LAB3: 2-D WAVES AND WAVE PACKETS

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Problem 1. We consider the following wave equation:

$$u_{tt} - u_{xx} = 0, \quad x \in (0,1),\ t \in (O,T), \quad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \tag{0.1}$$

with null boundary conditions at the two endpoints

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{0.2}$$

We consider the following initial data:

$$u^0(x) = \exp(-\gamma(x-0.5)^2)\exp(i\xi^0 x).$$

The initial velocity under consideration is $u^1 = 0$.

Here $\gamma = h^{-0.9}$ and $\xi^0 = \eta^0/h$, with $\eta^0 \in (0,\pi)$. Implement the following fully discrete scheme of the wave equation with these initial data and $\eta^0 = \pi/10$, $\eta^0 = \pi/2$, $\eta^0 = 3\pi/4$ and $\eta^0 = 9\pi/10$.

- implicit midpoint: $\frac{u_j^{k+1}-2u_j^k+u_j^{k-1}}{dt^2} - 0.5\frac{u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}}{dx^2} - 0.5\frac{u_{j+1}^{k-1}-2u_j^{k-1}+u_{j-1}^{k-1}}{dx^2} = 0$, with $\mu = 0.5$.

Task 1. For $h = 1/100$, plot the solution.

Task 2. For $\eta^0 = 9\pi/10$, take a vector of mesh sizes $h = 1/40, 1/60, 1/80, 1/100, 1/125, 1/150$ plot the observability constant

$$C = \frac{h\sum_{j=0}^{N}\left(\left|\frac{u_j^1-u_j^0}{dt}\right|^2 + \left|\frac{u_{j+1}^0-u_j^0}{h}\right|^2\right)}{dt\sum_{k=0}^{K+1}\left|\frac{u_N^k}{h}\right|^2}.$$

Problem 2: Implement the finite difference scheme for the 2-d wave equation:

$$u''_{j,k}(t) - \frac{u_{j+1,k}(t) - 2u_{j,k}(t) + u_{j-1,k}(t)}{h_x^2} - \frac{u_{j,k+1}(t) - 2u_{j,k}(t) + u_{j,k-1}(t)}{h_y^2} = 0, \quad 1 \le j \le N_x, \quad 1 \le k \le N_y,$$

with initial data

$$u^0(x) = \exp(-\gamma(x-0.5)^2 - \gamma(y-0.5)^2)\exp(i\xi^0 x + i\eta^0 y).$$

The initial velocity under consideration is $u^1 = 0$.

The simplest idea is to write the system as

$$U''(t) + AU(t) + U(t)B* = 0,$$

where $U(t) = (u_{j,k}(t))_{j,k}$. Then, since A and B are symmetric and positive definite matrices, one can decompose the matrices A like $A = P_A\Lambda_A(P_A)*$, where Λ_A is the diagonal matrix containing the eigenvalues of A and P_A is the matrix whose columns are the eigenvectors of A (similarly for B). Then by taking $V(t) = (P_A)*U(t)P_B$, V verifies the problem:

$$V''(t) + \Lambda_A V(t) + V(t)\Lambda_B = 0.$$

But this problem can be written as

$$V''(t) + \Lambda_{AB}.*V(t) = 0, \tag{0.3}$$

where $\Lambda_{AB}(j,k) = \Lambda_A(j,j) + \Lambda_B(k,k)$ and $C.*D = (c_{j,k}d_{j,k})_{j,k}$.

Observe that (0.3) is a decoupled system of $N_x \times N_y$ harmonic oscillators, whose solution is

$$V(t) = \sum_{\pm} \frac{1}{2}\Big(V^0 \pm V^1./i\sqrt{\Lambda_{AB}}\Big) \exp(\pm it\sqrt{\Lambda_{AB}}),$$

where $U^0 = (P_A) * U^0 P_B$ and $U^1 = (P_A) * U^1 P_B$.

Then $U(t) = P_A V(t)(P_B)*$.