Lecture 5

In this lecture we consider the uniqueness theory of 2nd order fully nonlinear degenerate elliptic PDE

$$F(-,u,Du,D^2u) = 0, \quad \text{on } \Omega, \qquad (*)$$

for the Dirichlet problem. This is achieved by means of a <u>comparison principle</u> for sub- and super-solutions. Before doing that, we consider the existence method of Perron (adapted to $(*)$ by H.Ishii) which applies to $(*)$ when it satisfies the comparison principle. We refrain from giving proofs herein.

<u>Terminology</u>: The continuous $u \in C^0(\Omega)$ is a <u>Viscosity Subsolution</u> of $(*)$ when

$$(p,X) \in J^{2,+}u(x) \implies F(x,u(x),p,X) \geq 0.$$

We then write

$$F(-,u,Du,D^2u) \geq 0 \quad , \text{ on } \Omega. \qquad \odot$$

Similarly, u is a <u>Viscosity Supersolution</u> when

$$(p,X) \in J^{2,-}u(x) \implies F(x,u(x),p,X) \leq 0.$$

We write

$$F(-,u,Du,D^2u) \leq 0, \quad \text{on } \Omega. \qquad \odot$$

We say that <u>$(*)$ satisfies the Comparison Principle</u> when for any subsolution u and supersolution v

in $C^0(\overline{\Omega})$ for which $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω.

We begin with Perron's existence method.

Theorem [Perron method] Suppose that the Dirichlet problem for $(*)$:

$$\begin{cases} F(\cdot, u, Du, D^2u) = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega, \end{cases} \qquad (\star)$$

where $\Omega \Subset \mathbb{R}^n$, $g \in C^0(\partial\Omega)$, satisfies the Comparison Principle. Suppose also there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of $(\star)$ (that is, ⊙ and ⊙ hold and also $\overline{u}|_{\partial\Omega} = \underline{u}|_{\partial\Omega} = g$). Then,

$$\boxed{v(x) := \sup\Big\{ w(x) : \underline{u} \leq w \leq \overline{u} \ \& \ w \text{ subsolution of } (\star) \Big\}}$$

is a solution of $(\star)$.

The essential point in the proof (which we refrain from giving) is the next application of the Stability Theorem in the obvious 1-sided version of subsolutions.

Lemma. Let $\mathcal{U} \subseteq C^0(\Omega)$ be a set of subsolution to $(*)$. Then,

$$v := \sup\{u : u \in \mathcal{U}\}$$

is also a subsolution.

The proof is based on the following lemma, which is an extension of the approximation Lemma of jets.

Lemma. [1-sided approximation of jets]

Let $u \in USC(\Omega)$, that is u is Upper Semi-Continuous (which means $u(x) \geq \limsup_{z \to x} u(z)$, $\forall x \in \Omega$). Let also $(p, X) \in J^{2,+}u(x)$ for some $x \in \Omega$ and suppose there is $(u_m)_1^\infty \subseteq USC(\Omega)$ such that

$$(\star) \quad \boxed{\begin{cases} \forall y \in \Omega, \exists\, \tilde{y}_m \to y : \; u_m(\tilde{y}_m) \longrightarrow u(y) \quad \text{and} \\ \forall\, y_m \to y, \; \limsup_{m\to\infty} u(y_m) \leq u(y). \end{cases}}$$

Then, there exist $(x_m, p_m, X_m) \longrightarrow (x, p, X)$ as $m \to \infty$ such that

$$(p_m, X_m) \in J^{2,+}u_m(x_m).$$

Remark: The condition $(\star)$ is known as Γ-CONVERGENCE of $-u_m$ to $-u$:

$$\boxed{-u_m \xrightarrow{\Gamma} -u, \quad \text{as } m \to \infty.}$$

Γ-convergence is introduced by De Giorgi as a type of convergence of functionals in Calculus of Variations and its utility is that minima converge to minima.

Accordingly, we have the following theorem, which

Remark:

The Perron method is a method of existence for equations which support the comparison principle and hence enjoy uniqueness.

Now we turn to the comparison principle for fully nonlinear elliptic (and parabolic) PDE.

Motivation and the C^2 case.

Let $u, v \in C^2(\Omega)$ be a subsolution and a supersolution of the PDE

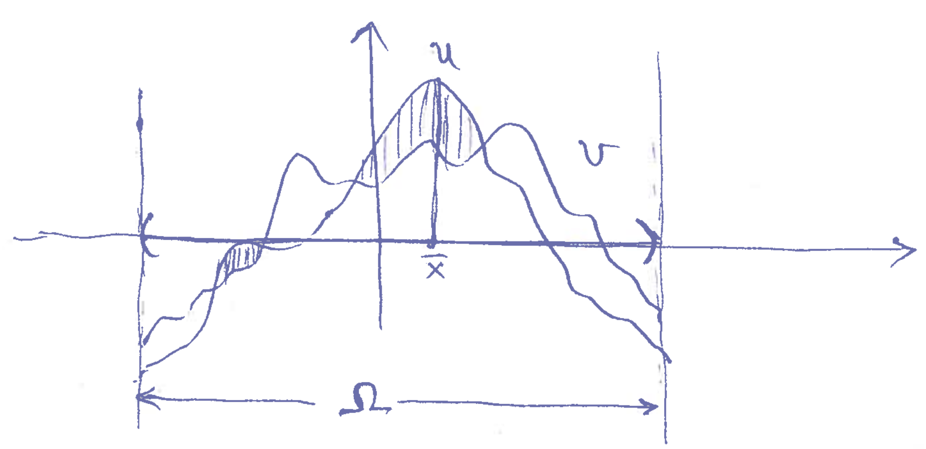
$$F(\cdot, u, Du, D^2u) = 0, \qquad (*)$$

and suppose $(*)$ is degenerate elliptic and $\Omega \Subset \mathbb{R}^n$. Suppose

$$u \le v \quad \text{on } \partial\Omega$$

and we need to show that

$$u \le v \quad \text{on } \overline{\Omega}.$$



Suppose for the sake of contradiction that $u > v$ somewhere inside Ω. Then there is $\bar{x} \in \Omega$ such that $u - v$

has a maximum at $\bar{x}$. Then

$$D(u-v)(\bar{x})=0,$$
$$D^2(u-v)(\bar{x})\leq 0.$$

Suppose also that the function

$$r \mapsto F(x,r,p,X) \qquad (\triangle)$$

is strictly decreasing. Then, F is called proper. Since u is subsolution and v a supersolution, we have

$$\begin{aligned} F(\bar{x},u(\bar{x}),Du(\bar{x}),D^2u(\bar{x})) \geq 0 &\geq F(\bar{x},v(\bar{x}),Dv(\bar{x}),D^2v(\bar{x})) \\ &\geq F(\bar{x},v(\bar{x}),Du(\bar{x}),D^2u(\bar{x})). \end{aligned} \qquad (\#)$$

Since $u(\bar{x})>v(\bar{x})$, $(\#)$ and $(\triangle)$ give a contradiction. Hence, $u\leq v$ on $\overline{\Omega}$, as claimed.

Problem in the C^0 case:

The above technique does not apply to general viscosity solutions, since $J^{2,+}$ MAY BE EMPTY AT POINTS OF MAXIMA OF $u-v$.

Resolution:

The idea is to double the number of variables and instead of u,v, consider instead $w(x,y):=u(x)-v(y)$ on $\Omega\times\Omega\subseteq\mathbb{R}^n\times\mathbb{R}^n$ and then penalise the doubling, in order to reduce maxima on the diagonal $\{y=x\}\subseteq\Omega\times\Omega$. The tool is to consider $u(x)-v(y)-\frac{\alpha}{2}|x-y|^2$ and let $\alpha\to\infty$.

For simplicity we restrict to the case of decoupled dependence on x:

$$F(x,r,p,X) \equiv F(r,p,X) - f(x)$$

Theorem. [Comparison Principle for Viscosity Solutions]

Let $f \in C^0(\overline{\Omega})$, $F \in C^0(\mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n))$ and suppose F satisfies $(\Omega \Subset \mathbb{R}^n)$:

$$X \le Y \Rightarrow F(r,p,X) \le F(r,p,Y), \qquad \text{(I)}$$

$$r \le s \Rightarrow F(r,p,X) - F(s,p,X) \le \gamma(r-s). \qquad \text{(II)}$$

Then, if u a subsolution and v a supersolution (in $C^0(\overline{\Omega})$) of

$$\boxed{F(w, Dw, D^2 w) = f, \quad \text{on } \Omega,}$$

we have: $u \le v$ on $\partial\Omega$ implies $u \le v$ on $\overline{\Omega}$.

Sketch of Proof.

The first step is

Lemma: For $v, u \in C^0(\overline{\Omega})$ and $\alpha > 0$, set

$$M_\alpha := \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \left\{ u(x) - v(y) - \frac{\alpha}{2}|x-y|^2 \right\}.$$

Since $M_\alpha < \infty$, we can choose an asymptotically maximising sequence as $\alpha \to \infty$, say (x_α, y_α):

$$\lim_{\alpha \to \infty} \left(M_\alpha - \left[u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \right] \right) = 0.$$

Then, if $x_\alpha \to \hat{x}$ (along a sequence $(\alpha_k)_1^\infty$), we have

$$\lim_{\alpha\to\infty} \alpha |x_\alpha - y_\alpha|^2 = 0,$$

$$\lim_{\alpha\to\infty} M_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_\Omega \{u - v\}.$$

The 2nd step is to assume (for the sake of contradiction) that $u > v$ somewhere in Ω. Then,

$$M_\alpha \geq (u-v)(\bar{x}) \equiv \delta > 0$$

for some $\bar{x} \in \Omega$ and $\alpha > 0$. We now choose (x_α, y_α) in the "argmax set", that is

$$M_\alpha = u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 .$$

Since $M_\alpha > 0$ and $u - v \leq 0$ on $\partial\Omega$, necessarily x_α, y_α are not on $\partial\Omega$ (for α large). Hence, $(x_\alpha, y_\alpha) \in \Omega \times \Omega$.

Consider first the <u>simpler case when $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$</u>. Then, at maxima $(\hat{x}, \hat{y})$ of

$$(x,y) \mapsto u(x) - v(y) - \frac{\alpha}{2}|x-y|^2 =: W(x,y),$$

in $\Omega \times \Omega$, we have $DW(\hat{x},\hat{y}) = 0$, $D^2 W(\hat{x},\hat{y}) \leq 0$, that is

$$\begin{bmatrix} Du(\hat{x}) \\ -Dv(\hat{y}) \end{bmatrix} = \begin{bmatrix} \alpha(\hat{x} - \hat{y}) \\ \alpha(\hat{y} - \hat{x}) \end{bmatrix}, \qquad (\boxdot)$$

$$\begin{bmatrix} D^2 u(\hat{x}) & 0 \\ 0 & -D^2 v(\hat{y}) \end{bmatrix} \leq \alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. \qquad (\boxdot\boxdot)$$

By testing $(\boxed{\cdot\cdot})$ on vectors $(\xi,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$, we get

$$D^2u(\hat{x})\leq D^2v(\hat{y})$$

and also

$$Du(\hat{x})=Dv(\hat{y})=\alpha(\hat{x}-\hat{y}).$$

Then, (I) & (II) give the desired contradiction. For the general case we apply the following <u>deep</u> theorem:

<u>Theorem</u> [<u>Theorem on Sums</u>]

Let $u,v\in C^0(\Omega)$, $\Omega\subseteq\mathbb{R}^n$, $\Phi\in C^2(\Omega\times\Omega)$ and suppose

$$F(x,y):=u(x)-v(y)-\Phi(x,y)$$

has a local maximum $(\hat{x},\hat{y})\in\Omega\times\Omega$:

$$F\leq F(\hat{x},\hat{y})\,,\quad\text{near }(\hat{x},\hat{y}).$$

Then, for $\varepsilon>0$, there are $X,Y\in\mathbb{S}(n)$ such that

$$\big(D_x\Phi(\hat{x},\hat{y}),X\big)\in\overline{J}^{2,+}u(\hat{x})\,,$$
$$\big(D_y\Phi(\hat{x},\hat{y}),Y\big)\in\overline{J}^{2,-}v(\hat{y})\,,$$

and

$$\begin{pmatrix}X & 0\\ 0 & Y\end{pmatrix}\leq D^2_{(x,y)}\Phi(\hat{x},\hat{y})+O(\varepsilon).$$

In the last statement, $\overline{J}^{2,\pm}$ are the <u>JET CLOSURES</u>:

$$\overline{J}^{2,+}u(x):=\Big\{(p,X)\;\Big|\;\exists\,(x_m,p_m,X_m)\to(x,p,X):(p_m,X_m)\in J^{2,+}u(x_m)\Big\}$$

The sketch of the proof is complete. $\square$