SOME CONVERGENCE AND SUPERCONVERGENCE RESULTS IN FEM

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Basic Terminology

Consider the problem: Find $u \in V$ such that

$$a(u, v) = F(v) \qquad \forall v \in V \ , \qquad (*)$$

where V, a, F satisfy the assumptions of Lax-Milgram lemma.

For each finite-dimensional subspace $V_h \subset V$, the associated discrete solution $u_h \in V_h$ satisfies

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h \ . \qquad (\dagger)$$

Let there exist a family $\{V_h\}_{h \to 0}$ such that

$$\lim_{h \to 0} \|u - u_h\|_V = 0 \ ,$$

where $\| \cdot \|_V$ is the norm of the Hilbert space V. Then we say that the *associated family of discrete problems is convergent.*

Céa's Lemma

Céa's Lemma: *There exists a constant $C > 0$ independent of the subspace V_h such that*

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V \quad .$$

Consequently, a sufficient condition for convergence is that there exists a family $\{V_h\}_{h \to 0}$ of subspaces of the space V such that, for each $v \in V$,

$$\lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0 \ ,$$

i.e., the union $\bigcup_{h>0} V_h$ is dense in V w.r.t. the $\| \cdot \|_V$-norm.

P r o o f : Let w_h be an arbitrary element in V_h. It is clear that $a(u - u_h, w_h) = 0$. Hence, from the V-ellipticity and continuity of $a(\cdot, \cdot)$, we have for any $v_h \in V_h$ $(w_h = u_h - v_h)$, that

$$C_2 \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) =$$

$$= a(u - u_h, u - u_h) + a(u - u_h, u_h - v_h) =$$

$$= a(u - u_h, u - v_h) \leq C_1 \|u - u_h\|_V \|u - v_h\|_V$$

and the estimate in Céa's lemma follows with $C = C_1 / C_2$.

Céa's lemma shows that the problem of estimating the error $\|u - u_h\|_V$ is reduced to the following problem in approximation theory: Evaluate the distance

$$d(u, V_h) = \inf_{v_h \in V_h} \|u - v_h\|_V$$

between a function $u \in V$ and the subspace $V_h \subset V$.

Assuming appropriate smoothness of u and certain regularity of FE triangulations, we will show that the distance $d(u, V_h)$ can be bounded by a constant times h^q for some exponent $q > 0$ and for sufficiently small $h > 0$.

Definition: *A set of triangulations* $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ *of* $\overline{\Omega}$ *is said to be a family of triangulations if for any* $\varepsilon > 0$ *there exists* $\mathcal{T}_h \in \mathcal{F}$ *with* $h < \varepsilon$.

Definition: *If there exist* $h_0 > 0$ *and integers* $k, q \geq 0$ *such that, for any* $\mathcal{T}_h \in \mathcal{F}$ *with* $h \in (0, h_0)$, *it holds that*

$$\|u - u_h\|_{k,\Omega} \leq C(u)h^q \ ,$$

where $C(u)$ *is independent of* h, *then we say that the order (rate) of convergence, in the norm* $\|\cdot\|_{k,\Omega}$, *is* q.

Sometimes we may simply write $\|u - u_h\|_{k,\Omega} = \mathcal{O}(h^q)$ for $h \to 0$.

Mesh Regularity Definitions

Definition: *A family of triangulations* $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ *is said to be* *regular* *if there exists a constant* $\kappa > 0$ *such that for any triangulation* $\mathcal{T}_h \in \mathcal{F}$ *and for any element* $K \in \mathcal{T}_h$ *there exists a ball* $\mathcal{B}_K \subset K$ *of radius* ϱ_K *such that*

$$\frac{\varrho_K}{h_K} \geq \kappa \ ,$$

where $h_K = \operatorname{diam} K$*.*

If, moreover,

$$\frac{\varrho_K}{h} \geq \kappa$$

holds then the family is said to be *strongly regular.*

- Any strongly regular family is regular.
- The diameter of any element from any $\mathcal{T}_h$, which belong to a strongly regular family, must be proportional to h, i.e.

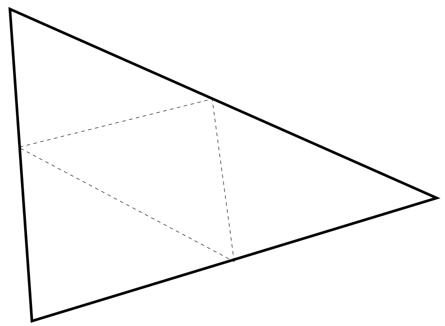
$$Ch \leq \operatorname{diam} K \leq h \quad \forall K \in \mathcal{T}_h.$$

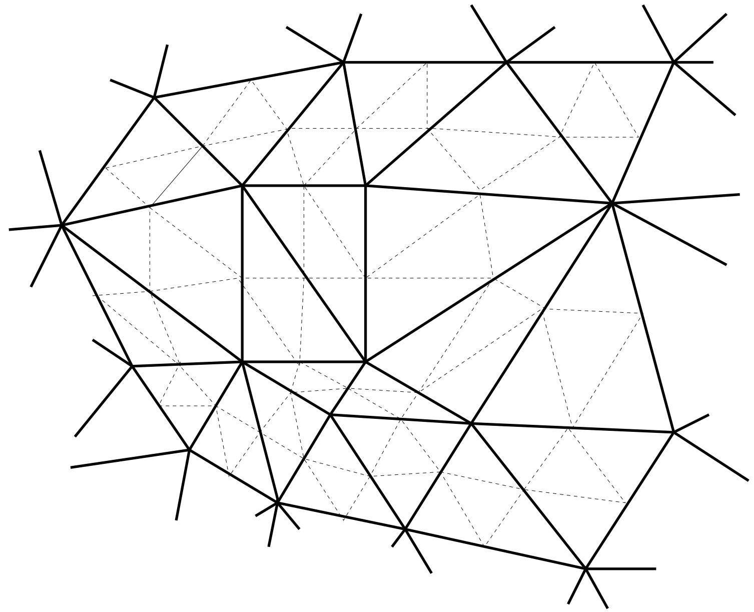
  However, this need not be true for regular families.
- Roughly speaking, the regularity of $\mathcal{F}$ only means that elements from $\mathcal{T}_h \in \mathcal{F}$ do not "degenerate" when $h \to 0$. It is not difficult to prove that the regularity of the family $\{\mathcal{T}_h\}_{h\to 0}$ of triangular elements is equivalent to *Zlámal's minimum angle condition:* There exists a constant α_0 such that

$$\alpha_K \geq \alpha_0 > 0$$

  for all $K \in \mathcal{T}_h$ and all the triangulations, where α_K denotes the smallest angle of K.

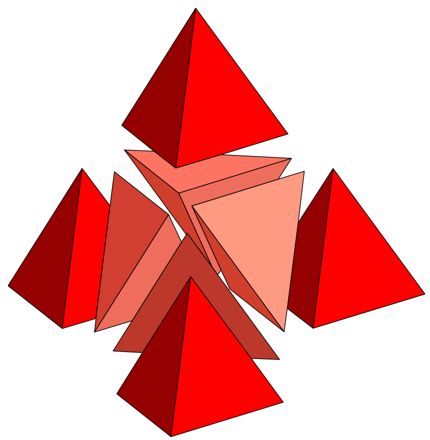
Some Construction of Regular Family of Triangulations

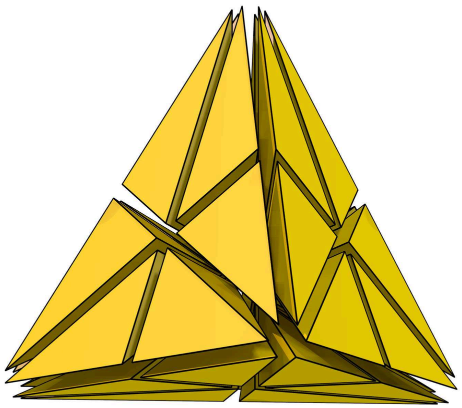


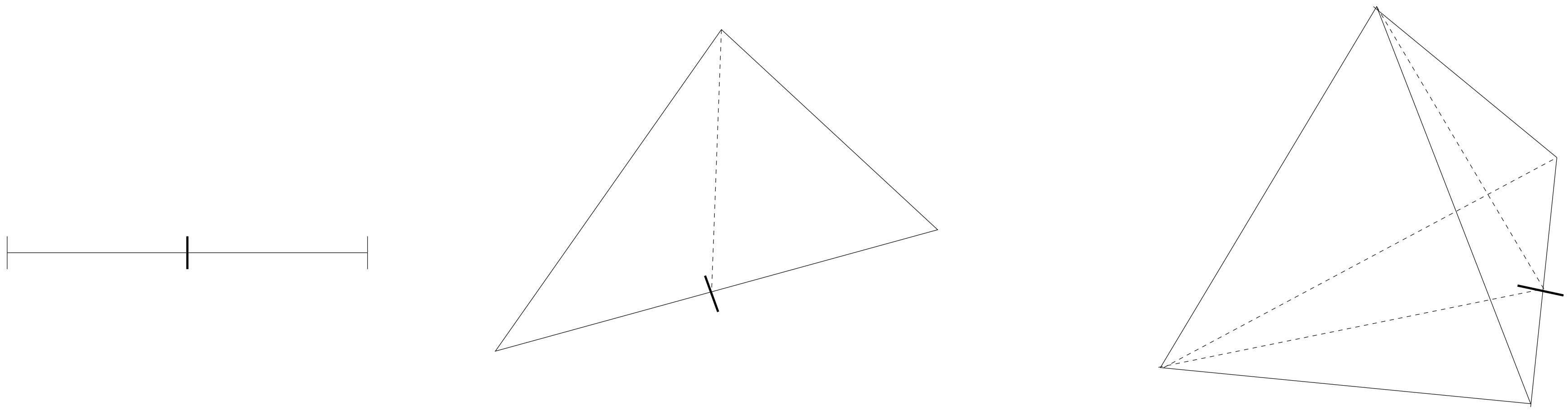


Usage of global 2D red refinement (infinite number of times) leads to a (strongly) regular family of triangulations due to the similarity effect.

3D Case, Bisections - No Similarity







However, there is a constructive proof of existence of a strongly regular family of face-to-face partitions of an arbitrary polyhedron into tetrahedra.

M. Křížek. An equilibrium finite element method in three-dimensional elasticity, Apl. Mat. 27 (1982), 46–75.

Note also that the regularity of a family of triangulations is only a sufficient condition for the convergence of u_h to u with respect to the energy norm. There are more general conditions guaranteeing the convergence, see lecture no. 3.

Bramble-Hilbert Lemma

The *seminorm* $|v|_{k,\Omega} := \left(\sum_{|i|=k} \|D^i v\|_{0,\Omega}^2 \right)^{1/2}$, $v \in H^k(\Omega)$.

Bramble-Hilbert Lemma: Let $\Omega \subset R^d$ be a bounded domain with a Lipschitz boundary and let Ψ be a continuous linear form on $H^{k+1}(\Omega)$ ($k \geq 0$ is an integer), i.e.,

$$|\Psi(v)| \leq c\|v\|_{k+1,\Omega} \qquad \forall v \in H^{k+1}(\Omega) .$$

If

$$\Psi(p) = 0 \qquad \forall p \in P_k(\Omega)$$

then there is a constant $C > 0$ such that

$$|\Psi(v)| \leq C\,|v|_{k+1,\Omega} \qquad \forall v \in H^{k+1}(\Omega) .$$

For the proof see [Ciarlet-1978].

In what follows, we shall shortly demonstrate how to derive the order of convergence in the $H^1(\Omega)$-norm when solving a second order elliptic problem by the linear triangular finite elements.

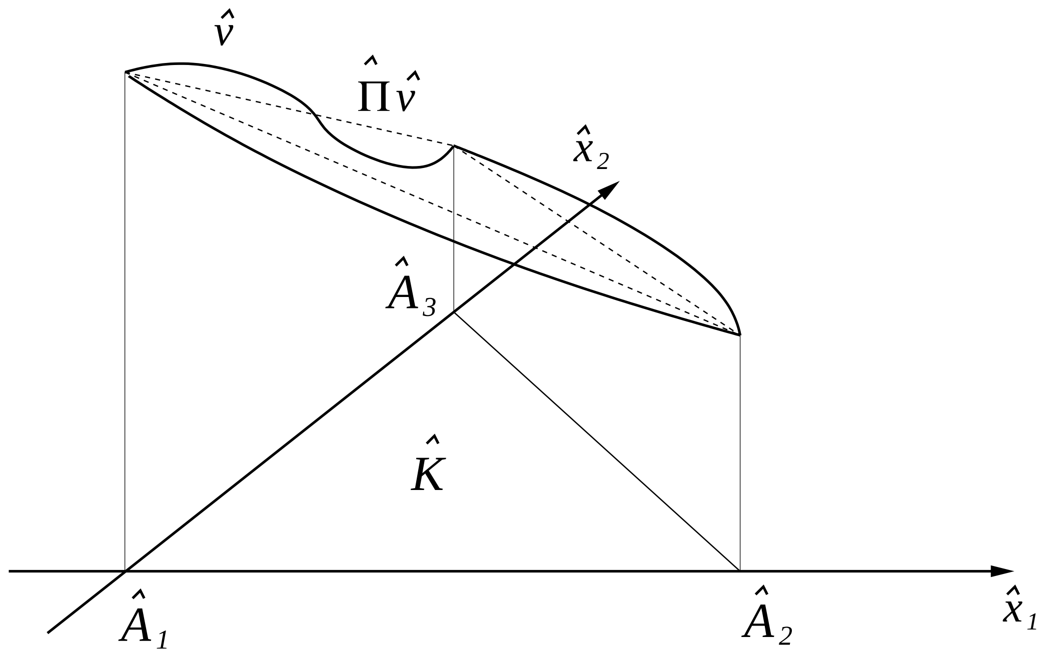
For $u \in H^2(\Omega)$ we shall construct an interpolation $\Pi_h u$ of u which belongs to V_h.

Then we will show that this interpolation is close to u with respect to the $H^1(\Omega)$-norm, and employ the Céa's lemma in order to estimate the error $u - u_h$ by $u - \Pi_h u$.

Let $\hat{K}$ be the *reference triangle* with vertices $\hat{A}_1 = (0,0)^T$, $\hat{A}_2 = (1,0)^T$, $\hat{A}_3 = (0,1)^T$. Any $\hat{v} \in H^2(\hat{K})$ is continuous and thus we can uniquely define a linear function $\hat{\Pi}\hat{v} \in P_1(\hat{K})$ by

$$(\hat{\Pi}\hat{v})(\hat{A}_i) = \hat{v}(\hat{A}_i) \ , \quad i = 1,2,3 \ , \quad \hat{v} \in H^2(\hat{K}) \ , \qquad (1)$$

where $\hat{\Pi}$ is called the *interpolation operator* and $\hat{\Pi}\hat{v}$ is called *$P_1(\hat{K})$-interponant of $\hat{v}$.*



Lemma 1: There exists a constant $C_1 > 0$ such that

$$\|\hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq C_1 \|\hat{v}\|_{2,\hat{K}} \qquad \forall \hat{v} \in H^2(\hat{K}) \ . \tag{2}$$

P r o o f : Since all the norms in each finite dimensional space are equivalent (here it is the space $P_1(\hat{K})$), we have

$$\|\hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq C_2 \|\hat{\Pi}\hat{v}\|_{C(\hat{K})} \qquad \forall \hat{v} \in H^2(\hat{K}) \ . \tag{3}$$

From (1) and Sobolev imbedding theorem[a] it follows that

$$\|\hat{\Pi}\hat{v}\|_{C(\hat{K})} \leq \|\hat{v}\|_{C(\hat{K})} \leq C_3 \|\hat{v}\|_{2,\hat{K}} \tag{4}$$

since $\hat{\Pi}\hat{v}$ is linear. Now using (3) and (4) we produce (2). ■

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[a] We have algebraic imbedding $H^2(\Omega) \subset C(\overline{\Omega})$, and also the topological imbedding $\|v\|_{C(\overline{\Omega})} \leq C\|v\|_{H^2(\Omega)}$ for $d = 2$.

Lemma 2: There exists a constant $C_4 > 0$ such that

$$\|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq C_4 \, |\hat{v}|_{2,\hat{K}} \qquad \forall \hat{v} \in H^2(\hat{K}) \; . \qquad (5)$$

P r o o f : Let us set the form

$$\Psi(\hat{v}) = \left(\hat{v} - \hat{\Pi}\hat{v}, \frac{\hat{\psi}}{\|\hat{\psi}\|_{1,\hat{K}}} \right)_{1,\hat{K}} \; , \qquad \hat{v} \in H^2(\hat{K}) \; , \qquad (6)$$

where $\hat{\psi} \in H^1(\hat{K})$ is fixed for a while and $\hat{\psi} \neq 0$.

We immediately find that

$$\Psi(\hat{p}) = 0 \qquad \forall \hat{p} \in P_1(\hat{K}) \; .$$

Using the Cauchy-Schwarz inequality, (6) and Lemma 1, we obtain

$$|\Psi(\hat{v})| \leq \|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq \|\hat{v}\|_{1,\hat{K}} + \|\hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq (1 + C_5)\|\hat{v}\|_{2,\hat{K}}$$

and thus by the Bramble-Hilbert lemma

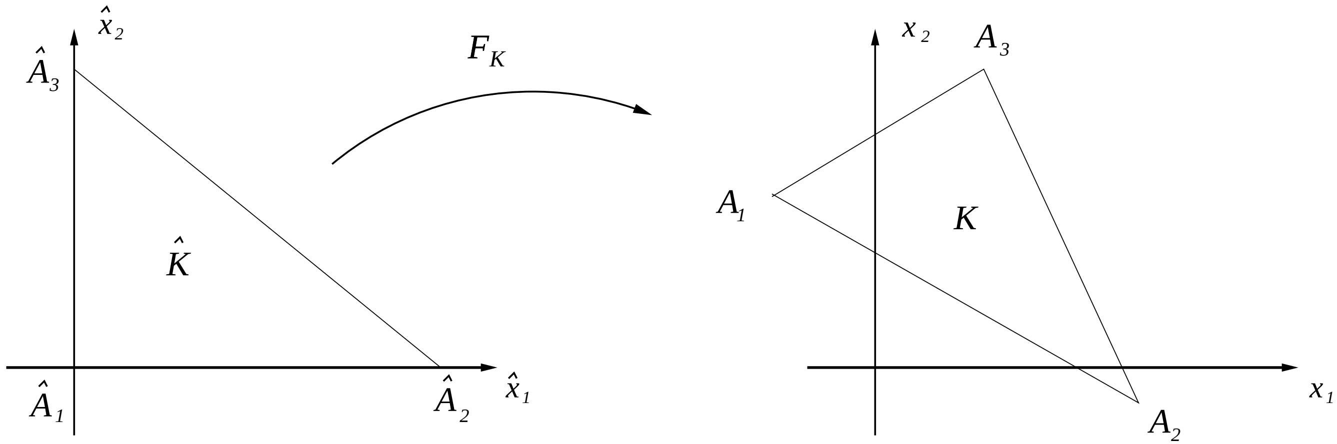
$$|\Psi(\hat{v})| \leq C_6 \, |\hat{v}|_{2,\hat{K}} \quad \forall \hat{v} \in H^2(\hat{K}) \ .$$

Finally we observe that

$$\|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} = \sup_{\hat{\psi} \neq 0, \ \hat{\psi} \in H^1(\hat{K})} \frac{\left| \left(\hat{v} - \hat{\Pi}\hat{v}, \hat{\psi} \right)_{1,\hat{K}} \right|}{\|\hat{\psi}\|_{1,\hat{K}}} \leq C_6 \, |\hat{v}|_{2,\hat{K}} \ \ .$$

■

Consider now an arbitrary triangle K with vertices $A_i, \;\; i = 1, 2, 3.$



Define an affine one-to-one mapping $F_K : \hat{K} \to K$ by

$$F_K(\hat{x}) = B_K \hat{x} + A_1 \; , \qquad \hat{x} = (\hat{x}_1, \hat{x}_2)^T \in \hat{K} \; , \tag{7}$$

where $B_K = (A_2 - A_1, A_3 - A_1)$ as $F_K(\hat{A}_i) = A_i, \; i = 1, 2, 3$, and B_K is a nonsingular 2×2 matrix (as A_i are not lying in a straight line).

For every $v \in L^2(K)$ and a.e. $\hat{x} \in \hat{K}$ let us set

$$\hat{v}(\hat{x}) = v(x) \ , \tag{8}$$

where $x = F_K(\hat{x})$. Thus we have a one-to-one correspondence between $\hat{v}$ and v.

Lemma 3: For any $k \in \{0, 1, 2, \dots\}$ there exists a constant $C_7 > 0$ such that for any triangle K and any $\hat{v} \in H^k(\hat{K})$, $v \in H^k(K)$ related via (8), it holds that

$$|v|_{k,K} \leq C_7 \|B_K^{-1}\|^k |\det B_K|^{1/2} \, |\hat{v}|_{k,\hat{K}} \ , \tag{9}$$

$$|\hat{v}|_{k,\hat{K}} \leq C_7 \|B_K\|^k |\det B_K|^{-1/2} \, |v|_{k,K} \ , \tag{10}$$

where $\|\cdot\|$ stands for the Euclidean norm.

For the proof see [Ciarlet-1978].

Now we can define the interpolant $\Pi_K v \in P_1(K)$ for $v \in H^2(K)$ by

$$(\Pi_K v)(A_i) = v(A_i) \ , \quad i = 1, 2, 3 \ . \tag{11}$$

Lemma 4: Let $\mathcal{F} = \{\mathcal{T}_h\}$ be a regular family of triangulations. Then there exists $C_8 > 0$ such that, for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$ with $h_K \leq 1$,

$$\|v - \Pi_K v\|_{1,K} \leq C_8 h_K |v|_{2,K} \qquad \forall v \in H^2(K) \ . \tag{12}$$

P r o o f : From (7) we obtain

$$\|B_K\| \leq 2\max(\|A_2 - A_1\|, \|A_3 - A_1\|) \leq 2h_K \ . \tag{13}$$

Since meas $K = \frac{1}{2}|\det B_K|$, it follows from the definition of mesh regularity that

$$\frac{1}{2}|\det B_K| = \text{meas } K \geq \text{meas } \mathcal{B}_K = \pi\varrho_K^2 \geq \pi\kappa^2 h_K^2$$

Denote by B_K^* the matrix of cofactors of entries of B_K. As the entries of B_K^* can be estimated by h_K, we arrive at

$$\|B_K^{-1}\| = \frac{1}{|\det B_K|}\|B_K^*\| \leq \frac{1}{2\pi\kappa^2 h_K^2} h_K \leq C_9 h_K^{-1} \ . \tag{14}$$

Writing $v - \Pi_K v$ in (9) instead of v, we get by (14) (with $k = 1, 0$) that

$$
\begin{aligned}
|v - \Pi_K v|_{1,K} &\leq C_7 \|B_K^{-1}\| |\det B_K|^{1/2} \|\hat{v} - \widehat{\Pi_K v}\|_{1,\hat{K}} \\
&\leq C_{10} h_K^{-1} |\det B_K|^{1/2} \|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} \ , \\
|v - \Pi_K v|_{0,K} &\leq |\det B_K|^{1/2} \left| \hat{v} - \widehat{\Pi_K v} \right|_{0,\hat{K}} \\
&\leq h_K^{-1} |\det B_K|^{1/2} \|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} \qquad (15)
\end{aligned}
$$

when $h_K \leq 1$. In above we use the fact that $\widehat{\Pi_K v} = \hat{\Pi}\hat{v}$.

Applying now (5), (10) (with $k=2$) and (13) in above, we get

$$\|v - \Pi_K v\|_{1,K} \leq C_{11} h_K^{-1} |\det B_K|^{1/2} \|\hat{v} - \hat{\Pi}\hat{v}\|_{1,\hat{K}} \leq C_{12} h_K^{-1} |\det B_K|^{1/2} |\hat{v}|_{2,\hat{K}}$$

$$\leq C_{13} h_K^{-1} |\det B_K|^{1/2} \|B_K\|^2 |\det B_K|^{-1/2} |v|_{2,K} \leq C_{14} h_K^{-1} h_K^2 |v|_{2,K} \ ,$$

which proves the lemma. ■

The seminorms occurring in (9) and (10) cannot be replaced by the norms. This makes clear why we have introduced the seminorm and used the Bramble-Hilbert lemma.

Theorem 1: Let $\mathcal{F} = \{\mathcal{T}_h\}$ be a regular family of triangulations of polygon $\bar{\Omega} \subset R^2$, let the solution $u \in V$ of problem $(*)$ belong to $H^2(\Omega)$ and let $V \subset H^1(\Omega)$ be equipped with the norm $\|\cdot\|_V = \|\cdot\|_{1,\Omega}$. Then there exist constants $h_0, C > 0$ such that, for any $\mathcal{T}_h \in \mathcal{F}$ with $h \in (0, h_0)$, the following error estimate holds

$$\|u - u_h\|_{1,\Omega} \leq Ch\, |u|_{2,\Omega} \ , \qquad (16)$$

where the discrete solution u_h of problem $(\dagger)$ is from the space of linear elements

$$V_h = \{v \in V \mid \ v|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\} \ . \qquad (17)$$

P r o o f : Let us define the linear interpolant $\Pi_h u \in V_h$ of u over the whole domain Ω by

$$\Pi_h u|_K = \Pi_K u \qquad \forall K \in \mathcal{T}_h \ .$$

Squaring and summing (12) for $v = u$, we see

$$\|u - \Pi_h u\|_{1,\Omega}^2 = \sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{1,K}^2 \leq C_{15} \sum_{K \in \mathcal{T}_h} h_K^2 \, |u|_{2,K}^2 \leq C_{15} h^2 \, |u|_{2,\Omega}^2 \ .$$

Further, from Céa's lemma we get that

$$\|u - u_h\|_{1,\Omega} \leq C_{16} \|u - \Pi_h u\|_{1,\Omega} \leq Ch \, |u|_{2,\Omega} \ .$$

■

The next theorem gives a sufficient condition for convergence of linear finite element approximations when no regularity assumption on the solution $u \in V \subset H^1(\Omega)$ is prescribed.

Theorem 2: Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of a polygon $\bar{\Omega} \subset R^2$ and let the space $C^\infty(\overline{\Omega}) \cap V$ be dense in V with respect to the norm $\|\cdot\|_V = \|\cdot\|_{1,\Omega}$. Then

$$\lim_{h \to 0} \|u - u_h\|_{1,\Omega} = 0 \ ,$$

where the discrete solution u_h of (†) belongs to the space

$$V_h = \{v \in V \mid \ v|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\} \ .$$

P r o o f : Let $\varepsilon > 0$ be given. Because of the density assumption there exists $w \in C^\infty(\overline{\Omega}) \cap V$ which satisfies the inequality

$$\|u - w\|_{1,\Omega} \leq \frac{\varepsilon}{2} \ . \qquad (18)$$

Squaring and summing (12) for $v = w$, we get as in Theorem 1, that for $\Pi_h w \in V_h$,

$$\|w - \Pi_h w\|_{1,\Omega} \leq Ch|w|_{2,\Omega} \ .$$

Hence, for a sufficiently small h we obtain

$$\|w - \Pi_h w\|_{1,\Omega} \leq \frac{\varepsilon}{2} \ . \qquad (19)$$

Employing again Céa's lemma, (18), and (19), we find that

$$\|u - u_h\|_{1,\Omega} \leq C\|u - \Pi_h w\|_{1,\Omega} \leq C\left(\|u - w\|_{1,\Omega} + \|w - \Pi_h w\|_{1,\Omega}\right) \leq C\varepsilon \ ,$$

where C depends only on the bilinear form $a(\cdot, \cdot)$. ■

Aubin-Nitsche Trick

Here, using the so-called *Aubin-Nitsche trick,* we prove that $\|u - u_h\|_{0,\Omega} = \mathcal{O}(h^2)$ under some H^2-regularity assumption.

Theorem 3: Let the assumptions of Theorem 1 be satisfied and let $a(\cdot,\cdot)$ be symmetric. Let the solution z of the problem

$$a(z,v) = (e,v)_{0,\Omega} \qquad \forall v \in V$$

belong to $H^2(\Omega)$ for any $e \in L^2(\Omega)$ and

$$\|z\|_{2,\Omega} \le c\|e\|_{0,\Omega}\ , \tag{20}$$

where $c > 0$ is independent of e. Then there exist $h_0, C > 0$ such that, for any $\mathcal{T}_h \in \mathcal{F}$ with $h \in (0, h_0)$, the following error estimate holds

$$\|u - u_h\|_{0,\Omega} \le Ch^2|u|_{2,\Omega}\ .$$

P r o o f : For the error $e = u - u_h \in V \subset L^2(\Omega)$ we define $z \in V$ and $z_h \in V_h$ by

$$\begin{aligned} a(z, v) &= (e, v)_{0,\Omega} \qquad \forall v \in V \ , \\ a(z_h, v_h) &= (e, v_h)_{0,\Omega} \qquad \forall v_h \in V_h \ . \end{aligned} \tag{21}$$

As $z \in H^2(\Omega)$ we have, by Theorem 1 and (20),

$$\|z - z_h\|_{1,\Omega} \leq C_1 h \|z\|_{2,\Omega} \leq c C_1 h \|e\|_{0,\Omega} \ . \tag{22}$$

Owing to the symmetry of $a(\cdot, \cdot)$ we obtain

$$a(z_h, e) = a(u - u_h, z_h) = 0 \ , \tag{23}$$

where the last equality follows from $(*)$ and $(\dagger)$. Thus from (21), (23), (22), and (16) we get

$$\|u - u_h\|_{0,\Omega}^2 = \|e\|_{0,\Omega}^2 = (e, e)_{0,\Omega} = a(z, e) = a(z - z_h, e)$$

$$\leq C_2 \|z - z_h\|_{1,\Omega} \|e\|_{1,\Omega} \leq C_3 h \|e\|_{0,\Omega} C_4 h |u|_{2,\Omega} \ ,$$

which proves the theorem. ∎

On Optimality of Estimates

When solving the Poisson equation with the homogeneous Dirichlet boundary conditions in a convex polygon by linear elements, then

$$\|u - u_h\|_{L^p(\Omega)} = \begin{cases} \mathcal{O}(h^2) & \text{for } p \in [2, \infty)\ , \\ \mathcal{O}(h^2 |\ln h|) & \text{for } p = \infty\ , \end{cases}$$

$$\|\nabla u - \nabla u_h\|_{(L^p(\Omega))^2} = \mathcal{O}(h) \qquad \text{for } p \in [2, \infty)\ ,$$

provided the corresponding family of triangulations is strongly regular and the second derivatives of u belong to $L^p(\Omega)$. These error estimates are optimal (see [Rannacher, Scott]).

Increasing Accuracy of FE Approximations

During the development of FEM it has been found that the rate of convergence of FE approximations at some exceptional points of a solution domain exceeds the possible global rate. This phenomenon has become known as *superconvergence.*

A systematic study of superconvergence phenomena started in the 70-th. Initially only the superconvergence at nodal points, and also at the Gauss-Legendre, Jacobi and Lobatto, etc points was addressed.

However, at the present time, the term "superconvergence" is used in a more broad sense than before.

One may recover the Galerkin solution or its derivatives by means of various *post-processings* and produce an acceleration of convergence. This is also called superconvergence by many authors if the post-processing is *easily computable.* After such a post-processing, one often can get an increase of accuracy not only at some (isolated) points, but also in some subdomain (local superconvergence) or even in the whole solution domain (global superconvergence).

Extrapolation techniques are also efficient tools to increase the accuracy of FE approximations.

In what follows, we always assume that the family of triangulations used is regular. However, in order to obtain the highest possible order of convergence, one usually imposes extra assumptions upon the geometrical structure of triangulations and some regularity of the true solution. We shall demonstrate this in several examples.

Some Basic References on Superconvergence

M. Křížek, P. Neittaanmäki. Superconvergence phenomenon in the finite element method arising from averaging gradients, Numer. Math. 45 (1984), 105–116.

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J. Brandts, M. Křížek. Gradient superconvergence on uniform simplicial partitions of polytopes. IMA J. Numer. Anal. 23 (2003), 489–505.

L. B. Wahlbin. Superconvergence in Galerkin Finite Element Methods. Springer Lecture Notes in Mathematics 1605, Springer-Verlag New York, 1995.

Natural Superconvergence

Consider the one-dimensional Dirichlet problem

$$-(a(x)u')' + c(x)u = f(x) \ , \qquad x \in \Omega = (0,1) \ ,$$

$$u(0) = u(1) = 0 \ ,$$

where $a(x) \geq \text{const} > 0$ and $c(x) \geq 0$ for every $x \in \Omega$.

The functions a, c, f are supposed to be sufficiently smooth.

For a uniform partition $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$, let $h := x_{i+1} - x_i$ and $K_i := [x_i, x_{i+1}]$.

The Galerkin method finds

$$u_h \in V_h = \{v_h \in H_0^1(\Omega) \mid v_h|_{K_i} \in P_k(K_i),\ i = 0, \dots, m\}$$

so that

$$(au_h', v_h')_{0,\Omega} + (cu_h, v_h)_{0,\Omega} = (f, v_h)_{0,\Omega} \quad \forall v_h \in V_h \ .$$

For $k \geq 2$, this method exhibits $\mathcal{O}(h^{2k})$-superconvergence at nodal points

$$\max_{1 \leq i \leq m} |u(x_i) - u_h(x_i)| = \mathcal{O}(h^{2k})$$

when $u \in H^{k+1}((0,1))$. Moreover, on any segment K_i there are $k-1$ interior points (the Lobatto points), where $u - u_h$ is $\mathcal{O}(h^{k+2})$, i.e., one order better than the optimal global rate of convergence

$$\max_{x \in \overline{\Omega}} |u(x) - u_h(x)| = \mathcal{O}(h^{k+1}) \ .$$

At the k Gaussian points of each K_i, the derivative u_h' possesses $\mathcal{O}(h^{k+1})$-superconvergence instead of $\mathcal{O}(h^k)$.

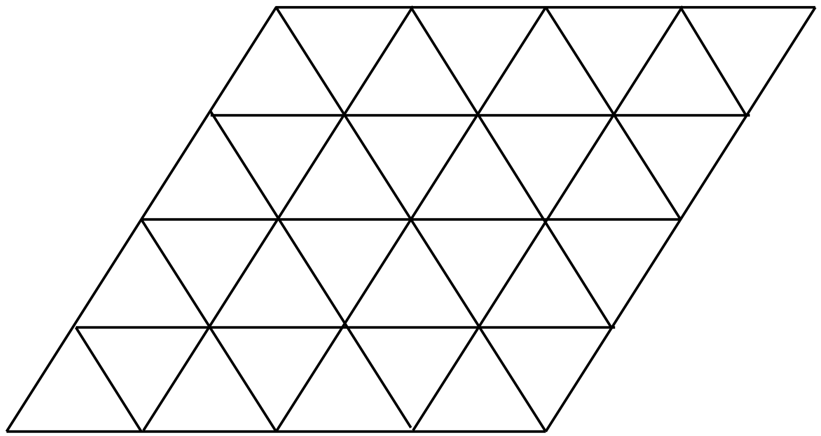
If $a \equiv 1$, $c \equiv 0$ and $k = 1$ then for arbitrary partition it may even be proved that $u(x_i) = u_h(x_i)$ for $i = 1, \ldots, m$, *i.e. there is no discretization error at nodes.*

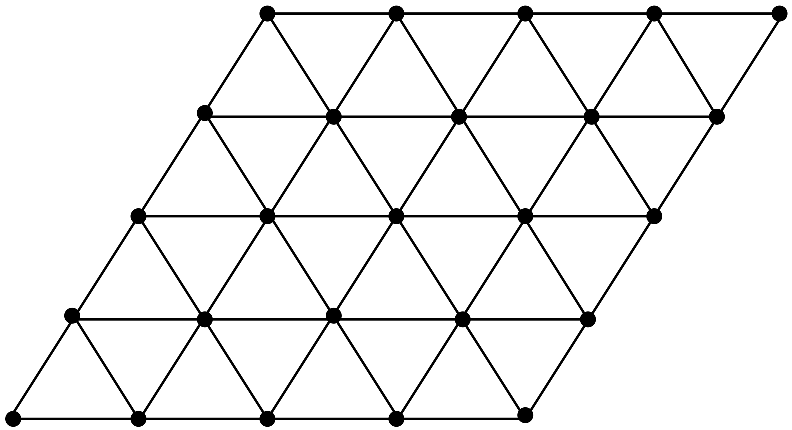
I. Hlaváček, M. Křížek. On exact results in the finite element method. Appl. Math. 46 (2001), 467 - 478.

Let $\overline{\Omega} \subset R^2$ be a convex polygon and let u be the variational solution of the problem

$$-\Delta u = f \quad \text{in} \quad \Omega\ , \qquad u = 0 \quad \text{on} \quad \partial\Omega\ .$$

Denote by $\mathcal{N}_h$ the set of nodal points of a triangulation $\mathcal{T}_h$ and let u_h be the corresponding Galerkin approximation based on the linear triangular elements. *Further, assume that $\mathcal{T}_h$ consists of equilateral triangles only.*





If u is smooth enough then

$$\max_{x \in \mathcal{N}_h} |u(x) - u_h(x)| = \mathcal{O}(h^4) \ .$$

Note that

$$\max_{x \in \overline{\Omega}} |u(x) - u_h(x)| = \mathcal{O}(h^2)$$

is the optimal global rate for this kind of triangulation.

For the quadratic triangular elements, i.e., when

$$u_h \in V_h = \{v_h \in H_0^1(\Omega) \mid v_h|_K \in P_2(K) \quad \forall K \in \mathcal{T}_h\} \ ,$$

an analogous result holds.

The rate of convergence at the nodal points and also at the midpoints of sides is $\mathcal{O}(h^4)$ for *uniform triangulations* (i.e., when any two adjacent triangles from $\mathcal{T}_h$ form a parallelogram) whereas the optimal global rate in the $C(\overline{\Omega})$-norm is only $\mathcal{O}(h^3)$.

Post-Processing

Solving the general elliptic problem for $d = 2$ by the standard linear triangular elements, we obtain

$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h) \ ,$$

provided all the assumptions of Theorem 1 are fulfilled. Hence,

$$\|\nabla u - \nabla u_h\|_{0,\Omega} = |u - u_h|_{1,\Omega} = \mathcal{O}(h)$$

and it can be shown that this estimate is optimal. For simplicity, assume that all $\mathcal{T}_h$ are uniform and that u_h is already known. We present a simple post-processing technique which recovers ∇u.

M. Křížek, P. Neittaanmäki. Superconvergence phenomenon in the finite element method arising from averaging gradients, Numer. Math. 45 (1984), 105–116.

I. Hlaváček, M. Křížek. On a superconvergent finite element scheme for elliptic systems, Parts I, II, III. Apl. Mat. 32 (1987).

Since ∇u_h is a piecewise constant vector field, we may define, at any nodal point $x \in \mathcal{N}_h$, the so-called *averaged gradient* by

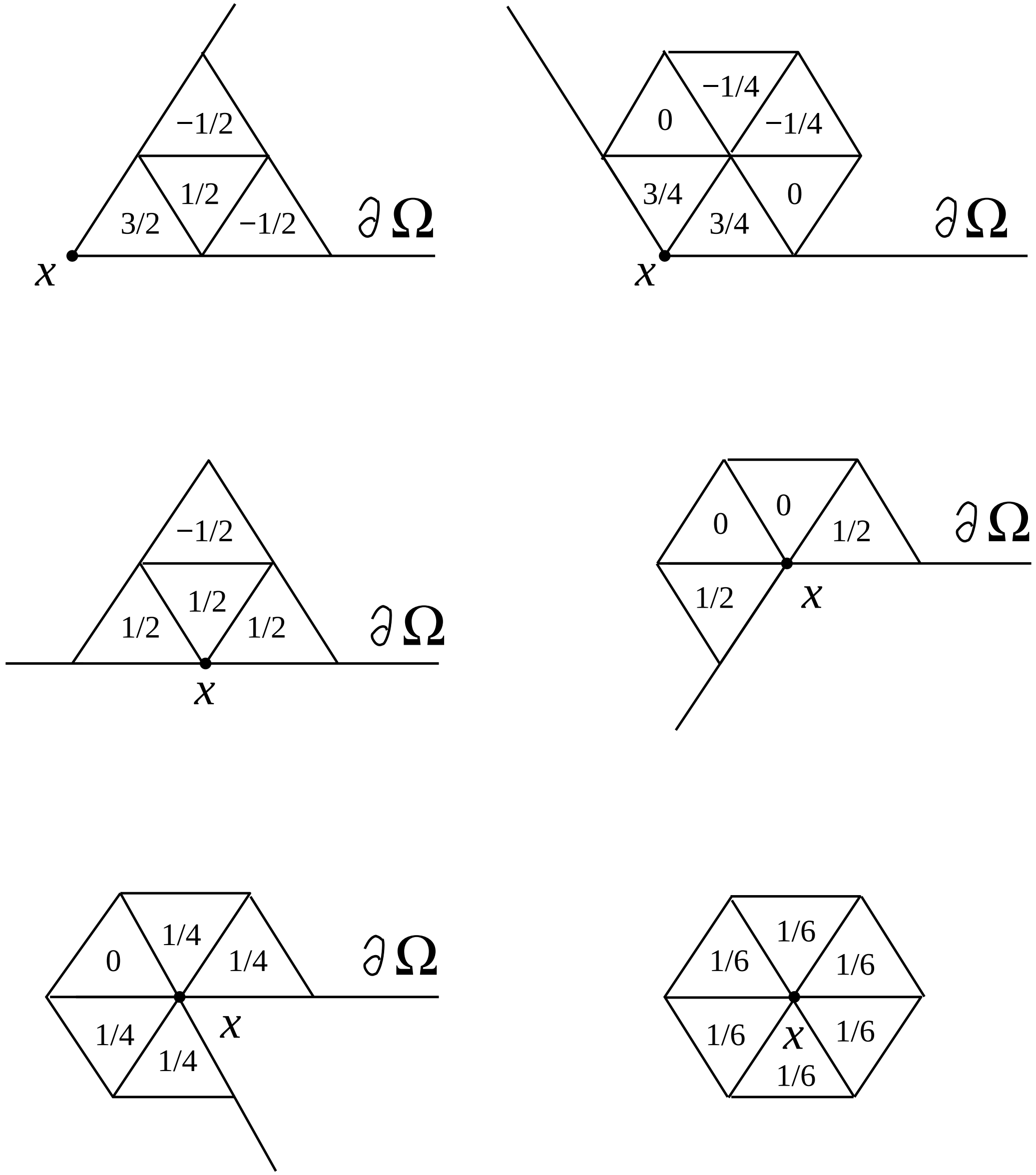
$$\big(G_h(\nabla u_h)\big)(x) = \sum_{j=1}^{m} w_j^x \nabla u_h|_{K_j^x} \ . \qquad (*)$$

Here $m = m(x) \in \{1, \ldots, 6\}$ is the number of triangles of $\mathcal{T}_h$ which "surround" the node x and w_j^x are weights (real numbers) whose values are also marked in the next figure in the corresponding triangles K_j^x.

For instance if $x \in \mathcal{N}_h$ is any interior nodal point of $\mathcal{T}_h$ then

$$\big(G_h(\nabla u_h)\big)(x) = \sum_{j=1}^{6} \frac{1}{6} \nabla u_h|_{K_j^x} \ ,$$

where K_j^x are the triangles surrounding the nodal point x.


−1/2
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x
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0
−1/4
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0
3/4
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1/2
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1/4
∂Ω
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x
1/4
1/6
1/6
1/6
1/6
x
1/6
1/6

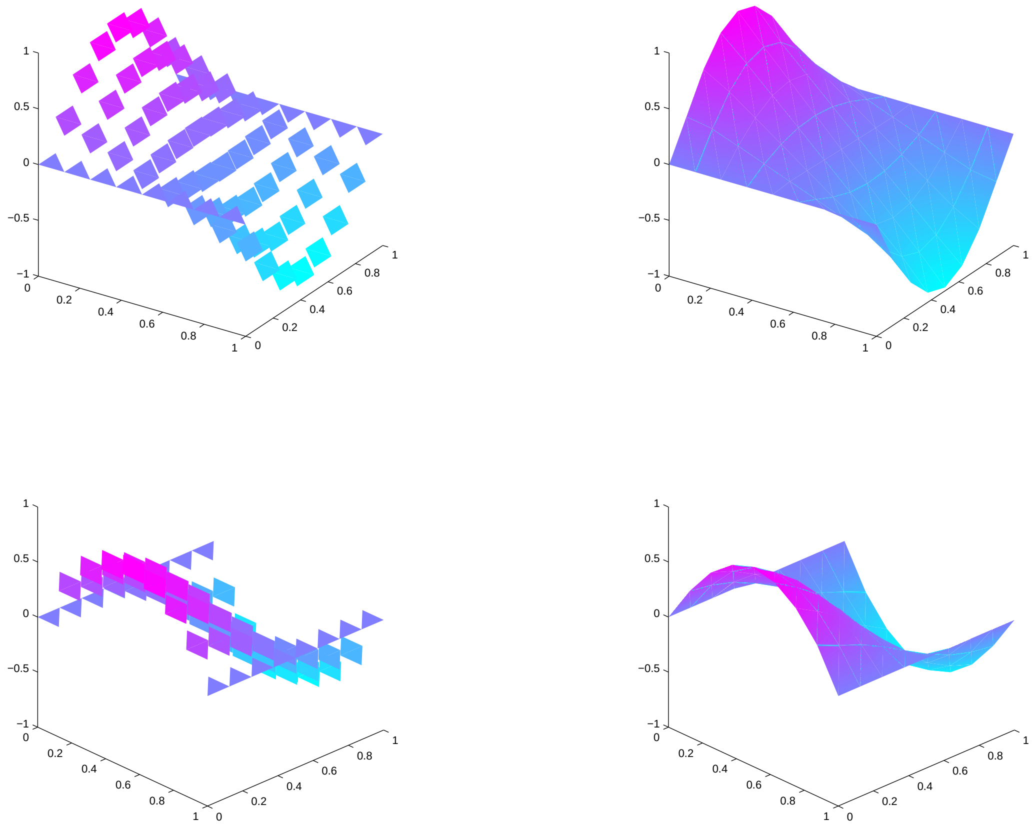
The relation $(*)$ enables us to define a piecewise linear continuous vector field still denoted by $G_h(\nabla u_h)$, for which it holds that

$$\|\nabla u - G_h(\nabla u_h)\|_{0,\Omega} = \mathcal{O}(h^2)$$

when $u \in H^3(\Omega)$.

Note that the weights w_j^x from the above figure are not uniquely determined. The error estimate in above remains valid also for some other values of w_j^x.

I. Hlaváček, M. Křížek. On a superconvergent finite element scheme for elliptic systems, Parts I, II, III. Apl. Mat. 32 (1987).



The x and y-components of the gradient ∇u_h and of the averaged gradient $G_h(\nabla u_h)$ of piecewise linear FE solution u_h on a uniform mesh.

Let us emphasize that the number of arithmetic operations for evaluating the averaged gradient by $(*)$ is asymptotically much smaller than for obtaining u_h. Only $\mathcal{O}(m)$ operations are necessary for computing $G_h(\nabla u_h)$ whilst $\mathcal{O}(m^2)$ are usually needed for u_h, where $m = \dim V_h$.

If the original field ∇u is continuous then for practical purposes it is, obviously, better to work with the continuous field $G_h(\nabla u_h)$ than the discontinuous field ∇u_h.

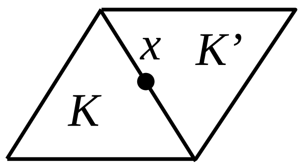
Let us mention that the post-processing yields quite good results even though the true solution is not in $H^3(\Omega)$.

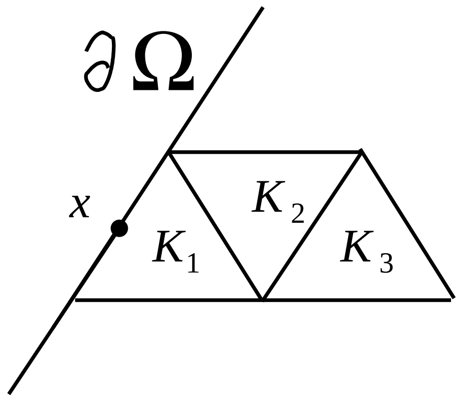
Another easily computable post-processing method for the linear triangular elements is based on the fact that the tangential component of ∇u_h is a superconvergent approximation to the tangential component of ∇u at the midpoints of all sides.

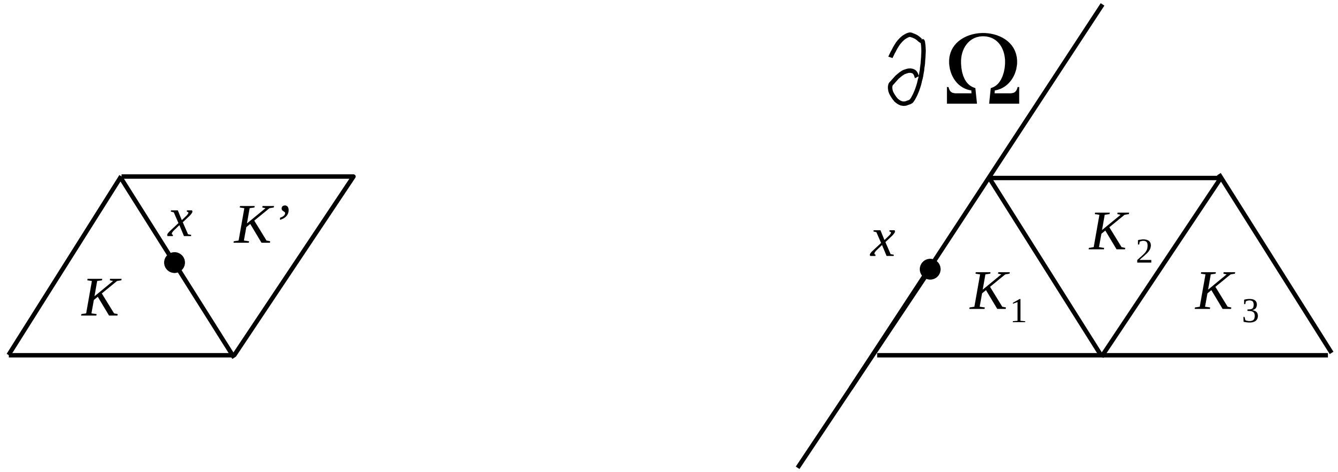
In order to recover also the normal component of the gradient we set

$$(g_h(\nabla u_h))(x) = \frac{1}{2}\nabla u_h|_K + \frac{1}{2}\nabla u_h|_{K'} \ ,$$

where $K,\ K'$ are adjacent triangles from a uniform triangulation $\mathcal{T}_h$ and x is the midpoint of their common side $K \cap K'$.







For midpoints lying on $\partial\Omega$ we set

$$(g_h(\nabla u_h))\,(x) = \nabla u_h|_{K_1} + \frac{1}{2}\nabla u_h|_{K_2} - \frac{1}{2}\nabla u_h|_{K_3}\ ,$$

where the position of the triangles K_1, K_2, K_3 is marked in above and x is the midpoint of the side $K_1 \cap \partial\Omega$.

Now for the above recovered point gradients we may uniquely determine the discontinuous piecewise linear vector field (still denoted by $g_h(\nabla u_h)$)

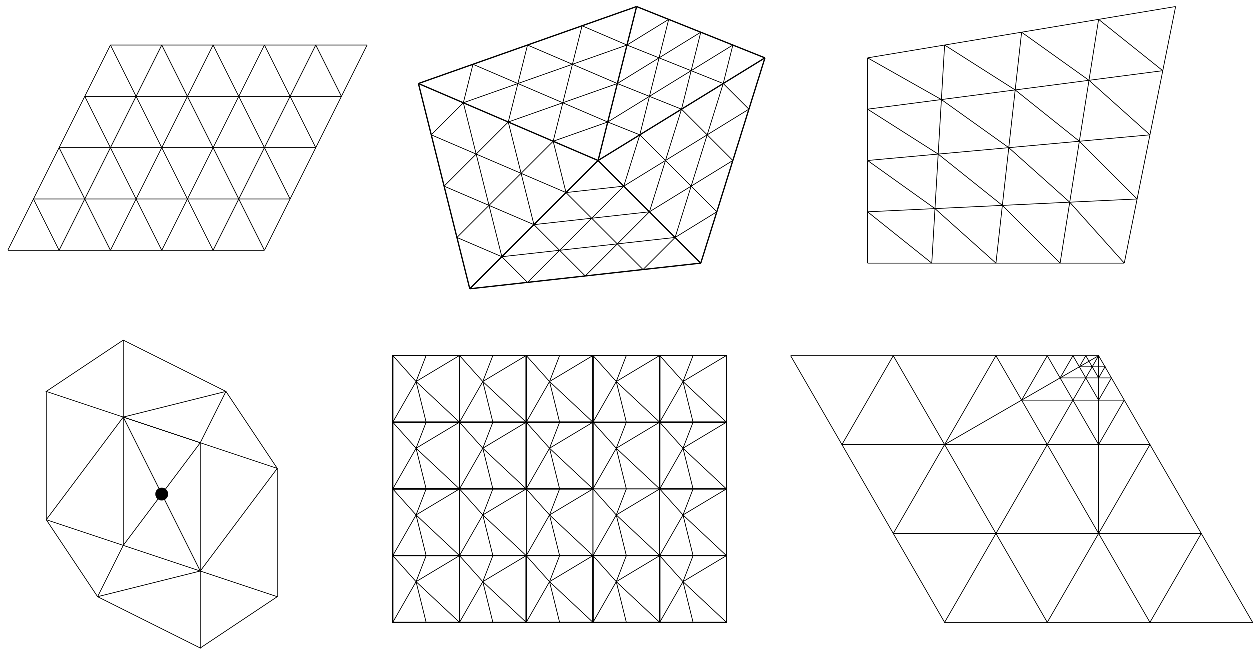
$$g_h(\nabla u_h) \in \left(L^\infty(\Omega)\right)^2 \ ,$$

in each triangle $K \in \mathcal{T}_h$ to be the linear interpolant to the three recovered gradient values at the midpoints. It may be proved again that

$$\|\nabla u - g_h(\nabla u_h)\|_{0,\Omega} = \mathcal{O}(h^2)$$

provided $u \in H^3(\Omega)$.

The averaging operators G_h and g_h can sucessfully be applied to several kinds of nonuniform triangulations:



Meshes exhibiting superconvergence phenomena: uniform, piecewice uniform, quasiuniform, locally symmetric, locally periodic, and self-similar.

However, modification of the operators G_h and g_h for an arbitrary irregular triangulation seems to be still an open problem (i.e. the choice of "optimal" weights).

The averaging operators G_h and g_h can be modified to elliptic systems with various boundary conditions, three-dimensional problems, time-dependent problems, etc.

Richardson Extrapolation

A suitable linear combination of Galerkin solutions corresponding to different triangulations may also increase the accuracy.

As an example consider the problem consisting of Poisson equation with Dirichlet boundary condition, solved by the linear triangular elements over uniform triangulations.

Put

$$\tilde{u}_h = \frac{1}{3}(4u_{h/2} - u_h) \ ,$$

where $u_{h/2}$ is the Galerkin approximation over the refinement of $\mathcal{T}_h$ by midlines.

Using a certain expansion theorem it can be shown that

$$\max_{x \in \mathcal{N}_h} |u(x) - \tilde{u}_h(x)| = \mathcal{O}(h^4)$$

when u is smooth enough.

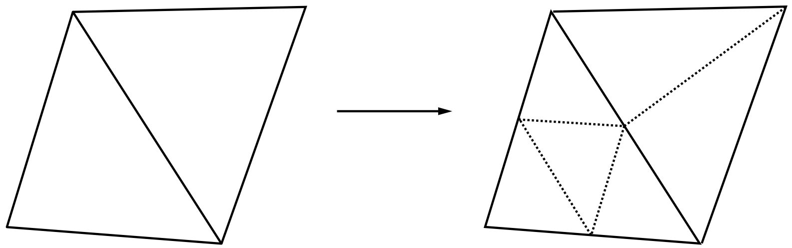
Hence, the accuracy of the extrapolation $(4u_{h/2} - u_h)/3$ is much better than that of $u_{h/2}$.

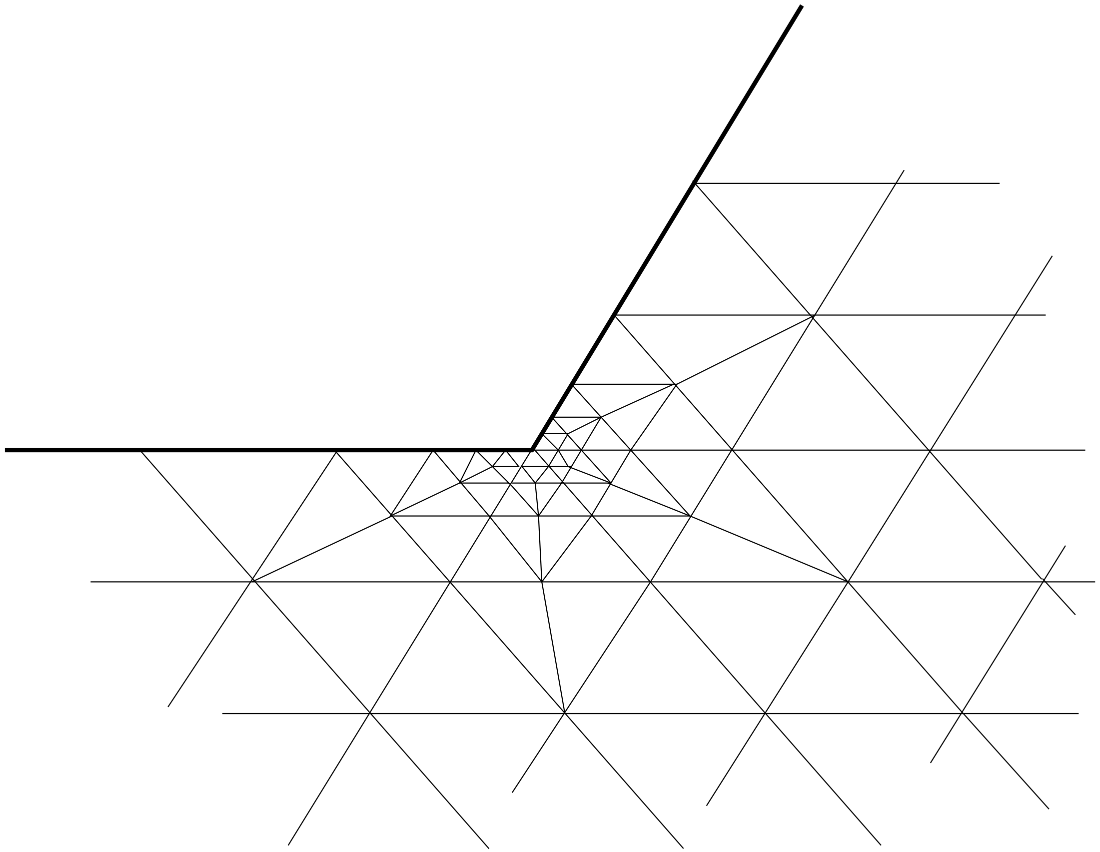
Adaptivity

The solution of BVP may involve the so-called point singularities. These occur e.g. at reentrant corners. Also at those points where one type of boundary condition changes into another, the solution need not be sufficiently smooth.

It is natural to make refinements close to “singular points” in order to increase the accuracy.

Figures in below illustrate a widely used technique for triangular elements based on a division by means of medians and midlines (red-green refinements).





Appropriate refinements should also be done at those parts of $\overline{\Omega}$ where the data (coefficients, right-hand side, boundary conditions) are changing considerably. At these parts we can expect big changes of the solution and its derivatives. Note that the quality of a finite element approximation can essentially depend upon the choice of a triangulation. For the linear triangular elements we have proved that

$$\|u - u_h\|_{1,\Omega}^2 \leq C \sum_{K \in \mathcal{T}_h} h_K^2 |u|_{2,K}^2$$

provided $u \in H^2(\Omega)$. So it is clear that we would like to balance the size of h_K with that of $|u|_{2,K}$. In particular, it is natural to choose h_K small, where $|u|_{2,K}$ is large. However, the behaviour of the solution u is a priori not known.

One way to overcome this difficulty is to use an adaptive triangulation refinement involving a posteriori error estimation. The domain Ω is initially triangulated and a local error in each element is evaluated. If, for a particular element, this is greater than a prescribed tolerance, then the elements (and adjacent elements) are subdivided, which causes a local refinement.