# Lecture 4

In this lecture we prove existence of viscosity solution to the Dirichlet problem for the $\infty$-Laplacian:

$$\begin{cases} \Delta_\infty u := Du \otimes Du : D^2 u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \qquad (*)$$

where $\Omega \Subset \mathbb{R}^n$ and $g \in W^{1,\infty}(\Omega)$. To this end, we use the stability property of viscosity theory and approximate the $\infty$-Laplacian by the p-Laplacian

$$\Delta_p u := D_i\left(|Du|^{p-2} D_i u\right) = 0 \qquad \binom{*}{*}$$

as $p \to \infty$. Formally, by expanding $\binom{*}{*}$ we have

$$|Du|^{p-2} D^2_{ii} u + (p-2)|Du|^{p-4} D_i u \, D_j u \, D^2_{ij} u = 0$$

and by normalising, we get

$$Du \otimes Du : D^2 u + \frac{|Du|^2}{p-2} \Delta u = 0. \qquad (✡)$$

Hence, as $p \to \infty$, (✡) gives $(*)$. On the other hand, the p-Laplacian has a unique minimising weak solution (which is $C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$) for boundary values $u = g$ on $\partial\Omega$, which minimises the p-Dirichlet functional

$$E_p(u,\Omega)=\int_\Omega |Du|^p$$

in $W_0^{1,p}(\Omega)+g$. To do this, we need to know that

$$\boxed{\begin{array}{l}\text{WEAK SOLUTIONS OF THE } p\text{-LAPLACIAN}\\ \text{ARE VISCOSITY SOLUTIONS AS WELL!}\end{array}}$$

This (and a more general fact) is true, and will be proved later. For, we have

**Theorem** [EXISTENCE FOR THE $\infty$-LAPLACIAN]
The problem $(*)$ has a viscosity solution $u_\infty \in W^{1,\infty}(\Omega)$.

Proof. Set

$$F_p(q,X):=q\otimes q:X+\frac{|q|^2}{p-2}X:I,$$

$$F_\infty(q,X):=q\otimes q:X.$$

Then, $F_p\to F_\infty$ locally uniformly as $p\to\infty$. The PDE

$$F_p(Du_p,D^2u_p)=0$$

has a solution $u_p\in C^{1,\alpha}(\Omega)\cap C^0(\overline{\Omega})$, with $u_p=g$ on $\partial\Omega$. Moreover, it is known by standard estimates$^{(*)}$ that $u_p\to u_\infty$ in $C^0(\overline{\Omega})$, to some $u_\infty\in W^{1,\infty}(\Omega)$ with $u_\infty=g$ on $\partial\Omega$. By stability of viscosity solutions $u_\infty$ solves $F_\infty(Du_\infty,D^2u_\infty)=0$, and the theorem follows. $\square$

The next result completes the existence for the $\infty$-Laplacian.

**Theorem** [Minimisers are Viscosity Solutions]
Let $u \in W^{1,\infty}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$, be a $\boxed{\text{local minimiser}}$ of the functional

$$E(u,\Omega) := \int_\Omega f(Du)$$

where $f \in C^2(\mathbb{R}^n)$ is convex. Then $u$ is a Viscosity Solution of the Euler–Lagrange PDE (expanded):

$$\boxed{f_{pp}(Du) : D^2u = 0,} \quad \text{on } \Omega.$$

(Notation: $f_p(p) \equiv Df(p)$, $f_{pp}(p) \equiv D^2f(p)$)

Remark:

EVEN THOUGH THE E.-L. PDE $\mathrm{Div}(f_p(Du)) = 0$ IS IN DIVERGENCE FORM AND $f$ IS SMOOTH CONVEX, WEAK SOLUTIONS MAY NOT EXIST BECAUSE UNLESS

$$|f_p(q)| \leq C \cdot |q|^{p-1}$$

THE FUNCTIONAL $E$ MAY NOT BE GATEAUT DIFFERENTIABLE!!!
BUT VISCOSITY SOLUTIONS EXIST!

**Proof.** Assume for the sake of contradiction that

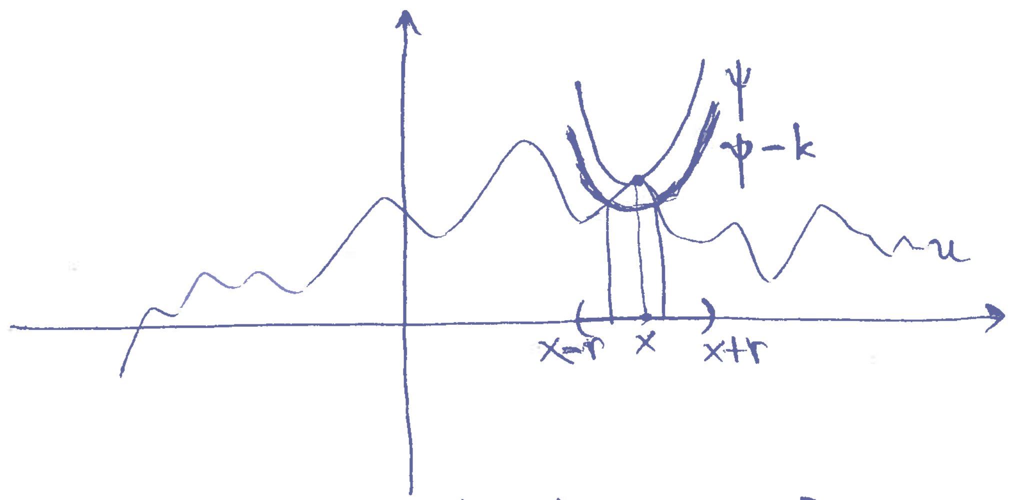
$$u-\psi < 0 = (u-\psi)(x)$$

near some $x \in \Omega$ for some $\psi \in C^2(\mathbb{R}^n)$, but there is $c>0$ such that

$$D_i\big(f_{p_i}(D\psi)\big)(x) \leq -c < 0.$$

Hence, since $\psi \in C^2(\mathbb{R}^n)$, there is an $r>0$ such that

$$D_i\big(f_{p_i}(D\psi)\big) \leq -\frac{c}{2}, \text{ on } \mathbb{B}_r(x) \subseteq \Omega. \qquad (\star)$$




By sliding $\psi$ downwards to $\psi-k$ for some $k>0$ small, we have

$$\Omega^+ := \{u-\psi+k>0\} \subseteq \mathbb{B}_r(x),$$

$$u = \psi - k, \text{ on } \partial\Omega^+.$$

We now multiply $(\star)$ by $(u-\psi+k)^+ \big(=\max\{u-\psi+k, 0\}\big)$ and integrate by parts:

$$\int_{\Omega^+} D_i(u-\psi) f_{p_i}(D\psi) \geq \frac{c}{2}\int_{\Omega^+} |u-\psi+k|.$$

Since $f$ is convex we have

$$\int_{\Omega^+} f(Du) - \int_{\Omega^+} f(D\psi) \geq \int_{\Omega^+} D(u-\psi)\cdot f_p(D\psi)$$
$$\geq \frac{c}{2}\int_{\Omega^+} |u-\psi+k|$$

Since $u$ a minimiser and $u=\psi+k$ on $\partial\Omega$, we have

$$\frac{c}{2}\int_{\Omega^+} |u-\psi+k| \leq \int_{\Omega^+} f(Du) - \int_{\Omega^+} f(D(\psi+k))$$
$$\leq 0.$$

Hence, $|\Omega^+|=0$ and the contradiction implies that $u$ is a viscosity solution of

$$f_{pp}(Du) : D^2 u = 0,$$

on $\Omega$. $\square$

<u>Remark:</u> $u$ may have infinite energy and need not even be a global minimiser.

(*) <u>The standard estimates:</u>

The boundary condition is an apriori energy bound:

$$E_p(u_p,\Omega) \le E_p(g,\Omega) < \infty,$$

and

$$\|Du_p\|_{L^p(\Omega)} \le \|Dg\|_{L^p(\Omega)} \le \|Dg\|_{L^\infty(\Omega)} \cdot |\Omega|^{\frac{1}{p}}.$$

For $q \le p$, we have

$$\|Du_p\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|Du_p\|_{L^p(\Omega)},$$

or

$$\boxed{\|Du_p\|_{L^q(\Omega)} \le |\Omega|^{1+\frac{1}{q}-\frac{1}{p}} \|Dg\|_{L^\infty(\Omega)}.} \qquad \odot$$

By Poincaré inequality,

$$\boxed{\|u_p\|_{L^q(\Omega)} \le C\left(\|Du_p\|_{L^q(\Omega)} + \|g\|_{W^{1,\infty}(\Omega)}\right).} \qquad (:)$$

Hence, for any $q > n$ fixed, we have that there is $u_\infty \in L^q(\Omega)$ such that

$$\boxed{\begin{cases} u_p \to u_\infty, & \text{in } L^q(\Omega), \\ u_p \longrightarrow u_\infty, & \text{in } L^q(\Omega), \end{cases}} \qquad (\therefore)$$

along a sequence $(u_{p_m})_1^\infty$. Moreover, by a diagonal argument the limit $u_\infty$ is the same for all $q > n$. By Morrey's estimate, $(\therefore)$ implies

$$\boxed{u_p \to u_\infty \quad \text{in } C^\alpha(\overline{\Omega}),}$$

for $\alpha = 1 - n/q$, and hence $u_p \to u_\infty$ locally uniformly.

Moreover, by letting $p \to \infty$ in $\odot$ we get

$$\|Du_\infty\|_{L^q(\Omega)} \le \liminf_{p\to\infty} \|Du_p\|_{L^q(\Omega)} \qquad (∷)$$
$$\le |\Omega|^{1+\frac{1}{q}} \|Dg\|_{L^\infty(\Omega)},$$

by the weak LSC (lower semi-continuity) of the $L^q$ norm (as a convex functional). By letting $q \to \infty$ to (∷) we deduce

$$\|Du_\infty\|_{L^\infty(\Omega)} \le |\Omega| \cdot \|Dg\|_{L^\infty(\Omega)}$$

and hence $u_\infty \in W^{1,\infty}(\Omega)$, as desired.