Semilinear evolution PDEs: special solutions, initial value problem, and stability

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Abstract:

Among the many equations that arise in applied mathematics, a number of them are semilinear evolution PDEs. Of course, each of these equations has its own particularities, but general methods have been developed with a wide range of potential applications. The aim of this course is to present a selection of basic results and techniques that anyone studying a semilinear evolution PDE is likely to find useful. For the sake of simplicity, we will mostly consider two model equations: the semilinear heat (NLH) and the Schrödinger (NLS) equations. However, the methods that we will present are useful to the analysis of other equations. We will indicate possible extensions and references.

Program:

1. *Special solutions.* Certain equations possess special solutions. Stationary solutions are one example but, depending on the invariants of the equation, other solutions may exist, like standing waves, traveling waves and self-similar solutions. Equations (NLH) or (NLS) then reduce to a semilinear elliptic equation. In one space dimension or under symmetry assumptions, the existence of solutions can be studied by elementary ODE techniques (shooting arguments). In more general situations, one can often apply variational arguments (global minimization, constrained minimization, mountain pass theorem).

2. *Initial value problem.* The existence of local solutions for (NLH) follows from the simplest fixed-point argument using either an explicit kernel (in the whole space) or the theory of semigroups (in a bounded domain). Certain care must be taken, though, on such issues as uniqueness and the blowup alternative. For (NLS), local existence also follows from a simple fixed-point argument, provided one makes use of the Strichartz estimates. We will also establish simple sufficient conditions for global existence and for finite time blowup.

3. *Stability of special solutions.* It may be useful to decide whether or not a given special solution is stable (in an appropriate sense) with respect to small perturbations of its initial value. Indeed, an unstable solution is unlikely to be seen, either experimentally or numerically. For (NLH), if the linearized operator at a given stationary solution has positive first eigenvalue, then the solution is stable. If the first eigenvalue is negative, then the stationary solution is unstable. In general, there are local stable and unstable manifolds. For (NLS), the stability of standing waves cannot be decided by those arguments because all eigenvalues of the linearized operator are purely imaginary. This corresponds to center manifolds, for which other techniques are required. We will give sufficient conditions for stability and instability, based on dynamical systems arguments and on variational (invariant sets and constrained minimization) arguments.

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