$L^1$ stability theory and rigorous error estimates for some 1D hyperbolic systems of balance laws. (2).

Laurent Gosse (with Debora Amadori)

IAC-CNR (Bari and then Rome)
### Plan of the talk

1. **Introduction**

2. **The example of the position-dependent Jin-Xin model**
   - Setting up the scheme
   - Setting up a nonlinear $L^1$ functional with "big data"
   - The main theorem
   - Elements of proof: Lyapunov functional
   - Comparison with Kuznetsov estimate
   - Origin of the discrepancy of Time-Splitting scheme
   - Numerical validation on locally damped wave eqn.

3. **Extension: weakly nonlinear (repulsive) Poisson coupling**
Let’s recall the general methodology!

Consider \( U = (u, a) \), \( V = (v, b) \) 2 WFT approximations of:

\[
\partial_t u + \partial_x f(u) - g(u) \partial_x a = 0 \quad \partial_t a = 0.
\]

**Key tool:**

Lyapunov functional \( t \mapsto \Phi(U, V)(t) \) unif. equivalent to \( L^1 \) norm:

\[
\mathcal{O}(1)(\|u(t) - v(t)\|_{L^1} + \|a - b\|_{L^1}) \leq \Phi(U, V)(t) \leq \Phi(U, V)(0) \leq \mathcal{O}(1)(\|u(0) - v(0)\|_{L^1} + \|a - b\|_{L^1})
\]

Here both the \( \mathcal{O}(1) \) are indep. on \( t \), and possibly made explicit.

Let \( v(0) = u^{\Delta x}(0) \), \( b = a^{\Delta x} \) be suitably defined (piecewise constant, with jumps at the points \( j \Delta x \)), the decay of \( \Phi \) yields ...
Main tool: A Lyapunov-type functional for WFT

\[ \|u_0 - u^{\Delta x}(0)\|_{L^1} \leq \Delta x \ \text{TV}\{u_0\}, \ \|a - a^{\Delta x}\|_{L^1} \leq \Delta x \ \text{TV}\{a\} = \Delta x \|k\|_{L^1} \]

Therefore, we end up with

\[ \|u(t) - u^{\Delta x}(t)\|_{L^1} \leq O(1) \left( \|u(0) - u^{\Delta x}(0)\|_{L^1} + \|a - a^{\Delta x}\|_{L^1} \right) \]

\[ \leq O(1) \Delta x \left( \text{TV}\{u_0\} + \|k\|_{L^1} \right) \]

**Remarks**

- Above estimate only depends on \( \|k\|_{L^1} \) and not on \( \text{TV}\{k\} \)

- In the scalar case, with \( f' > 0 \) and possibly accretive source, this approach is followed to study a Godunov–type scheme (combined with the WB approx.) : the presence of Godunov projections, at \( t_j = j \Delta t \), results in an additional term in the error estimate, growing linearly in time.
Semilinear 2 $\times$ 2 space-dependent Jin-Xin model

\[\begin{align*}
\partial_t \rho + \partial_x J &= 0 \\
\partial_t J + \partial_x \rho &= -k(x)(A(\rho) - J)
\end{align*}\]

- $k \in L^1 \cap BV(\mathbb{R})$, $k(x) \geq 0$
- $A \in C^1(\mathbb{R})$, $|A'| < 1$ (→ sub-characteristic condition).

Diagonal variables $f^\pm : \rho = f^+ + f^-$, $J = f^+ - f^-$, and set $a(x) : a'(x) = k(x)$. Then rewrite the system as

\[\begin{align*}
\partial_t f^- - \partial_x f^- - G(f^-, f^+) \partial_x a &= 0 \\
\partial_t f^+ + \partial_x f^+ + G(f^-, f^+) \partial_x a &= 0 \\
\partial_t a &= 0
\end{align*}\]

Observe that $A(\rho) \equiv 0$ gives the locally damped wave eqn.
The example of the position-dependent Jin-Xin model

Setting up the scheme

The Well-Balanced Approximation

\[
\begin{align*}
\partial_t f^- - \partial_x f^- - G \partial_x a &= 0 \\
\partial_t f^+ + \partial_x f^+ + G \partial_x a &= 0 \\
\partial_t a &= 0
\end{align*}
\]

- Characteristic speeds $\pm 1, 0$ (→ super-LD field)
- $\lambda = 0 \leadsto 0$-wave: stationary solutions, $\partial_x f^\pm = G(f^-, f^+)\partial_x a$.
  **Notice**: $J = f^+ - f^- = \text{const.}$ (always) across 0-waves
- Computational grid with CFL=1, $\Delta x > 0$, $x_j = j\Delta x$, $j \in \mathbb{Z}$
- $f^{\pm}_{\Delta x}(x, 0) = f^{\pm}_0(x_j+)$, $a_{\Delta x}(x) = a(x_j)$ $x \in (x_j, x_{j+1})$.
  At each $x = x_j$ solve a Riemann problem for the $3 \times 3$ system.
- There are positively invariant domains $D$ for WB-WFT algo.
Visualization of WB-WFT algorithm with CFL=1

Figure 3. Schematic view of a WB approximate solution: circles indicate Riemann problems studied in Prop. 1. Since the Courant number is 1, constant states always lie in between them.

\[ \text{CFL}=1 \rightarrow \text{no need of a projection operator, WFT=Godunov.} \]
Uniform in time BV bounds

Let \( k \geq 0, k \in L^1 \Rightarrow a(x) = \int^x k(y) \, dy \) is monotone and bounded.

\[
\text{BV bounds for (WB) approximations :}
\]

\[
\text{TV } f_{\Delta x}^\pm(t, \cdot) \leq \text{TV } f_0^\pm + 4 C_0 \| k \|_{L^1}
\]

where \( D \) is an invariant domain and \( C_0 \) the "Maxwellian gap",

\[
C_0 = \max\{|G(f^-, f^+)|; (f^-, f^+) \in D\}.
\]

Notice :

The BV estimate depends on \( \| k \|_{L^1} = \text{TV } a \) and not on time : oscillations (or discontinuities) in \( k \) don’t develop/grow in \( f_{\Delta x}^\pm(t, \cdot) \).
The example of the position-dependent Jin-Xin model

The main theorem

**Error estimate via $L^1$ stability**

For $x_1 < x_2$ and $2t \leq x_2 - x_1$, consider the $L^1$ error estimate,

$$I(t) = \int_{x_1 + t}^{x_2 - t} |f_{\Delta x}^\pm(t, x) - f^\pm(t, x)|\,dx.$$ 

**Theorem:** Under smallness restriction $4 \, C_1 \|k\|_{L^1} < 1$, then

$$I(t) \leq K \cdot I(0) + \Delta x \cdot \|k\|_{L^1} \cdot R_1 \left( C_0, K, \|k\|_{L^1}, \text{TV} \{f_0^\pm\} \right)$$

(in particular, init. data $f_0^\pm \in L^1 \cap BV(\mathbb{R})$ may be big), where

$$K = \frac{1}{1 - 4 \, C_1 \|k\|_{L^1}} \geq 1, \quad C_0 = \max_D |G|, \quad C_1 = \frac{4}{3 \log(\frac{3}{2})}.$$ 

Here $\|k\|_{L^1} = \|k\|_{L^1(x_1, x_2)}$, $\text{TV} \{f_0^\pm\} = \text{TV} \{f_0^\pm; (x_1, x_2)\}$. 
$I(t) \leq K \cdot I(0) + \Delta x \cdot \|k\|_{L^1} \cdot R_1$

$$K = \frac{1}{1 - 4 C_1 \|k\|_{L^1}}, \quad R_1 = R_1 (C_0, K, \|k\|_{L^1}, TV \{f_0^\pm\})$$

- Estimate uniform in time because $\Phi$ decays,
- $I(0) \leq \Delta x \ TV \{f_0^\pm\} \rightarrow$ the error is $O(1) \Delta x$
- The estimate holds as long as $\|k\|_{L^1} < \frac{1}{4C_1}$ : (pessimistic ?)

$$K \rightarrow \infty \text{ as } \|k\|_{L^1(x_1,x_2)} \text{ approaches } \frac{1}{4C_1}.$$  

- Stiff relaxation regime $\rightarrow$ limit $\partial_t \rho + \partial_x A(\rho) = 0$, so accuracy in $\sqrt{\Delta x}$. It’s normal to have a restriction on the size of $|k(x)|$. 

Comments
A BLY-type functional handling ”big data”

Let $U = (f_1^-, f_1^+, a)$ and $V = (f_2^-, f_2^+, b)$.

- Solve the $3 \times 3$ Riemann problem between $U(t, x)$ and $V(t, x)$
- Size of transversal waves: $q_{\pm 1} = \Delta(f_{\pm})$, $q_0(x) = b(x) - a(x)$

Figure 8. Interaction between a “transversal Riemann problem” (left) and a $-1$-wave resulting in the new Riemann problem (right) illustrating the simplest situation described by Prop. 2,

For $x_1 < x_2$ and $t \leq T = (x_2 - x_1)/2$:

$$t \mapsto \Phi[U, V](t) = \sum_{i=-1}^{1} \int_{x_1 + t}^{x_2 - t} |q_i(x)| W_i(x) dx.$$
Convenient choice of the weights

Interaction potentials are useful only for \( i = 0 \to " \text{big data} \),

\[
W_{\pm 1}(x) = 1 + \kappa_1 \cdot A_{\pm 1}(x), \\
W_0(t, x) = 1 + \kappa_1 \cdot A_0(t, x) + \kappa_2 \cdot (Q(U) + Q(V))
\]

together with,

\[
A_{-1}(x) = \sum_{x_\alpha < x} \sigma_0^\alpha, \quad A_1(x) = \sum_{x_\alpha > x} \sigma_0^\alpha, \\
A_0(t, x) = \sum_{x_\alpha < x} |\sigma_1^\alpha| + \sum_{x_\alpha > x} |\sigma_{-1}^\alpha|.
\]

- The sums above extend over jumps in both \( U \) and \( V \).
- If \((a(\cdot), b(\cdot))\) have sufficiently small total variation, then

\[
1 \leq W_{\pm 1}(x) \leq K, \quad 1 \leq W_0(t, x) \leq K_0.
\]
Uniform equivalence with the $L^1$ norm

**Lemma**: If $\kappa_1 \geq 2.K.C_1$ then, outside interaction times:

$$\frac{d\Phi[U, V]}{dt} \leq 0.$$  

The functional actually decreases in time thanks to the semi-linear structure of the problem. By equivalence with the $L^1$ norm, a uniform bound is easily deduced. Accordingly, define:

$$I(t) = \| (f^-, f^+)(t, \cdot) \|_{L^1(x_1+t, x_2-t)}$$

If $\kappa_2$ big enough, $\Phi[U, V]$ decreases at interactions, too, and so,

$$I(t) \leq \Phi[U, V](t) + (2C_0 - 1) \int_{x_1+t}^{x_2-t} |a - b| \, dx$$

$$\leq K.I(0) + (2C_0(K + 1) + K_0 - 1) \int_{x_1}^{x_2} |a - b| \, dx.$$
A second error estimate: via Kuznetsov method

A complementary estimate can be obtained via Kuznetsov method, for $k \in L^1 \cap BV$, but without restriction on $\|k\|_{L^1}$.

**Theorem**: If $k \in L^1 \cap BV(\mathbb{R})$, WB-WFT algo. also satisfies:

$$I(t) \leq I(0) + \sqrt{\Delta x} \cdot t \cdot 2R_2$$

(observe that $t$ appears now!), where

$$R_2(t, x_1, x_2) = \sqrt{C_0 \|k\|_{L^1} A(t)} + \sqrt{\Delta x} C_0 \|k\|_{L^1} \|k\|_{L^\infty}$$

and (remember the estimate in the scalar case),

$$A(t) = \frac{32}{C_0 t} \text{TV} \{f_0^\pm\} + \text{TV} \{k\}.$$ 

Here $\| \cdot \|_{L^1}, \| \cdot \|_{L^\infty}, \text{TV} \{ \cdot \}$ are referred to $(x_1, x_2)$.
Comparison

\( I(t) \leq K \cdot I(0) + \Delta x \cdot \| k \|_{L^1} \cdot R_1, \)

\( I(t) \leq I(0) + \sqrt{\Delta x} \cdot t \cdot 2R_2 \)

- \( R_1 \left( C_0, K, \| k \|_{L^1}, \text{TV} \{ f_0^\pm \} \right), \quad K = (1 - 4 \text{Lip}(g) \| k \|_{L^1})^{-1} \)
- \( R_2 \left( C_0, \text{Lip}(g), \| k \|_{L^1}, \| k \|_{L^\infty}, \text{TV} \{ k \}, \text{TV} \{ f_0^\pm \}, t, \Delta x \) \)

Notice that:

(1) (WB, funct.) insensitive to \( t, \text{TV} \{ k \} \), but for small \( \| k \|_{L^1} \),

(2) (K., entropy) OK if \( \| k \|_{L^1} \) large, but grows with \( t, \text{TV} \{ k \} \).

Numerical examples confirm that \( I(t) \) isn't sensitive to \( \text{TV} \{ k \} \).
The semilinear 1D damped wave equation

We (temporarily) restrict ourselves to, (cf. G-Toscani scheme)

\[ \partial_{tt} u - \partial_{xx} u + 2k(x)g(\partial_t u) = 0 \]  

\text{(*)}

For \( J = \partial_t u, \rho = -\partial_x u \), it rewrites as a 2 \times 2 semilinear system,

\[
\begin{cases}
\partial_t \rho + \partial_x J = 0 \\
\partial_t J + \partial_x \rho = -2k(x)g(J)
\end{cases}
\]

- \( k \in L^1(\mathbb{R}), \quad k(x) \geq 0 \)
- \( g \in C^1(\mathbb{R}), \quad g(0) = 0, \quad g \) strictly increasing

For \( g(J) = J \), this case is included in the preceding WB analysis.
The example of the position-dependent Jin-Xin model

Origin of the discrepancy of Time-Splitting scheme

Time-Splitting Approximation for damped wave eqn

For $\Delta t > 0$, define $(f^-, f^+) := (f^-, f^+)^{\Delta t}$ as follows:

1. on $[0, \Delta t)$, $f^\pm$ given by exact soln of the linear problem

   $\partial_t f^- - \partial_x f^- = 0, \quad \partial_t f^+ + \partial_x f^+ = 0.$

2. At time $t = \Delta t$ the solution is updated by the source term:

   $$(f^-, f^+) (\Delta t +, x) = O_{\Delta t} \left( (f^-, f^+) (\Delta t -, x); k(x) \right),$$

   where $O_t$ denotes the solution operator of the ODE

   $$\begin{cases}
y' = k g(z - y) \\
z' = -k g(z - y)
\end{cases}, \quad k > 0.$$

N.B. Along ODE: $y + z = const., \quad |z - y|' = -2k|g(z - y)| \leq 0.$
Invariant domains for damped wave by Time-Splitting

**Figure 5.** Invariant domain for both homogeneous and complete system
The example of the position-dependent Jin-Xin model

Origin of the discrepancy of Time-Splitting scheme

BV bounds for TS approximation of damped wave eqn

\[
\partial_t \left( \text{TV } f^+(t, \cdot) + \text{TV } f^-(t, \cdot) \right) \leq 2 \| g(J) \|_\infty \text{TV } \{ k \} . \tag{TS}
\]

**Proof:** Derive in \( x \) the system, set \( u = \partial_x (f^-) \), \( v = \partial_x (f^+) \), so

\[
\begin{cases}
\partial_t u - \partial_x u = g'(J)(v - u)k + g(J)k_x \\
\partial_t v + \partial_x v = -g'(J)(v - u)k - g(J)k_x ,
\end{cases}
\]

thus classical stability theory for space-dependent ODE implies:

\[
\partial_t (|u| + |v|) + \partial_x (|v| - |u|) = g'(J)k \left(-|u| - |v| + v \text{ sgn } u + u \text{ sgn } v \right) \leq 0
\]

\[
+ g(J)k_x \left\{ \text{sgn}(u) - \text{sgn}(v) \right\}[2mm] \leq 2\| g(J) \| |k_x | .
\]

By integrating in \( x \), we get (TS) : total variation may grow with \( t \).
Therefore the total variation of \( f^\pm \) **possibly increases at a linear rate**, proportional to \( TV \{k\} \). This weaker BV-bound has consequences when it comes to insert it inside an error estimate because it yields several time components.

**Reason**: the ODE is "sensitive" to the changing values of \( k \).

**Lemma**: Let \((y_1, z_1)(t), (y_2, z_2)(t)\) be 2 solns of ODE,

\[
y' = k \, g(z - y), \quad z' = -k \, g(z - y)
\]

corresponding to \( k = k_1, k_2 \) respectively. Assume that \((y_1, z_1)(0) = (y_2, z_2)(0)\). Then

\[
|y_1(t) - y_2(t)| + |z_1(t) - z_2(t)| \leq 2\|g\|_\infty |k_2 - k_1| \, t.
\]

Nothing exotic, only classical ODE theory typical for scattering problems ...
An illustrative numerical example

- $g(J) = J$ (→ easier to find exact stat. solns)
- smooth $k_\alpha(x) = \sin^2(\alpha \pi x) \chi\{|x| < 1\}$, with $\alpha \in \mathbb{N}$

  Notice: $\|k_\alpha\|_{L^1} = 1$, while $TV\ k_\alpha$ increases with $\alpha$

- $f^+_b = 1, \quad f^-_b = 0.4, \quad \Delta t = \Delta x = 2^{-6}$

- Visualize **Pointwise errors**:

  $$e^\Delta t_{\pm}(t, x_j) = (f^{\pm})^\Delta t(t, x_j) - f^{\pm}(x_j), \quad (TS)$$
  $$e^{\Delta x}_{\pm}(t, x_j) = (f^{\pm})^{\Delta x}(t, x_j) - f^{\pm}(x_j), \quad (WB)$$

  where $f^{\pm}(x)$ are steady solutions corresponding to data above.
$L^1$ stability and error estimates

The example of the position-dependent Jin-Xin model

Numerical validation on locally damped wave eqn.

**Figure 8.** Conversion of a Cauchy problem in $\mathbb{R}$ into a BVP on $(-1, 1)$. 
$L^1$ stability and error estimates

The example of the position-dependent Jin-Xin model

Numerical validation on locally damped wave eqn.

Black curve (WB) insensitive to oscillations of $k_\alpha$. Blue/Green curves stand for simple split-scheme (depending if ODE step is after/before free transport) and carry those oscillations. Red curve is for 2nd order Strang-splitting (smaller oscillations).
Errors on $\rho$ (left) and $J$ (right): the errors on $f^{\pm}$ cancel each other when added, but reinforce when subtracted (for $J$). Even for Strang-splitting, the oscillations persist on $J$ (but much smaller).
An elementary test on the "inertial approximation"

Consider the system $\partial_t \rho + \partial_x J = 0$, $\partial_t J + \partial_x \rho + \rho \partial_x \phi = -k(x)J$, where $-\partial_{xx} \phi = \rho - d(x) + \text{B.C.}$, both fcts $d(x)$ and $k(x)$ being discontinuous "doping profile" and damping term, respectively.

Errors on $\rho$ (left) and $J$ (right) for discont. $k(x)$ and $\partial_x \phi \equiv 0$. 

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