phi(t_2, \Phi(t_1, x)) = \Phi(t_1 + t_2, x)
\end{cases}
\]

\[
\forall \, x \in M \text{ and } \forall \, t_1, t_2 \in T
\]
Examples

1. Discrete dynamical systems
Examples

1. Discrete dynamical systems

\[ f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in \left[ \frac{1}{2}, 1 \right]
\end{cases} \]
Examples

I. Discrete dynamical systems

\[ f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in [\frac{1}{2}, 1] 
\end{cases} \]

\[ T = \mathbb{N} \text{ (discrete time)} \]
\[ M = [0, 1] \text{ (state space)} \]
\[ \Phi : T \times M \to M \\
(n, x) \mapsto \Phi(n, x) = f^n(x) \]
Examples

1. Discrete dynamical systems

\[ f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in [\frac{1}{2}, 1] 
\end{cases} \]

\[ T = \mathbb{N} \text{ (discrete time)} \]

\[ M = [0, 1] \text{ (state space)} \]

\[ \Phi : T \times M \rightarrow M \]

\[ (n, x) \mapsto \Phi(n, x) = f^n(x) \]
Examples

1. Discrete dynamical systems

\[ f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in \left[ \frac{1}{2}, 1 \right]
\end{cases} \]

The dynamics of the tent map is chaotic!
Examples

2. Continuous dynamical systems: ODEs
Examples

2. Continuous dynamical systems: ODEs

\[
\begin{aligned}
\text{IVP} & \left\{ \begin{array}{l}
\frac{dx}{dt} = f(x) \\
x(0) = x_0
\end{array} \right. \\
[f \in C^1(\mathbb{R}^n)]
\end{aligned}
\]

\[\Phi(t, x_0) : \text{solution of the (IVP)}\]
Examples

2. Continuous dynamical systems: ODEs

\[ \begin{align*}
\text{(IVP)} & \quad \begin{cases} 
\frac{dx}{dt} = f(x) \\
x(0) = x_0
\end{cases} \quad [f \in C^1(\mathbb{R}^n)] \\
\Phi(t, x_0) & \text{: solution of the (IVP)}
\end{align*} \]

\[ T = \mathbb{R} \quad \text{(continuous time)} \]

\[ M = \mathbb{R}^n \quad \text{(state space)} \]

\[ \Phi : T \times M \to M \]

\[ (t, x_0) \mapsto \Phi(t, x_0) \]
Examples

2. Continuous dynamical systems: ODEs

\[ \begin{align*}
\text{(IVP)} \quad & \begin{cases}
\frac{dx}{dt} = f(x) \\
x(0) = x_0
\end{cases} \\
& [f \in C^1(\mathbb{R}^n)]
\end{align*} \]

\[ \Phi(t, x_0) : \text{solution of the (IVP)} \]

\[ T = \mathbb{R} \quad \text{(continuous time)} \]

\[ M = \mathbb{R}^n \quad \text{(state space)} \]

\[ \Phi : T \times M \to M \]

\[ (t, x_0) \mapsto \Phi(t, x_0) \]

\[ \frac{dx}{dt} = \sigma(y - x) \]

\[ \frac{dy}{dt} = rx - y - xz \]

\[ \frac{dz}{dt} = xy - bz \]

Lorenz equations
Examples

3. Infinite dimensional dynamical systems

(a) Partial differential equations
Examples

3. Infinite dimensional dynamical systems

(a) Partial differential equations

**Cahn-Hilliard equation**

\[ \frac{\partial u}{\partial t} - \Delta \left( -\nu \Delta u - u + u^3 \right) = 0 \]

\[ \Omega \subset \mathbb{R}^n, \; n = 1, 2, 3 \]
Examples

3. Infinite dimensional dynamical systems

(a) Partial differential equations

Cahn-Hilliard equation

\[
\frac{\partial u}{\partial t} - \Delta \left( -\nu \Delta u - u + u^3 \right) = 0
\]

\[\Omega \subset \mathbb{R}^n, \ n = 1, 2, 3\]

\[T = [0, \infty) \ (\text{continuous time})\]

\[M = L^2(\Omega) \ (\text{infinite dimensional state space})\]

\[\Phi : T \times M \rightarrow M\]

\[(t, x_0) \mapsto \Phi(t, x_0) \ (\text{semigroup})\]
Examples

3. Infinite dimensional dynamical systems

(a) Partial differential equations

Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} - \Delta \left( -\nu \Delta u - u + u^3 \right) = 0$$

$$\Omega \subset \mathbb{R}^n, \ n = 1, 2, 3$$

$$T = [0, \infty) \ (\text{continuous time})$$

$$M = L^2(\Omega) \ (\text{infinite dimensional state space})$$

$$\Phi : T \times M \to M$$

$$(t, x_0) \mapsto \Phi(t, x_0) \ (\text{semigroup})$$
Examples

3. Infinite dimensional dynamical systems

(a) Partial differential equations

\begin{align*}
\frac{\partial u}{\partial t} - \Delta \left( -\nu \Delta u - u + u^3 \right) &= 0 \\
\Omega &\subset \mathbb{R}^n, \ n = 1, 2, 3 \\
T &= [0, \infty) \ (\text{continuous time}) \\
M &= L^2(\Omega) \ (\text{infinite dimensional state space}) \\
\Phi : T \times M &\to M \\
(t, x_0) \mapsto \Phi(t, x_0) \ (\text{semigroup})
\end{align*}
Examples

3. Infinite dimensional dynamical systems

(b) Delay differential equations
Examples

3. Infinite dimensional dynamical systems

(b) Delay differential equations

Wright’s equation

\[ y'(t) = -\alpha y(t - 1)[1 + y(t)] \]
Examples

3. Infinite dimensional dynamical systems

(b) Delay differential equations

Wright’s equation

\[ y'(t) = -\alpha y(t - 1)[1 + y(t)] \]

\[ T = [0, \infty) \text{ (continuous time)} \]

\[ M = C[-1, 0] \text{ (infinite dimensional state space)} \]

\[ \Phi : T \times M \rightarrow M \]
\[ (t, x_0) \mapsto \Phi(t, x_0) \text{ (semigroup)} \]
Examples

3. Infinite dimensional dynamical systems

(b) Delay differential equations

\[ y_0(t) = -(t + 0.8)^4 \]

\[ y(t) \]

\[ y'(t) = -\frac{12}{5} y(t - 1)[1 + y(t)] \]

\[ \infty \text{-dimensional dynamical system} \]

Phase Space: \( C[-1, 0] \)
What kind of solutions are we interested in?
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

*Henri Poincaré*

**Compact invariant sets**

Exploit smoothness, boundedness and low dimensionality.
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré

Compact invariant sets
Exploit smoothness, boundedness and low dimensionality.

- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.
- Global attractors.
Equilibrium solutions
Equilibrium solutions

ODEs
\[ \frac{dx}{dt} = f(x) \]

\[ \mathcal{E} = \{ x \in \mathbb{R}^n \mid f(x) = 0 \} \]
Equilibrium solutions

\textbf{ODEs} \quad \frac{dx}{dt} = f(x) \quad \textbf{PDEs} \quad \frac{\partial u}{\partial t} = E(u)

\mathcal{E} = \{x \in \mathbb{R}^n \mid f(x) = 0\} \quad \mathcal{E} = \{u \in L^2(\Omega) \mid E(u) = 0\}
Equilibrium solutions

**ODEs** \[ \frac{dx}{dt} = f(x) \]

\[ \mathcal{E} = \{ x \in \mathbb{R}^n \mid f(x) = 0 \} \]

**PDEs** \[ \frac{\partial u}{\partial t} = E(u) \]

\[ \mathcal{E} = \{ u \in L^2(\Omega) \mid E(u) = 0 \} \]
Periodic solutions
Periodic solutions

**Discrete dynamical system**

\[ x_k = f^k(x_0) = x_0 \]
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]

\( x_0 \in \mathbb{R}^n \)
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]

\[ f \rightarrow \bullet x_1 \]

\[ \bullet x_0 \in \mathbb{R}^n \]
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]

\( x_0 \in \mathbb{R}^n \)
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]
Periodic solutions

Discrete dynamical system

\[ x_k = f^k(x_0) = x_0 \]
Periodic solutions

**Discrete dynamical system**

\[ x_k = f^k(x_0) = x_0 \]

**ODEs**

\[ \frac{dx}{dt} = f(x) \]

\[ x(t + T) = x(t) \in \mathbb{R}^n \]

\[ \forall \ t \in \mathbb{R} \]
Periodic solutions

\[ \frac{\partial u}{\partial t} = E(u) \]

\[ u(t + T, \cdot) = u(t, \cdot) \in L^2(\Omega) \]

\[ \forall t \in \mathbb{R} \]

\[ u_0 = u(0, \cdot) \in L^2(\Omega) \]
Periodic solutions

**PDEs** \[
\frac{\partial u}{\partial t} = E(u)
\]
\[u(t + T, \cdot) = u(t, \cdot) \in L^2(\Omega)\]
\[\forall \ t \in \mathbb{R}\]

\[u_0 = u(0, \cdot) \in L^2(\Omega)\]

**Delay equations** \[
\frac{dy}{dt} = \mathcal{F}(y(t), y(t - \tau))
\]
\[y(t + T) = y(t) \in \mathbb{R}\]
\[\forall \ t \in \mathbb{R}\]

\[x_0 = x_0(t) \in C[-\tau, 0]\]
Connecting orbits

**ODEs**

\[ \frac{dx}{dt} = f(x) \]

\[ \lim_{t \to \pm \infty} x(t) = x^\pm \in \mathbb{R}^n \]

- **Homoclinic orbit**: \( x^+ = x^- \)
- **Heteroclinic orbit**: \( x^+ \neq x^- \)
Chaos
(for discrete dynamical systems)
Chaos
(for discrete dynamical systems)

Consider a metric space \((M, d)\).

**Definition:** \(f : M \rightarrow M\) has *sensitive dependence on initial conditions* if there exists \(\delta > 0\) such that, for any \(x \in M\) and any neighborhood \(N\) of \(x\), there exists \(y \in N\) and \(n \geq 0\) such that \(d \left[ f^n(x), f^n(y) \right] > \delta\).
Chaos
(for discrete dynamical systems)

Consider a metric space \((M, d)\).

**Definition:** \(f : M \rightarrow M\) has **sensitive dependence on initial conditions** if there exists \(\delta > 0\) such that, for any \(x \in M\) and any neighborhood \(N\) of \(x\), there exists \(y \in N\) and \(n \geq 0\) such that \(d[f^n(x), f^n(y)] > \delta\).

**Definition:** \(f : M \rightarrow M\) is said to be **topologically transitive** if for any pair of open sets \(U, V \subset M\) there exists \(k > 0\) such that \(f^k(U) \cap V \neq \emptyset\).
Chaos
(for discrete dynamical systems)

Consider a metric space \((M, d)\).

**Definition:** \(f : M \rightarrow M\) has **sensitive dependence on initial conditions** if there exists \(\delta > 0\) such that, for any \(x \in M\) and any neighborhood \(N\) of \(x\), there exists \(y \in N\) and \(n \geq 0\) such that \(d[f^n(x), f^n(y)] > \delta\).

**Definition:** \(f : M \rightarrow M\) is said to be **topologically transitive** if for any pair of open sets \(U, V \subset M\) there exists \(k > 0\) such that \(f^k(U) \cap V \neq \emptyset\).

**Definition:** \(f : M \rightarrow M\) is said to be **chaotic** on \(M\) if

1. \(f\) has sensitive dependence on initial conditions.
2. \(f\) is topologically transitive.
3. The periodic points of \(f\) are dense in \(M\).
A model for chaos: symbolic dynamics
A model for chaos: symbolic dynamics

Consider the sequence space on two symbols

\[ \Sigma_2 = \{ s = (s_0s_1s_2\cdots) \mid s_j = 0 \text{ or } 1 \} \]

with a metric \( d : \Sigma_2 \times \Sigma_2 \rightarrow [0, \infty) \) defined by

\[ d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \].
A model for chaos: symbolic dynamics

Consider the sequence space on two symbols

$$\Sigma_2 = \{ s = (s_0 s_1 s_2 \cdots) \mid s_j = 0 \text{ or } 1 \}$$

with a metric $d : \Sigma_2 \times \Sigma_2 \to [0, \infty)$ defined by

$$d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$ 

**Theorem:** The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ given by

$$\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$$

is chaotic.
A comparison tool: topological conjugacy

**Definition:** Let \( f : X \to X \) and \( g : Y \to Y \) be two maps. \( f \) and \( g \) are said to be *topologically conjugate* if there exists a homeomorphism \( h : X \to Y \) such that

\[
h \circ f = g \circ h.
\]
Definition: Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two maps. $f$ and $g$ are said to be \textit{topologically conjugate} if there exists a homeomorphism $h : X \rightarrow Y$ such that $h \circ f = g \circ h$. 

A comparison tool: topological conjugacy
A comparison tool: topological conjugacy

Definition: Let \( f : X \to X \) and \( g : Y \to Y \) be two maps. \( f \) and \( g \) are said to be topologically conjugate if there exists a homeomorphism \( h : X \to Y \) such that

\[
h \circ f = g \circ h.
\]

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics.
**Theorem:** The tent map $f : [0, 1] \rightarrow [0, 1]$ is chaotic.

$$f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in [\frac{1}{2}, 1] 
\end{cases}$$
Theorem: The tent map $f : [0, 1] \to [0, 1]$ is chaotic.

$$f(x) = \begin{cases} 2x, & \text{for } x \in [0, \frac{1}{2}) \\ 2(1-x), & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$I_0 = [0, \frac{1}{2})$ \hspace{1cm} $I_1 = [\frac{1}{2}, 1]$

Proof. Define $S : [0, 1] \to \Sigma_2$ by $S(x) = (s_0s_1s_2\cdots)$ where $s_i = 0$ if $f^i(x) \in I_0$ and $s_i = 1$ if $f^i(x) \in I_1$. 

Monday, June 27, 2011
Theorem: The tent map \( f : [0, 1] \rightarrow [0, 1] \) is chaotic.

\[
f(x) = \begin{cases} 
2x, & \text{for } x \in [0, \frac{1}{2}) \\
2(1 - x), & \text{for } x \in [\frac{1}{2}, 1]
\end{cases}
\]

\[
I_0 = [0, \frac{1}{2}) \quad I_1 = [\frac{1}{2}, 1]
\]

Proof. Define \( S : [0, 1] \rightarrow \Sigma_2 \) by \( S(x) = (s_0 s_1 s_2 \cdots) \)
where \( s_i = 0 \) if \( f^i(x) \in I_0 \) and \( s_i = 1 \) if \( f^i(x) \in I_1 \).

\( S : \text{homeomorphism} \)

\[
S \circ f = \sigma \circ S
\]

\[
[0, 1] \xrightarrow{f} [0, 1] \xrightarrow{S} \Sigma_2 \xrightarrow{\sigma} \Sigma_2
\]

Monday, June 27, 2011
Theorem: The tent map $f : [0, 1] \rightarrow [0, 1]$ is chaotic.
In practice, how to study a dynamical system?
In practice, how to study a dynamical system?

A standard approach is to get insight from numerical simulations to formulate new conjectures, and then attempt to prove the conjectures using pure mathematical techniques only. As one shall argue in this course, this strong dichotomy need not exist in the context of dynamical systems, as the strength of numerical analysis, functional analysis and even topology, can be combined to prove, in a rigorous mathematical sense, the existence of equilibria, periodic solutions, connecting orbits and even chaotic dynamics.
In practice, how to study a dynamical system?

A standard approach is to get insight from numerical simulations to formulate new conjectures, and then attempt to prove the conjectures using pure mathematical techniques only. As one shall argue in this course, this strong dichotomy need not exist in the context of dynamical systems, as the strength of numerical analysis, functional analysis and even topology, can be combined to prove, in a rigorous mathematical sense, the existence of equilibria, periodic solutions, connecting orbits and even chaotic dynamics.

Rigorous computations

The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré

**Compact invariant sets**

Exploit smoothness, boundedness and low dimensionality.

- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

*Henri Poincaré*

**Compact invariant sets**

Exploit smoothness, boundedness and low dimensionality.

- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.

\[ \mathcal{F}(x) = 0 \]
Ingredients of rigorous computations

**Numerical analysis.**
- Collocation, Galerkin and spectral methods.
- Continuation algorithms.
- Interval arithmetic, FFT.

**Analysis & Functional analysis.**
- Fixed point theorems.
- Analytic estimates.
- Parameterization of invariant manifolds.
Ingredients of rigorous computations

**Numerical analysis.**
- Collocation, Galerkin and spectral methods.
- Continuation algorithms.
- Interval arithmetic, FFT.

**Analysis & Functional analysis.**
- Fixed point theorems.
- Analytic estimates.
- Parameterization of invariant manifolds.

\[ \mathcal{F}(x) = 0 \iff T(x) = x \]
Let \((B, \| \cdot \|_B)\) be a Banach space, that is a complete normed vector space. Assume the existence of a contraction mapping \(T\) on \(B\), that is a mapping such that

1. \(T(B) \subset B\);
2. there exists \(0 < \kappa < 1\) such that, for every \(x, y \in B\),

\[
\|T(x) - T(y)\|_B \leq \kappa \|x - y\|_B.
\]

Then there exists a unique \(x^* \in B\) such that \(T(x^*) = x^*\).
Banach Fixed Point Theorem

Let \((B, \| \cdot \|_B)\) be a Banach space, that is a complete normed vector space. Assume the existence of a contraction mapping \(T\) on \(B\), that is a mapping such that

1. \(T(B) \subseteq B\);
2. there exists \(0 \leq \kappa < 1\) such that, for every \(x, y \in B\),

\[
\|T(x) - T(y)\|_B \leq \kappa \|x - y\|_B.
\]

Then there exists a unique \(x^* \in B\) such that \(T(x^*) = x^*\).
Banach Fixed Point Theorem

Let \((B, \| \cdot \|_B)\) be a Banach space, that is a complete normed vector space. Assume the existence of a contraction mapping \(T\) on \(B\), that is a mapping such that

1. \(T(B) \subset B\);

2. there exists \(0 < \kappa < 1\) such that, for every \(x, y \in B\),

\[
\| T(x) - T(y) \|_B \leq \kappa \| x - y \|_B.
\]

Then there exists a unique \(x^* \in B\) such that \(T(x^*) = x^*\).

\(\bigcirc\) can be verified by strict inequalities

\(\Rightarrow\) rigorous computations are possible !
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $\mathcal{F}(x) = 0$ in $X$. 
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $\mathcal{F}(x) = 0$ in $X$.
3. Construct (possibly with the help of the computer) $A \approx D\mathcal{F}(\bar{x})^{-1}$
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $\mathcal{F}(x) = 0$ in $X$.
3. Construct (possibly with the help of the computer) $A \approx D\mathcal{F}(\bar{x})^{-1}$
4. Verify that $A$ is injective
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $\mathcal{F}(x) = 0$ in $X$.
3. Construct (possibly with the help of the computer) $A \approx D\mathcal{F}(\bar{x})^{-1}$
4. Verify that $A$ is injective
5. Define $T(x) = x - A\mathcal{F}(x)$ so that $T(x) = x \iff \mathcal{F}(x) = 0$. 

Monday, June 27, 2011
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $F(x) = 0$ in $X$.
3. Construct (possibly with the help of the computer) $A \approx D\!F(\bar{x})^{-1}$
4. Verify that $A$ is injective
5. Define $T(x) = x - A\!F(x)$ so that $T(x) = x \iff F(x) = 0$.
6. Consider a closed ball $B_{\bar{x}}(r) \subset X$ of radius $r > 0$ centered at $\bar{x}$
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

1. Consider $X$ a Banach space
2. Assume we computed a numerical approximation $\bar{x}$ of $\mathcal{F}(x) = 0$ in $X$.
3. Construct (possibly with the help of the computer) $A \approx D\mathcal{F}(\bar{x})^{-1}$
4. Verify that $A$ is injective
5. Define $T(x) = x - A\mathcal{F}(x)$ so that $T(x) = x \iff \mathcal{F}(x) = 0$.
6. Consider a closed ball $B_{\bar{x}}(r) \subset X$ of radius $r > 0$ centered at $\bar{x}$
7. Solve for $r > 0$ so that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping.