An introduction to optimal control problem

The use of Pontryagin maximum principle

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ERC NUMERIWAVES – Course

Generalities

The aim of this course is to give basis to solve analytically or numerically optimal control problems.
In full generality, we consider a system governed by the dynamic:

$$\dot{x} = f(x, u) . \tag{1}$$

with $x \in \mathbb{R}^n$ is the state variable and $u \in \mathbb{R}^m$ is the control variable. The control problems is:

Problem (Control problem)

Given $T > 0$, $x^i, x^f \in \mathbb{R}^n$ does it exists $u : [0, T] \to \mathbb{R}^m$ such that systems (1) *steers x^i to x^f in time T.*

For optimal control problem, we consider a cost function:

$$J(x, u) = \int_0^T f_0(x(t), u(t))\, \mathrm{d}t + g(x(T), u(T))$$

and the aim is to find a control u which realize the control problem and minimize J.

References (books)

- E. Trélat Contrôle optimal : théorie et applications
- J.-M. Coron Control and Nonlinearity
- E. B. Lee and L. Markus Foundations of optimal control theory
- A. A. Agrachev and Y. L. Sachkov Control theory from the geometric viewpoint
- L. S. Pontryagin The maximum principle in optimal control
- A. D. Ioffe and V. M. Tihomirov Theory of extremal problems
- J.-L. Lions Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués
- M. Tucsnak and G. Weiss Observation and control for operator semi-groups
- H. O. Fattorini Infinite-dimensional optimization and control theory

1. Controllability results
   - Linear control problems
   - Nonlinear control problems
2. Optimal control problems
3. Numerical methods

Linear control problems I

Autonomous systems

We consider the system:

$$\dot{x} = Ax + Bu \qquad x(0) = x^i \,. \tag{2}$$

with $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n,m}(\mathbb{R})$.
Proving the controllability of the system (2) is equivalent as proving the observability of the adjoint system:

$$\dot{\varphi} = -A^*\varphi \qquad \varphi(T) = \varphi^f \,, \tag{3}$$

with the observation operator B^*.
That is to say, proving that there exists a constant $C > 0$ such that:

$$\|\varphi^f\|^2_{\mathbb{R}^n} \leqslant C \int_0^T \|B^*\varphi(t)\|^2_{\mathbb{R}^m} \,\mathrm{d}t \qquad (\varphi^f \in \mathbb{R}^n) \,.$$

Linear control problems II

Autonomous systems

Theorem (Kalman rank condition)

If

$$\operatorname{rank}\left[B\ AB\ \ldots\ A^{n-1}B\right] = n\,,$$

then for every $T >$ *and every* $x^i, x^f \in \mathbb{R}^n$*, there exists* $u \in L^2(0,T;\mathbb{R}^m)$ *such that system* (2) *steers* x^i *to* x^f *in time* T*. In addition, the control of minimal* L^2- *norm is given by:*

$$u(t) = B^*\varphi(t)$$

with φ *the solution of the adjoint problem* (3) *with final condition* $\hat{\varphi}^f$ *and where* $\hat{\varphi}^f$ *realise the minimum of:*

$$J(\varphi^f) = \frac{1}{2}\int_0^T \|B^*\varphi(t)\|^2_{\mathbb{R}^m}\,\mathrm{d}t + \langle x^i, \varphi(0)\rangle_{\mathbb{R}^n} - \langle x^f, \varphi^f\rangle_{\mathbb{R}^n}$$

(where here φ *is the solution of the adjoint problem* (3) *with final condition* φ^f*).*

Linear control problems III

Autonomous systems

Example

$$\ddot{x} = -x + u$$

Linear control problems

Non-autonomous systems

We consider here the system:

$$\dot{x} = A(t)x + B(t)u \tag{4}$$

and we assume that A and B are smooth.

Theorem (Kalman rank condition for non-autonomous linear systems)

Let us define for every $i \in \mathbb{N}$ the matrices $B_i \in \mathcal{M}_{n,m}(\mathbb{R})$ by:

$$B_0(t) = B(t)\,, \quad B_i(t) = \dot{B}_{i-1}(t) - A(t)B_{i-1}(t) \qquad (t \in [0, T])\,.$$

If there exists $\bar{t} \in [0, T]$ so that:

$$\mathrm{Span}\left\{B_i(\bar{t})u\,,\ u \in \mathbb{R}^m\,,\ i \in \mathbb{N}\right\} = \mathbb{R}^n$$

then the system (4) is controllable.

Nonlinear control problems I

Linear test

We consider here the general system

$$\dot{x} = f(x, u)\,. \tag{5}$$

Let assume that $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$. Let us consider a trajectory $(\bar{x}, \bar{u})$, i.e. $\dot{\bar{x}} = f(\bar{x}, \bar{u})$, the linearizing along this trajectory leads to the linear system:

$$\dot{x} = \partial_x f(\bar{x}, \bar{u})\, x + \partial_u f(\bar{x}, \bar{u})\, u\,. \tag{5'}$$

Nonlinear control problems II

Linear test

Theorem

If system (5') *is controllable, then the system* (5) *is* locally controllable along the trajectory $(\bar{x}, \bar{u})$.

That is to say:
For every $\varepsilon > 0$*, there exists* $\eta > 0$ *such that for every* $x^i, x^f \in \mathbb{R}^n$ *with*

$$\|\bar{x}(0) - x^i\|_{\mathbb{R}^n}, \|\bar{x}(T) - x^f\|_{\mathbb{R}^n} < \eta\,,$$

there exists a trajectory (x, u) *of* (5) *so that*

$$x(0) = x^i\,,\ x(T) = x^f \quad \textit{and} \quad \sup_{t \in [0,T]} \|u(t) - \bar{u}(t)\|_{\mathbb{R}^m} \leqslant \varepsilon\,.$$

Nonlinear control problems III

Linear test

Example (Spring)

$$\ddot{x} = -x - x^3 + u$$

Example (Brockett integrator)

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

Nonlinear control problems I

Systems without drift

Let us now consider the control system:

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i \,, \tag{6}$$

where the f_i are smooth vector fields on $\mathbb{R}^n$.
To these vector fields, we associated the Lie algebra $\mathrm{Lie}\{f_1, \dots, f_m\}$ generated by the iterated Lie brackets:

$$[f_i, f_j](x) = \partial_x f_j f_i(x) - \partial_x f_i f_j(x) \qquad (x \in \mathbb{R}) \,.$$

Nonlinear control problems II

Systems without drift

Theorem (Chow)

Let us assume:

$$\mathrm{rank}\,\mathrm{Lie}\{f_1, \dots, f_m\}(x) = n \qquad (x \in \mathbb{R}^n)\,.$$

Then,

1. *If the set of admissible controls $\Omega \subset \mathbb{R}^m$ is nonempty and contains 0 in its interior, then there exists $T > 0$ such that the system (6) is controllable.*
2. *If the admissible set Ω is $\mathbb{R}^m$, for every $T > 0$, the system (6) is controllable.*

Example (Brockett integrator)

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

Nonlinear control problems I

Drift systems

Let us now consider the control system with drift:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i \,, \tag{7}$$

where the f_i are smooth vector fields on $\mathbb{R}^n$ and we assume that $f(0) = 0$ (i.e. $(0,0)$ is an equilibrium point).
For such a system, it is also natural to consider the Lie algebra generated by $\{f_0, \dots, f_m\}$.
But the Lie algebra rank condition is not enough to obtain controllability,

Example

$$\dot{x}_1 = x_2^2$$
$$\dot{x}_2 = u$$

Nonlinear control problems II

Drift systems

To end up with this problem, we define the set $\mathrm{Br}\{f_0, \dots, f_m\}$of formal iterated Lie brackets generated by $\{f_0, \dots, f_m\}$. And for every $h \in \mathrm{Br}$, we define $\delta_i(h)$ the number of times that f_i appears in h.
Let us also define

$$\sigma(h) = \sum_{\pi \in \mathrm{Per}_m} \pi(h),$$

with Per_m the group of permutation of $\{0, \dots, m\}$ such that for every $\pi \in \mathrm{Per}_m$, $\pi(0) = 0$.
For $\theta \in [0, \infty]$, we will say that the control system (7) satisfies the Sussmann condition $S(\theta)$ if

1. it satisfies the Lie algebra rank condition at 0,
2. for every $h \in \mathrm{Br}$ with $\delta_0(h)$ odd and $\delta_i(h)$ even for every $i \in \{1, \dots, m\}$, $\sigma(h)(0)$ is in the span of the $g(0)$'s such that

$$g \in \mathrm{Br} \quad \text{and} \quad \begin{cases} \theta\delta_0(g) + \sum_{i=1}^m \delta_i(g) < \theta\delta_0(h) + \sum_{i=1}^m \delta_i(h) & \text{if } \theta \in [0, \infty), \\ \delta_0(g) < \delta_0(h) & \text{if } \theta = \infty. \end{cases}$$

Nonlinear control problems III

Drift systems

Theorem (Sussmann)

If for some $\theta \in [0,1]$*, the control system* (7) *satisfies the condition* $S(\theta)$*, then it is* small time locally controllable.

That is to say:
For every $\tau > 0$*, there exists* $\eta > 0$ *such that for every* $x^i, x^f \in \mathbb{R}^n$*, with*

$$\|x^i\|, \|x^f\| < \eta\,,$$

there exists $u : [0,\tau] \to \mathbb{R}^m$ *such that the trajectory of* (7) *satisfies:*

$$x(0) = x^i\,,\ x(\tau) = x^f \quad \text{and} \quad \|u(t)\| \leqslant \tau \qquad (t \in [0,\tau])\,.$$

Example

$$\dot{x}_1 = x_2^3\,, \quad \dot{x}_2 = u$$

Nonlinear control problems IV

Drift systems

We can extend this result to more general problems. Consider the general control system:

$$\dot{x} = f(x, u)\,, \tag{8}$$

with $f(0,0) = 0$.
To (8), we associate:

$$\dot{x} = f(x, y)\,, \quad \dot{y} = u\,. \tag{8'}$$

Theorem

If for some $\theta \in [0,1]$, the control system (8) (i.e. (8')) satisfies the condition $S(\theta)$, then it small time locally controllable.

Linear control problems I

Linear-quadratic theory

We consider the system:

$$\dot{x} = A(t)x + B(t)u\,, \qquad x(0) = x^i \tag{9}$$

and given $T > 0$, we want to minimize the cost:

$$J(u) = x(T)^\top Q x(T) + \int_0^T \left(x(t)^\top W(t)x(t) + u(t)^\top U(t)u(t)\right)\,\mathrm{d}t\,, \tag{10}$$

with x the solution of (9)
In the above, the matrices Q and $W(t)$ are symmetric nonnegative and $U(t)$ is positive. In addition, we assume that A, B, W and U are L^∞ in time.

Linear control problems II

Linear-quadratic theory

Theorem (Existence)

Assume there exists $\alpha > 0$ *such that:*

$$\int_0^T u^\top(t)U(t)u(t)\,\mathrm{d}t \geqslant \alpha \int_0^T u^\top(t)u(t)\,\mathrm{d}t \qquad (u \in L^2([0,T],\mathbb{R}^m))\,,$$

then there exists a unique control $u \in L^2([0,T];\mathbb{R}^n)$ *minimizing* J.

Linear control problems III

Linear-quadratic theory

Theorem (Optimality condition)

The trajectory x associated to the optimal control u is optimal if and only if there exists and adjoint vector p such that:

$$\dot{p} = -A(t)^\top p + W(t)x(t)\,, \quad p(T) = -Qx(T)\,.$$

More other, the optimal control u is:

$$u(t) = U(t)^{-1}B(t)^\top p(t)\,.$$

Example

$$\dot{x} = u\,, \quad x(0) = x^i$$

with the cost $J(u) = \displaystyle\int_0^T \left(x(t)^2 + u(t)^2\right)\,\mathrm{d}t.$

Linear control problems I

Time optimality

We consider the system:

$$\dot{x} = A(t)x + B(t)u\,, \qquad x(0) = x^i \tag{11}$$

Given $x^f \in \mathbb{R}^n$, the aim is to find the minimal time $T > 0$ such that there exists a control $u : [0, T] \to \mathbb{R}^m$ steering x^i to x^f in time T and such that:

$$u(t) \in \Omega \qquad (t \in [0, T])\,,$$

with Ω a compact set of $\mathbb{R}^m$.

Theorem (Existence)

If there exists a time $T > 0$ such that x^i can be steered to x^f with a control u with $u(t) \in \Omega$ then, the minimal time exists.

Linear control problems II

Time optimality

Theorem (Optimality condition)

Assume that the previous existence theorem hold and let us define T the minimal time. Then the control $u : [0, T] \to \mathbb{R}^m$ is a control in time T if and only if there exists $p(t)$ a non trivial solution of:

$$\dot{p} = -A(t)^\top p \,,$$

such that u satisfies:

$$p(t)^\top B(t)u(t) = \max_{v\in\Omega} p(t)^\top B(t)v \qquad (t \in [0, T] \;\; a.e.) \,.$$

Linear control problems III

Time optimality

Let us focus on the autonomous control system:

$$\dot{x} = Ax + Bu\,, \tag{12}$$

with $A \in R^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$.

Theorem (Bang-Bang property)

Assume that the pair (A,B) satisfies the Kalmann rank condition ant that $\Omega = [-1,1]$ then,

1. *If all the eigenvalues of A are real, then the time optimal control has at most $n-1$ commutations;*
2. *If A has a non real eigenvalue, then for every $N \in \mathbb{N}$, there exists points x^i and x^f such that the time optimal control has at least N commutations.*

Example

$$\ddot{x} = -x + u$$

Existence results I

General systems

Let us consider the control system:

$$\dot{x} = f(t, x, u)\,, \tag{13}$$

with a control u such that $u(t) \in \Omega$, with Ω a compact set of $\mathbb{R}^m$. The aim is to join a set $M^i \subset \mathbb{R}^n$ to a set $M^f \subset \mathbb{R}^n$ by minimizing the cost: with the cost:

$$J(u) = \int_0^{t(u)} f_0(t, x(t), u(t))\,\mathrm{d}t + g(t(u), x(t(u)))\,.$$

Existence results II

General systems

Theorem

- *Assume that* M^i *is accessible from* M^f*, i.e. for every* $x^f \in M^f$*, there exists* $x^i \in M^i$, $u : \mathbb{R}_+ \to \mathbb{R}^m$ *with* $u(t) \in \Omega$ *such that there exists* $t = t(u) > 0$ *for which the solution of* (13) *with initial condition* $x(0) = x^i$ *satisfies* $x(t(u)) = x^f$.
- *Assume in addition that for all of these control u, there exists a constant* $b > 0$ *(independent of u) such that:*

$$t(u) + \|x(t)\| \leqslant b \qquad (t \in [0, t(u)]) .$$

- *Assume also that for every* $s \in \mathbb{R}$ *and* $y \in \mathbb{R}^n$*, the set*

$$\left\{ \begin{pmatrix} f(s,y,v) \\ f_0(s,x,v) + \gamma \end{pmatrix} , \ v \in \Omega \, , \ \gamma \geqslant 0 \right\} \qquad \text{is convex.}$$

Then there exists a control u, an initial condition $x^i \in M^i$ *and a time* $t(u) \geqslant (u)$ *such that u steers* x^i *to* x^f *in time* $t(u)$ *and u minimize J.*

Existence results III

General systems

Remark

The boundedness property for u (u with values in a compact set Ω) and for x can be replaced by some growth conditions.
See for instance Harlt, Sethi and Vickson, A survey of the maximum principle for optimal control problems with state constraints, SIAM review, 1995.

Pontryagin maximum principle I

General result

Theorem (Pontryagin maximum principle)

If u is optimal, then there exists an application $p : [0, T] \to \mathbb{R}^n$ and $p_0 \leqslant 0$ such that:

1. *$(p(\cdot), p_0)$ is nontrivial;*
2. *for almost every $t \in [0, T]$, we have:*

$$\dot{x} = f(t, x, u)$$
$$\dot{p} = -\partial_x H(t, x, p, p_0, u)$$

with:

$$H(t, x, p, p_0, u) = \langle f(t, x, u), p \rangle_{\mathbb{R}^n} + p_0 f_0(t, x, u)$$

and the control u is such that:

$$\max_{v \in \Omega} H(t, x(t), p(t), p_0, v) = H(t, x(t), p(t), p_0, u(t)) \qquad (t \in [0, T] \ \ a.e.)\,.$$

Pontryagin maximum principle II

General result

Remark

One can also notice that:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t, x(t), p(t), p_0, u(t)) = \partial_t H(t, x(t), p(t), p_0, u(t)) \,.$$

and hence if f and f_0 are independent of t, then $H(t, x(t), p(t), p_0, u(t))$ is constant.

Pontryagin maximum principle I

Transversality conditions

Theorem (Transversality conditions)

In addition,

1. *If the final time T is not fixed, then,*

$$\max_{v\in\Omega} H(T, x(T), p(T), p_0, v) = -p_0 \partial_t g(T, x(T)) \,.$$

2. *If $x(0)$ and $x(T)$ are not fixed but satisfies $(x(0), x(T)) \in \mathcal{M}$ with $\mathcal{M}$ a sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$, then,*

$$(p(0), p(T) - p_0 \partial_x g(T, x(T))) \perp \mathrm{T}_{(x(0),x(T))}\mathcal{M} \,.$$

Pontryagin maximum principle II

Transversality conditions

Example

1. If $x(0)$ is free, then, $p(0) = 0$;
2. If we want periodic trajectories, i.e. $x(0) = x(T)$ and if $g = 0$, then, $p(0) = p(T)$.

Pontryagin maximum principle I

Applications

Example (Linear control problem)

$$\dot{x} = Ax + Bu$$

- T fixed, $x(0) = x^i$, $x(T) = x^f$, $\Omega = \mathbb{R}^m$, $g = 0$, $f_0(t, x, u) = \frac{1}{2}\|u\|^2$;
- T fixed, $x(0) = x^i$, $x(T) \in \mathbb{R}^n$, $\Omega = \mathbb{R}^m$, $g(t, x) = x^\top Qx$, $f_0(t, x, u) = x^\top Wx + u^\top Uu$;
- T free with $T > 0$, $x(0) = x^i$, $x(T) = x^f$, $\Omega = B_{\mathbb{R}^m}(0, 1)$, $g = 0$, $f_0(t, x, u) = 1$.

Pontryagin maximum principle II

Applications

Example (Brockett integrator)

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= \langle Mx, u\rangle \end{aligned} \qquad (x(0), y(0)) = (0, 0)$$

Minimize T such that $(x(T), y(T)) = (0, y^f)$, with $\Omega = B_{\mathbb{R}^m}(0, 1)$.
Assuming $M^\top \neq M$.

Example (Zermelo's problem)

$$\begin{aligned} \dot{x} &= v\cos(u) + c(y) \\ \dot{y} &= v\sin(u) \end{aligned} \qquad (x(0), y(0)) = (0, 0)\,.$$

- $T > 0$ free, $y(T) = 1$, $x(T) \in \mathbb{R}$, $f = 0$, $g(t, x) = x^2$.
  Assuming $c(y) > v$ for every y;
- Minimize T such that $y(T) = 1$ and $x(T) \in \mathbb{R}$.

Shooting methods I

We remind that the optimality system is:

$$\dot{x} = \partial_p H(t, x, p, p_0, u) \tag{14a}$$

$$\dot{p} = -\partial_x H(t, x, p, p_0, u) \tag{14b}$$

where:

$$H\left(t, x(t), p(t), p_0, u(t)\right) = \max_{v \in \Omega} H\left(t, x(t), p(t), p_0, v\right) \tag{15}$$

In general, p_0 can be chosen to -1 if it is not possible, this means that the admissible trajectories are independent of the cost function.

Let us assume that $u(t)$ is uniquely determined by (15), i.e., $u = u(t, x, p)$. Thus, writing $z = (x, p)^\top$, (14) can be written as:

$$\dot{z} = F(t, z)\,.$$

Shooting methods II

We chose an initial condition $z(0) = z^i$ and we look for z^i such that transversality conditions are fulfilled. The transversality condition can be expressed has the zeros of a function G, defined by $G(z^i) = R(z^i, z(T; z^i))$.

The aim is to find z^i such that:

$$G(z^i) = 0\,.$$

this can be done for instance with a Newton method.

Remark

If T is free, we can add T in the state variable, with the equation $\dot{T} = 0$ and rescale the problem on $[0, 1]$.

Remark

This procedure can be parallelized. Instead of having an unknown z^i we can take a subdivision $0 = t_0 < \cdots < t_N = T$ and take as unknown, $z(t_i)$. In addition to the transversality condition, we will have the continuity of z at points $t_1, \ldots, t_{N-1}$.

Shooting methods

Newton method

We recall the Newton method.

Given $G:\mathbb{R}^d\to\mathbb{R}^d$ the aim is to find $z\in\mathbb{R}^d$ such that $G(z)=0$. If z_k is close to z, then we have:

$$0=G(z)=G(z_k)+\mathrm{d}G(z_k)\cdot(z-z_k)+o_0(z-z_k)\,.$$

Then we chose

$$z_{k+1}=z_k+d_k\,,$$

with $d_k\in\mathbb{R}^d$ solution of:

$$G(z_k)+\mathrm{d}G(z_k)\cdot d_k=0\,.$$

Full discretization I

For this method, we tackle directly the optimization problem:

$$\min \quad J(T,x,u) \quad \left|\begin{array}{l} \dot{x} = f(t_i, x, u) \\ u(t) \in \Omega \\ x(0) = ?\,,\ x(T) = ? \end{array}\right. \tag{16}$$

More precisely, we will discretize it, i.e., given a subdivision $0 = t_0 < \cdots < t_N = T$, we consider the unknowns $x_i \simeq x(t_i)$ and $u_i \simeq u(t_i)$. And we write $X = (x_0, \dots, x_N)$ and $U = (u_0, \dots, u_N)$.
We also consider a quadrature formula for J and $\dot{x} = f(x,u)$, i.e. $J(T,x,u) \simeq \tilde{J}(T,X,U)$ and $\dot{x} = f(t,x,u)$ becomes constraints of the type $c_i(X,U) = 0$ for every $i \in \{1, \dots, N\}$.

Full discretization II

Thus the minimisation problem (16') becomes:

$$
\min \quad \tilde{J}(T, X, U) \\
\left| \begin{array}{l} c_i(X, U) = 0 \qquad i = 1, \dots, N \\ u_i \in \Omega \qquad i = 0, \dots, N \\ x_0 = ?\,,\ x_N = ? \end{array} \right. \tag{16'}
$$

which is a finite dimensional minimisation problem with constraints.
This problem can be solved with a penalty method or with a sequential quadratic programming method.

Remark

Other methods are based on Halminton-Jacobi equation.
We refer to Bardi and Capuzzo-Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equation, Birkhäuser, 1997.

Thanks for you attention