LAB2: 1-D TRANSPORT AND WAVE EQUATIONS

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Problem 1. We consider the following transport equation:

$$u_t + u_x = 0, \quad x \in (0,1),\ t \in (O,T), \quad u(x,0) = u^0(x), \tag{0.1}$$

with null boundary condition on the left endpoint

$$u(0,t) = 0, \quad t \in (0,T), \tag{0.2}$$

or periodic ones

$$u(0,t) = u(1,t), \quad t \in (0,T). \tag{0.3}$$

We consider the following three initial data:

$$u^0 = u^0_g(x) = \exp(-100(x-0.5)^2), \quad u^0 = u^0_s(x) = \sin(\pi x) \text{ or } u^0 = u^0_c(x) = \chi_{[1/4,3/4]}(x).$$

Implement the following five fully discrete schemes of the transport equation:

- explicit backward Euler: $\frac{u_j^{k+1}-u_j^k}{dt} + \frac{u_j^k - u_{j-1}^k}{dx} = 0$, with both boundary conditions (0.2) and (0.3) and $\mu = 0.5$ and $\mu = 1.5$. Here, $\mu := dt/dx$ is the Courant-Friedrich-Lewy (CFL) number. You can vary the space mesh size dx in order to obtain more accuracy. You will observe convergence for $\mu = 0.5$ and instability for $\mu = 1.5$.
- implicit backward Euler: $\frac{u_j^{k+1}-u_j^k}{dt} + \frac{u_j^{k+1} - u_{j-1}^{k+1}}{dx} = 0$, with both boundary conditions (0.2) and (0.3) and $\mu = 0.5$ and $\mu = 1.5$. You will observe convergence for both values of the CFL number.
- explicit forward Euler: $\frac{u_j^{k+1}-u_j^k}{dt} + \frac{u_{j+1}^k - u_j^k}{dx} = 0$, with both boundary conditions (0.2) and (0.3) and $\mu = 0.5$. You will observe the blow-up of the solution, due to the instability of the forward scheme. The fact that for this scheme you have to move its location of the null boundary condition from the left extreme to the right one is an additional indication of the instability of the approximation.
- explicit centered Euler: $\frac{u_j^{k+1}-u_j^k}{dt} + \frac{u_{j+1}^k - u_{j-1}^k}{2dx} = 0$, with periodic boundary conditions and $\mu = 0.5$ and $\mu = 1.5$.
- implicit centered Euler: $\frac{u_j^{k+1}-u_j^k}{dt} + \frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2dx} = 0$, with periodic boundary conditions and $\mu = 0.5$ and $\mu = 1.5$.
- Crank-Nicolson: $\frac{u_j^{k+1}-u_j^k}{dt} + 0.5\frac{u_{j+1}^k - u_{j-1}^k}{2dx} + 0.5\frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2dx} = 0$, with periodic boundary conditions and $\mu = 0.5$ and $\mu = 1.5$.

Set $\mathbf{u}^k := (u_j^k)_j$. To pass from one time $t^k = kdt$ to the next one t^{k+1}, you have to write the scheme as $\mathbf{u}^{k+1} = A\mathbf{u}^k$ for the explicit schemes and $B\mathbf{u}^{k+1} = A\mathbf{u}^k$ for the explicit ones and to compute the solution at all times in a loop. Moreover, for the implicit scheme at any time step you have to invert the matrix B, using the command inv(B) or $B\backslash$.

To visualize the numerical results, you have several possibilities. Among them, I suggest two:

- A static graphical result. If you stored the solution in the matrix U, whose k-th column $U(:,k)$ is the vector representing the solution at time t^{k-1}, x is the vector containing the space grid points $x = (ndx)_{0\le n\le N+1}$ ($(N+1)dx = 1$) and t is the vector containing the time grid points $t = (t^k)_{0\le k\le K+1}$, $(K+1)dt = T$, then you use the commands:

  $[X, Te] = \text{meshgrid}(x,t)$ and $\text{mesh}(X, Te, U')$.

  The first one transforms the vectors x and t into the matrices X and Te of equal size having on raws/columns copies of x/t. The command mesh does parametric plots of the points

$(X(i,j), Te(i,j), U'(i,j))$, $1 \leq i \leq K+2$, $1 \leq j \leq N+2$ in the $3-d$ space. Here U' is the Matlab notation for the transpose of the matrix U.

- A moving plot for which you have two possibilities. A loop of $2-d$ plots, showing successively the solution at time t^j, $0 \leq j \leq K+1$:

```
figure(1)
for j=1:K+2
plot(x,U(:,j))
title(['The numerical solution at time t=' num2str(t(j))])
axis([0 1 0 1])
pause(0.5)
end
```

A second possibility is to produce a video, following the sequence of Matlab commands:

```
fig=figure;
aviobj = avifile('SolutionTransport.avi')
for j=1:lt
plot(x,U(:,j))
title(['The numerical solution at time t=' num2str(t(j))])
axis([0 1 0 1])
F = getframe(fig);
aviobj = addframe(aviobj,F);
end
close(fig)
aviobj = close(aviobj);
```

Problem 2. We consider the following wave equation:

$$u_{tt} - u_{xx} = 0, \quad x \in (0,1),\ t \in (O,T), \quad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \tag{0.4}$$

with null boundary conditions at the two endpoints

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{0.5}$$

We consider the following three initial data:

$$u^0 = u_g^0(x) = \exp(-100(x-0.5)^2), \quad u^0 = u_s^0(x) = \sin(\pi x) \text{ or } u^0 = u_p^0(x) = \begin{cases} 2x, & x \in (0,1/2) \\ 2(1-x), & x \in (1/2,1). \end{cases}$$

The initial velocity under consideration is $u^1 = 0$ or $u^1 = u_x^0$ or $u^1 = -u_x^0$.

Implement the following three fully discrete schemes of the wave equation:

- explicit leapfrog: $\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{dt^2} - \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{dx^2} = 0$, with $\mu = 0.5$, $\mu = 1$ and $\mu = 1.5$. Here, $\mu := dt/dx$ is the CFL number. You will observe convergence for $\mu = 0.5$ and $\mu = 1$ and instability for $\mu = 1.5$.
- implicit leapfrog: $\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{dt^2} - \frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{dx^2} = 0$, with $\mu = 0.5$, $\mu = 1$ and $\mu = 1.5$.
- implicit midpoint: $\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{dt^2} - 0.5\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{dx^2} - 0.5\frac{u_{j+1}^{k-1} - 2u_j^{k-1} + u_{j-1}^{k-1}}{dx^2} = 0$, with $\mu = 0.5$, $\mu = 1$ and $\mu = 1.5$.