Rigorous computations for the study of dynamical systems

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In practice, how to study a dynamical system?
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A standard approach is to get insight from numerical simulations to formulate new conjectures, and then attempt to prove the conjectures using pure mathematical techniques only. As one shall argue in this course, this strong dichotomy need not exist in the context of dynamical systems, as the strength of numerical analysis, functional analysis and even topology, can be combined to prove, in a rigorous mathematical sense, the existence of equilibria, time-periodic solutions, connecting orbits and even chaotic dynamics.
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Rigorous computations

The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.
What kind of solutions are we interested in?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré

Compact invariant sets
Exploit smoothness, boundedness and low dimensionality.

- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.
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\[ \mathcal{F}(x) = 0 \]
Tools used in rigorous computations

Numerical analysis.
• Fourier expansions.
• Galerkin approximations.
• Collocation methods.
• Continuation algorithms.
• Interval arithmetic.
• Fast Fourier Transform (FFT).

Analysis & Functional analysis.
• Fixed point theorems.
• Analytic estimates.
• Parameterization of invariant manifolds.
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\[ \mathcal{F}(x) = 0 \iff T(x) = x \]
Interval arithmetic

The basic operations of interval arithmetic are, for two intervals \([a, b]\) and \([c, d]\) that are subsets of the real line \((-\infty, \infty)\),

\[
\begin{align*}
[a,b] + [c,d] &= [a + c, b + d] \\
[a,b] - [c,d] &= [a - d, b - c] \\
[a,b] \times [c,d] &= [\min (ac, ad, bc, bd), \max (ac, ad, bc, bd)] \\
[a,b] / [c,d] &= [\min (a/c, a/d, b/c, b/d), \max (a/c, a/d, b/c, b/d)]
\end{align*}
\]
Let $(B, \| \cdot \|_B)$ be a Banach space, that is a complete normed vector space. Assume the existence of a contraction mapping $T$ on $B$, that is a mapping such that

1. $T(B) \subset B$;
2. there exists $0 < \kappa < 1$ such that, for every $x, y \in B$, 
   \begin{equation*}
   \|T(x) - T(y)\|_B \leq \kappa \|x - y\|_B.
   \end{equation*}

Then there exists a unique $x^* \in B$ such that $T(x^*) = x^*$. 

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\(\langle\rightarrow\rangle\) can be verified using interval arithmetic!
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6. Consider a closed ball $B_{\bar{x}}(r) \subset X$ of radius $r > 0$ centered at $\bar{x}$
7. Solve for $r > 0$ so that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping.
Examples

• One dimensional example
• Infinite dimensional example