THE MAXIMUM ANGLE CONDITION IS NOT NECESSARY FOR CONVERGENCE OF FEM

Sergey Korotov

Basque Center for Applied Mathematics / IKERBASQUE

http://www.bcamath.org & http://www.ikerbasque.net

We show that the famous *maximum angle condition* in the finite element analysis is not necessary to achieve the optimal convergence rate when simplicial finite elements are used to solve elliptic problems. This condition is only sufficient. In fact, finite element approximations may converge even though some dihedral angles of simplicial elements tend to π.

A. Hannukainen, S. Korotov, M. Křížek. The maximum angle condition is not necessary for convergence of the finite element method. Numer. Math. 120 (2012), 79–88.

ANGLE CONDITIONS

Various angle conditions have several important roles in analysis of FEM. They enable us to derive the optimal order interpolation bounds and prove convergence of FEM, to derive various a posteriori error estimates, to perform regular mesh refinements, to preserve qualitative properties of smooth solutions in FE - simulations, etc.

• Note that only one obtuse triangle in a triangulation can completely destroy the discrete maximum principle.

In order to remind the situation with the convergence of FEM, we first consider a family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of conforming (face-to-face) triangulations of a polygonal domain.

In 1968 Miloš Zlámal introduced the following *minimum angle condition* which states that there should exist a constant α_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$0 < \alpha_0 \leq \alpha_K, \tag{1}$$

where α_K is the minimal angle of K.

Under this (sufficient) condition he derived the optimal order bounds of the interpolation error in the Sobolev H^1-norm (and H^2-norm) and therefore also of the discretization error for FEM applied to second (and fourth) order elliptic equation with some boundary conditions.

M. Zlámal. On the finite element method. Numer. Math. 12 (1968), 394–409.

The same condition was also introduced by Alexander Ženíšek for FEM applied to a system of linear elasticity equations of second order, published in 1969. This paper was submitted already on April 3, 1968, whereas Zlámal's paper on April 17, 1968.

Nevertheless, condition (1) is known as *Zlámal's minimum angle condition*, since paper by Ženíšek was published in Czech.

A. Ženíšek. The convergence of the finite element method for boundary value problems of a system of elliptic equations (in Czech). Apl. Mat. 14 (1969), 355–377.

The following four regularity conditions for families of simplicial partitions are commonly used in FE analysis. The constants c_i may depend on the dimension $d \in \{2, 3\}$. Let $h_K := \operatorname{diam} K$.

Condition 1 (minimum angle condition): There exists $c_1 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$, any $K \in \mathcal{T}_h$, and any dihedral angle α and, for $d = 3$, also any angle α within a triangular face of K, we have

$$\alpha \geq c_1. \tag{2}$$

Condition 2 (inscribed ball condition): There exists $c_2 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$ there exists a ball $b \subset K$ with radius r_K such that

$$r_K \geq c_2 h_K\,. \tag{3}$$

Condition 3: There exists $c_3 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$

$$\text{meas}_{\,d} K \geq c_3 \,\text{meas}_{\,d} B, \tag{4}$$

where $B \supset K$ is the circumscribed ball about K.

Condition 4: There exists $c_4 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $K \in \mathcal{T}_h$

$$\text{meas}_{\,d} K \geq c_4 h_K^d \,. \tag{5}$$

Theorem: The above four regularity conditions are equivalent for $d = 2, 3$.

J. Brandts, S. Korotov, M. Křížek. On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions. Comput. Math. Appl. 55 (2008), 2227–2233.

• There are also higher-dimensional (equivalent) analogues of Conditions 1–4.

In 1976, three research groups independently found that a weaker condition than the minimum angle condition can used in proofs of the optimal rate of the interpolation error which by the Céa's lemma yields also some rate of the discretization error.

They proposed the so-called *maximum angle condition:* There exists a constant γ_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$\gamma_K \leq \gamma_0 < \pi, \tag{6}$$

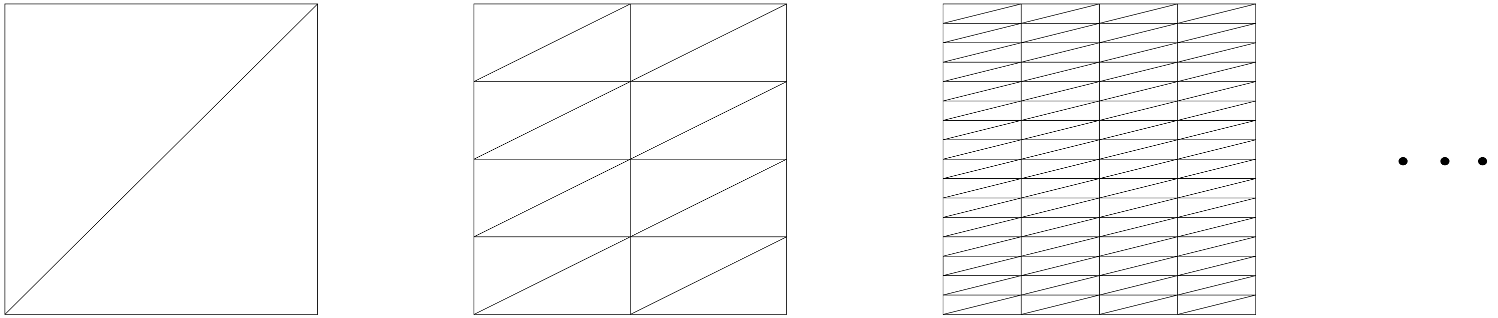
where γ_K is the maximum angle of K.

I. Babuška, A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.* 13 (1976), 214–226.

R. E. Barnhill, J. A. Gregory. Sard kernel theorems on triangular domains with applications to finite element error bounds. *Numer. Math.* 25 (1976), 215–229.

P. Jamet. Estimation de l'erreur pour des éléments finis droits presque dégénérés. *RAIRO Anal. Numér.* 10 (1976), 43–60.

Clearly, the minimum angle condition implies the maximum angle condition, since $\gamma_K \leq \pi - 2\alpha_K \leq \pi - 2\alpha_0 \equiv \gamma_0$. But the converse implication does not hold, see the figure in below.

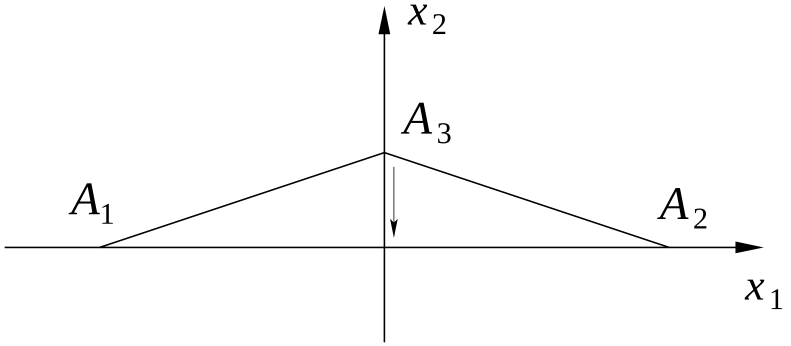


Note that John L. Synge already in 1957 proved the optimal order of nodal linear interpolation under condition (6), but without any application to FEMs.

J. L. Synge. The Hypercircle in Mathematical Physics. Cambridge Univ. Press, Cambridge, 1957.

In FEM literature there are examples showing that if the maximum angle condition does not hold then the linear triangular finite elements loose their optimal interpolation order.

The main idea: take $\varepsilon > 0$ and triangle K with $A_1 = (-1, 0)$, $A_2 = (1, 0)$, and $A_3 = (0, \varepsilon)$, hence, $\gamma_K \to \pi$ when $\varepsilon \to 0$.



I. Babuška, A. K. Aziz. On the angle condition in the finite element method. SIAM J. Numer. Anal. 13 (1976), 214–226.

G. Strang, G. Fix. An Analysis of the Finite Element Method. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973.

A. Ženíšek. The convergence of the finite element method for boundary value problems of a system of elliptic equations. Apl. Mat. 14 (1969), 355–377.

Consider the function $v(x_1, x_2) = x_1^2$ and its linear interpolant

$$(L_\varepsilon v)(x_1, x_2) = -\frac{x_2}{\varepsilon} + 1 \quad \text{on } K, \tag{7}$$

i.e.,

$$(L_\varepsilon v)(A_i) = v(A_i), \quad i = 1, 2, 3.$$

Using the standard Sobolev space notation, (7), and the fact $\frac{\partial v}{\partial x_2} = 0$, we find that

$$\|v - L_\varepsilon v\|_{1,K}^2 \geq \left| \frac{\partial L_\varepsilon v}{\partial x_2} \right|_{0,K}^2 = \frac{1}{\varepsilon^2} \operatorname{meas} K = \frac{1}{\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0. \tag{8}$$

We conclude that one badly shaped triangle in every triangulation $\mathcal{T}_h \in \mathcal{F}$ can yield an arbitrary large interpolation error in the Sobolev H^1-norm.

The maximum angle condition corresponding to 3D setting requires that all dihedral angles between faces and all angles between edges (within triangular faces) are bounded from above by a constant less than π. The optimal interpolation rate in the H^1-norm for linear elements is preserved under this condition.

M. Křížek. On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. 29 (1992), 513–520.

For tetrahedral elements similar examples as in (7) can also be constructed. Namely, if the maximal angle between two faces or the maximal angle between edges (within triangular faces) tends to π, then the interpolation error may tend to ∞ like in (8).

• Examples, analogous to the above mentioned, caused numerical analysts to believe that large angles of triangular elements (i.e., when the maximum angle condition is not satisfied) produce also large discretization error when solving second order elliptic problems by FEM. For instance, Babuška and Aziz state that the maximum angle condition is essential for convergence of FEM, whereas D'Azevedo and Simpson assert that it is necessary and sufficient for convergence.

I. Babuška, A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.* 13 (1976), 214–226.

E. F. D'Azevedo, R. B. Simpson. On optimal interpolation triangle incidences. *SIAM J. Sci. Statist. Comput.* 10 (1989), 1063–1075.

To the contrary, we show here that the finite element method may converge even when the maximum angle condition is violated for a quite large number of elements in the used partitions.

Let us emphasize that the Céa's lemma

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V$$

gives only an upper bound of the discretization error by means of the interpolation error.

Note that the discretization error can be, in some cases, of the same order as the interpolation error.

But in principle, the discretization error can also be much smaller than the interpolation error, as we will see later !

• First, we shall give illustrative two-dimensional examples showing that the theoretical and also practical convergence rate of the discretization error is of the optimal order $\mathcal{O}(h)$ in the H^1-norm even though the maximal angle over all triangles tends to π, i.e., the maximum angle condition is not necessary.

• Then we shall generalize some examples to simplicial elements of an arbitrary space dimension and introduce a more abstract formulation of our main result.

• Finally, we shall present some numerical results showing that large dihedral angles coming from 3D red refinement may, in general, deteriorate the convergence of FEM.

WHY IS THE MAXIMUM ANGLE CONDITION

NOT NECESSARY ?

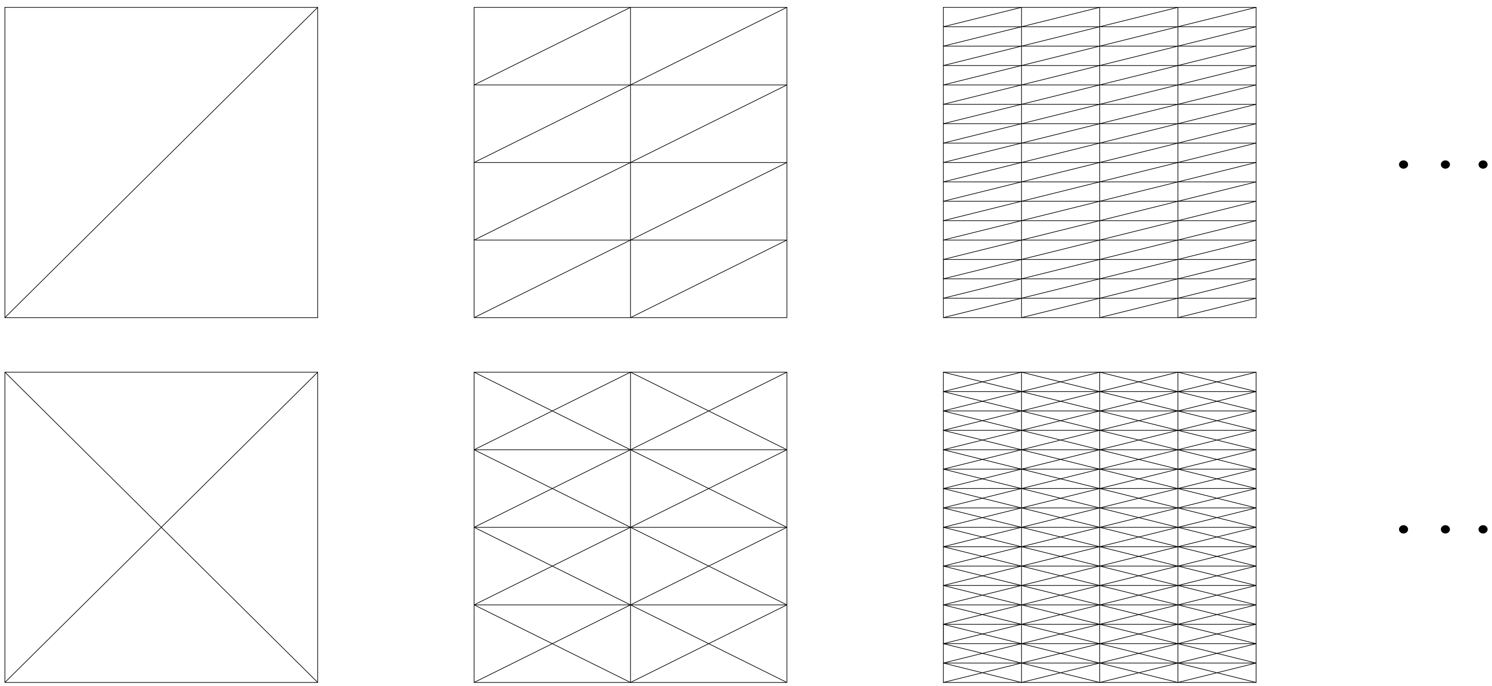
We now show that the discretization error can be very small, whereas the interpolation error is large.

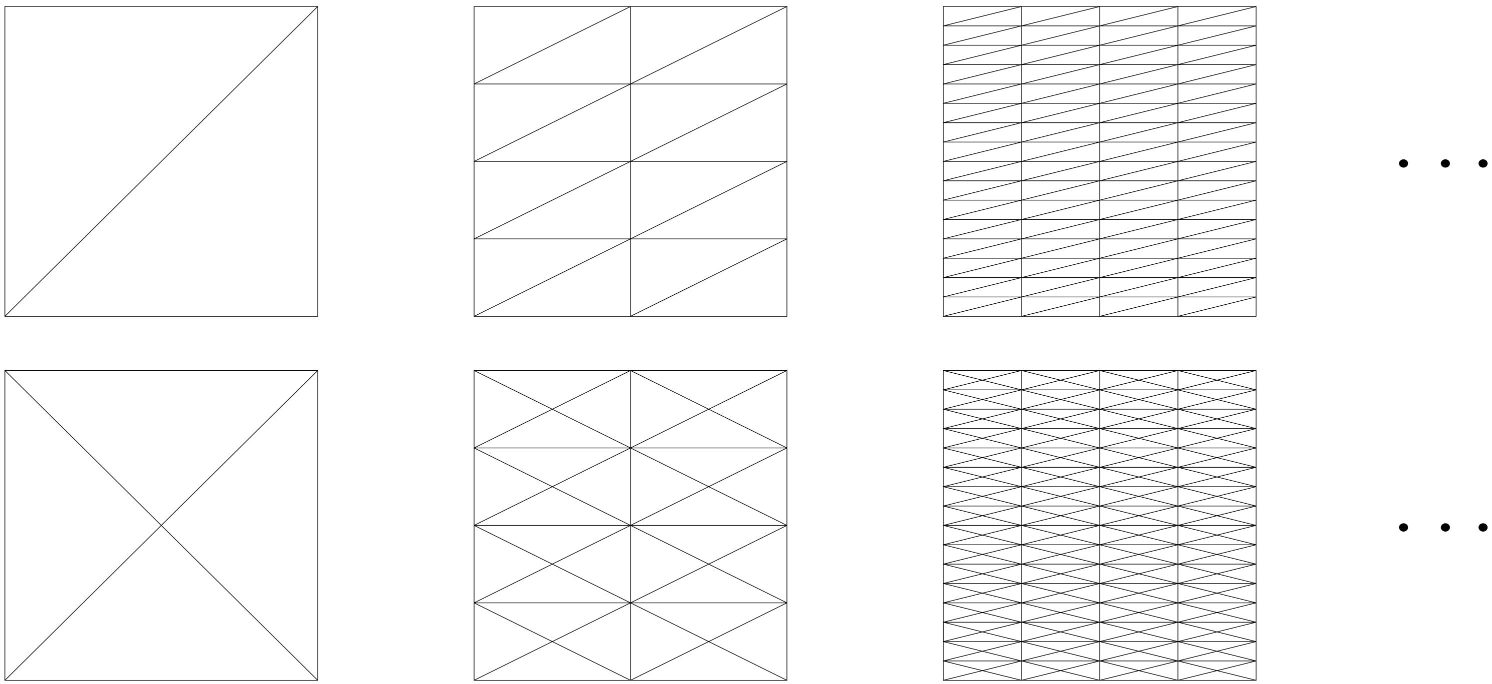
For simplicity, consider the Poisson equation with the homogeneous Dirichlet boundary conditions in the unit square $\Omega = (0, 1) \times (0, 1)$,

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{9}$$

where $f \in L^2(\Omega)$. Since Ω is convex, its weak solution is from the Sobolev space $H^2(\Omega)$ and thus continuous by the Sobolev imbedding theorem.

Example 1: Define two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of triangulations of $\overline{\Omega}$. To this end we first introduce uniform rectangular meshes of the given unit square consisting of congruent rectangles. Its horizontal sides are divided into 2^k equal parts and the vertical parts are divided into 4^k equal parts for $k = 0, 1, 2, \ldots$ To construct $\mathcal{F}_1$ we divide each rectangle by its diagonal with a positive slope, whereas for $\mathcal{F}_2$ we take both diagonals.





We observe that $\mathcal{F}_1$ satisfies the maximum angle condition with $\gamma_0 = \pi/2$ for all k, whereas for $\mathcal{F}_2$ we observe that $\gamma_K \to \pi$ for every second triangle from any $\mathcal{T}_h \in \mathcal{F}_2$. Let V_h and W_h be finite element spaces of continuous and piecewise linear functions over triangulations from $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Obviously,

$$V_h \subset W_h. \tag{10}$$

Denote by $u_h \in W_h$ the standard Galerkin approximation of the weak solution $u \in H^2(\Omega)$ of our PDE problem. Let $L_h u$ stands for the linear interpolant of u in V_h. Then by Céa's lemma and due to $V_h \subset W_h$ there exists a constant $C > 0$ such that

$$\|u - u_h\|_1 \le C \inf_{w_h \in W_h} \|u - w_h\|_1 \le C \inf_{v_h \in V_h} \|u - v_h\|_1 \le$$

$$\le C\|u - L_h u\|_1 \le C' h |u|_2 \text{ as } h \to 0, \qquad (*)$$

where the last inequality can be proved under the maximum angle condition for a constant $C' > 0$ independent of h.

• This example shows that the discretization error tends to 0 at least linearly in the H^1-norm even though the maximal angle of every second triangle from any $\mathcal{T}_h \in \mathcal{F}_2$ tends to π.

In Figure 1 we observe the practical rates of convergence on $\mathcal{F}_1$ and $\mathcal{F}_2$ for our test problem with RHS $f(x_1, x_2) = \pi^2 \sin \pi x_1 \sin \pi x_2$.

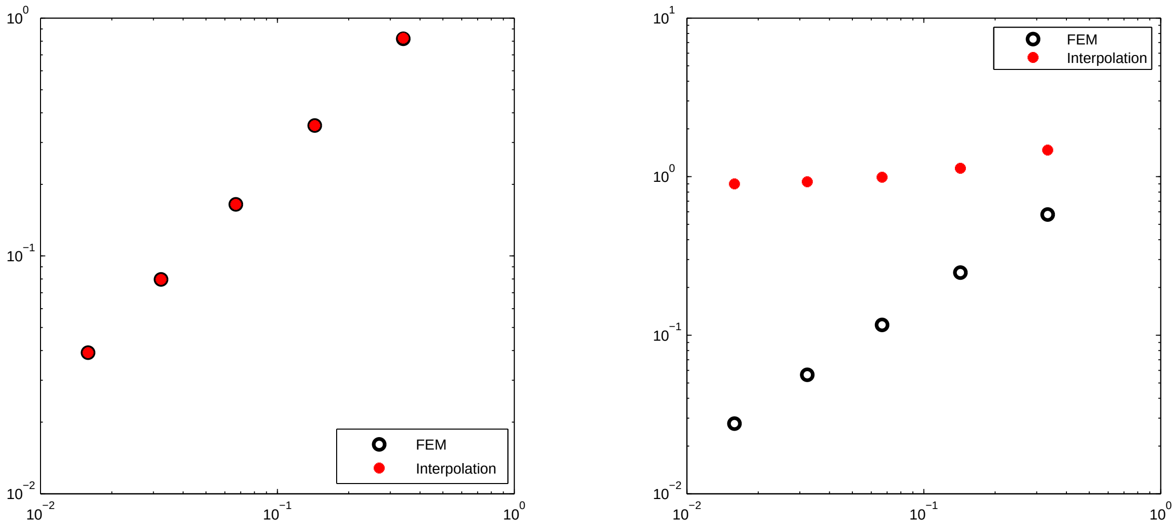
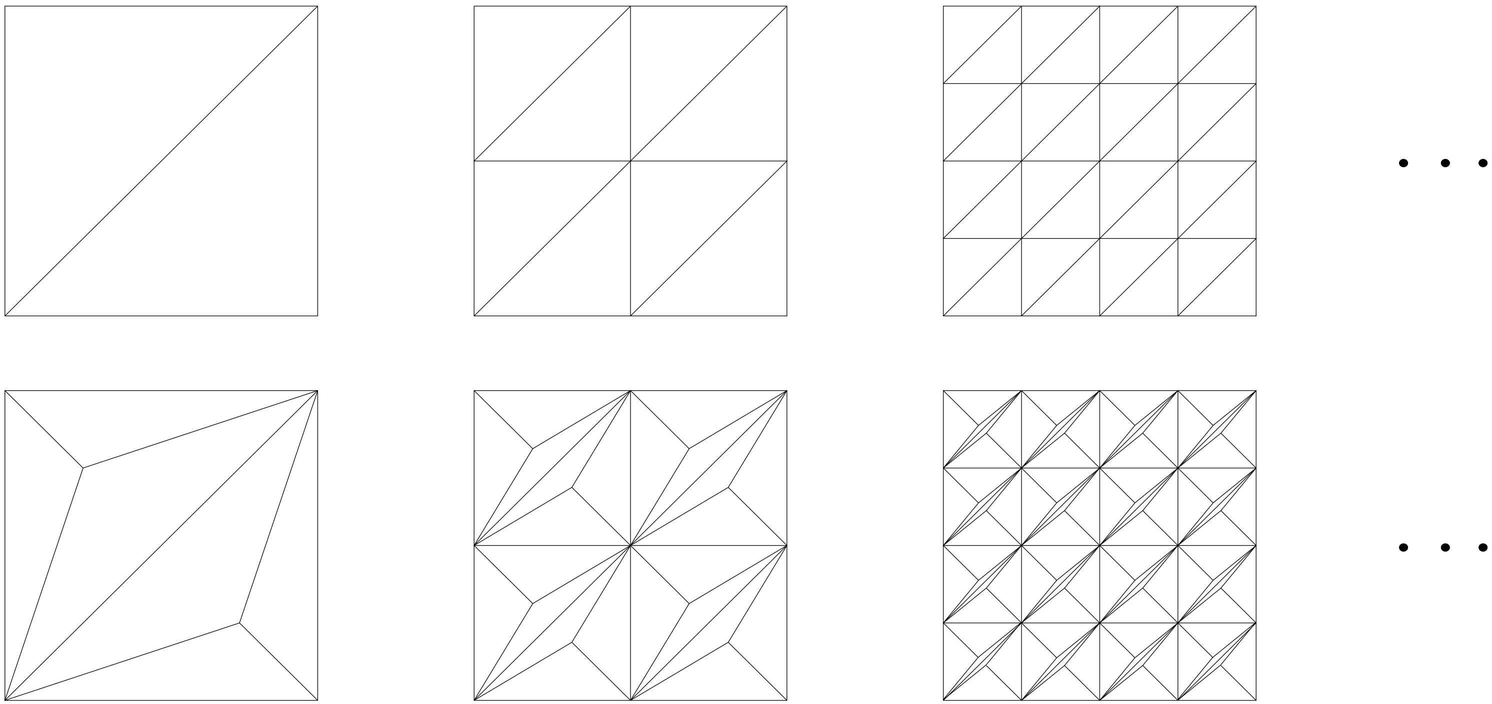


Figure 1: The practical convergence rates for the families $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right). The horizontal axis corresponds to the discretization parameter and the vertical axis corresponds to the H^1-norm of the discretization and interpolation errors. The difference between interpolation and discretization errors on the left figure is extremely small, which cannot be seen from the graph.

Example 2: Another supportive example is illustrated in below. In this case, the family $\mathcal{F}_3$ satisfies even the minimum angle condition and the maximal angle of every third triangle from any $\mathcal{T}_h \in \mathcal{F}_4$ tends to π. We can define V_h and W_h over triangulations from $\mathcal{F}_3$ and $\mathcal{F}_4$ as in the previous example so that $V_h \subset W_h$, and derive $(*)$ again.



SOME GENERALIZATIONS

• First, we give a natural generalization of Example 1 to arbitrary space dimension.

Let the unit d-cube $\Omega = (0,1)^d$, $d \in \{2,3,\dots\}$, be divided uniformly into congruent d-blocks. Consider for instance the d-block

$$B = (0,h_1) \times \cdots \times (0,h_d).$$

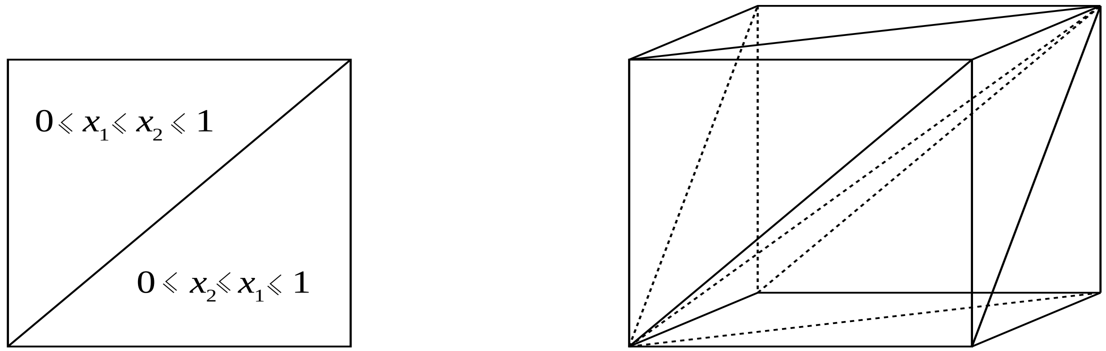
Without loss of generality we may assume that

$$h_1 \geq h_2 \geq \cdots \geq h_d,$$

where h_i^{-1} is integer for $i \in \{1,\dots,d\}$. Moreover, let

$$h_1 = h_1(k) = 2^{-k} \text{ and } h_d = h_d(k) = 4^{-k} \text{ for } k = 0,1,2,\dots$$

We will again consider two families $\mathcal{F}_5$ and $\mathcal{F}_6$ of nested simplicial partitions. Partitions from $\mathcal{F}_5$ are based on Kuhn's partition.



For instance, if $k = 0$ then $\overline{\Omega}$ is decomposed into $d!$ nonobtuse simplices defined as follows

$$K_\sigma = \{x = (x_1, \dots, x_d) \in \mathbf{R}^d \,|\, 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(d)} \le 1\}, \quad (11)$$

where σ ranges over all permutations of the numbers $1, 2, \dots, d$. For $k \ge 1$ all the resulting d-blocks are decomposed into d-simplices in a topologically similar way. None of the dihedral angles of these simplices is greater than $\pi/2$.

To define the family $\mathcal{F}_6$ we denote by G the centre of gravity of each d-block. Simplices (11) define Kuhn's partition of each $(d-1)$-dimensional facet of a given d-block B. Now we take the convex hull of G and each $(d-1)$-dimensional simplex from the boundary ∂B. This gives required d-simplices.

Some of such constructed sub-simplices have large dihedral angles tending to π as $k \to \infty$.

We can now consider our test problem in arbitrary space dimension. If its solution is smooth enough, the Lagrange interpolation operator is well defined and we may try to use a trick a la $(*)$ again. E.g. it is possible to do that in 3D as there is a 3D analogue of the maximum angle condition.

• In fact, a more universal statement, applicable also for nonsimplicial Lagrange and Hermite elements (possibly degenerating in various ways), can be formulated as follows.

Consider a general elliptic problem in a weak form: Find $u \in V$ such that

$$a(u, v) = F(v) \qquad \forall v \in V, \tag{12}$$

where V is a Hilbert space with the induced norm $\| \cdot \|_V$, $a(\cdot, \cdot)$ is a continuous V-elliptic bilinear form, and $F(\cdot)$ is a linear continuous functional over V. Then we have:

Theorem: *Let* $\{V_h\}_{h\to 0}$ *and* $\{W_h\}_{h\to 0}$ *be two families of finite element spaces such that* $V_h \subset W_h \subset V$. *Assume that for each* $v \in V$

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \tag{13}$$

i.e., the union $\bigcup_{h>0} V_h$ *is dense in* V. *Then*

$$\|u - u_h\|_V \to 0,$$

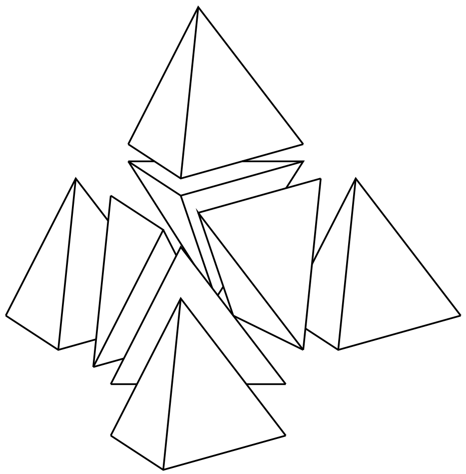
where $u_h \in W_h$ *is the standard finite element approximation of the weak solution* $u \in V$ *of elliptic boundary value problem (12).*

P r o o f : From Cea's lemma and (13), we obtain

$$\|u-u_h\|_V \le C \inf_{w_h \in W_h} \|u-w_h\|_V \le C \inf_{v_h \in V_h} \|u-v_h\|_V \to 0 \text{ as } h \to 0. \qquad \square$$

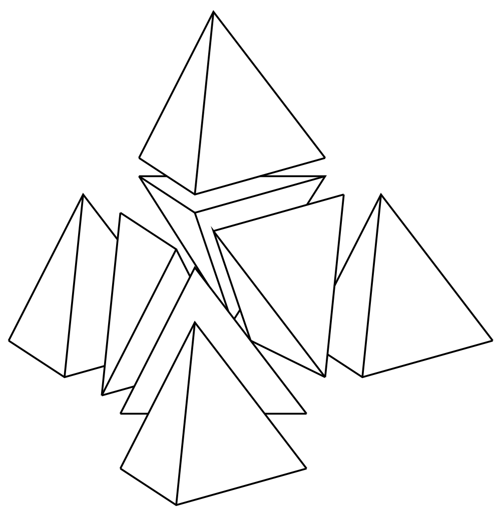
RED REFINEMENT TECHNIQUES

Now we show however that large dihedral angles of tetrahedra may have a bad influence on the convergence of FEM. Consider the standard 3D red refinement algorithm of a given tetrahedron into eight subtetrahedra. Using this algorithm recursively, we obtain a family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of tetrahedral partitions.

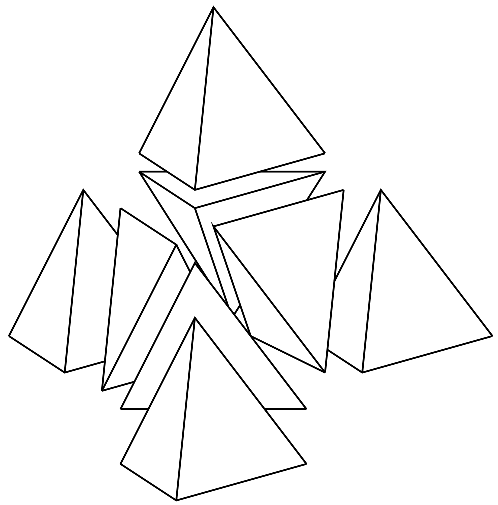


Proposition: *The maximal (minimal) dihedral angle of all tetrahedra $K \in \mathcal{T}_h \in \mathcal{F}$ forms a nondecreasing (nonincreasing) sequence as $h \to 0$.*

P r o o f : The proposition follows immediately from the fact that the four "exterior" subtetrahedra arising from the red refinement algorithm are similar to the original tetrahedron. □



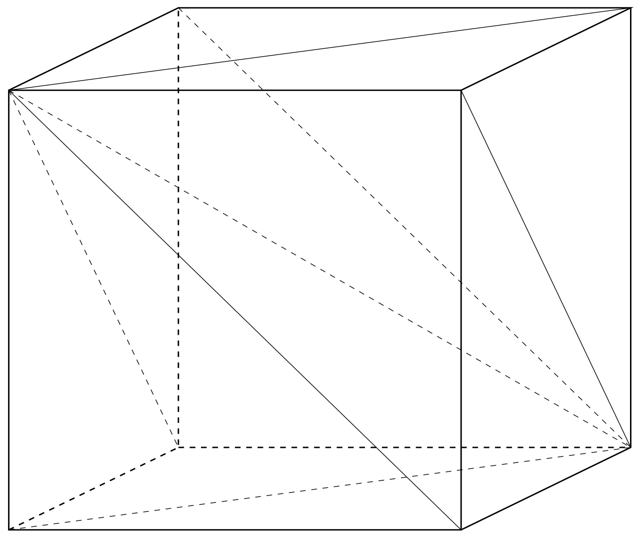
Further, we will use two different basic strategies, the longest-diagonal and shortest-diagonal refinement, for dividing the interior octahedron.



Let $\Omega = (0,1)^3$ and consider the problem

$$-\Delta u = \sin \pi x_1 \sin \pi x_2 \sin \pi x_3 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \qquad (14)$$

Let the initial partition of $\overline{\Omega}$ be the Kuhn division of the cube into six nonobtuse tetrahedra (see (11)).



b)

In the longest-diagonal refinement, the interor octahedron is divided by taking the longest diagonal as a common edge for all four resulting subtetrahedra. Zhang found that the inscribed ball condition (which is equivalent to the minimum angle condition) is not valid for the longest-diagonal red refinement algorithm.

Our numerical tests, moreover, show that even the maximum angle condition does not hold. The maximal dihedral angle of subsequent partitions is 90°, 135°, 144.74°, 161.57°, ..., i.e. some subtetrahedra degenerate quite fast.

S. Zhang. Successive subdivisions of tetrahedra and multigrid methods on tetrahedral meshes. Houston J. Math. 21 (1995), 541–556.

On the other hand, in the shortest-diagonal red refinement, the shortest edge is chosen as the common edge and the resulting subtetrahedra do not degenerate, i.e. all dihedral angles and also angles between edges are bounded from below by a positive constant. The longest-diagonal refinement will lead to considerably slower decay of h in comparison to the shortest-diagonal refinement. In practice, this means that certain value of h is obtained for smaller number of degrees of freedom for the shortest-diagonal refinement, compared to the longest-diagonal refinement. Therefore, the usage of h as a measure for convergence is not quite correct in this example. The optimal convergence rate $\mathcal{O}(h)$ could be theoretically and also practically obtained only for the shortest-diagonal red refinement algorithm.

In Figure 2, we have visualized the convergence rate in the H^1-norm for the problem (14). The practical rate of convergence for the longest-diagonal refinement seems to be $\mathcal{O}(h^{1/2})$ and for the shortest-diagonal refinement $\mathcal{O}(h)$. However, when degrees of freedom are compared, the longest-diagonal refinement performs considerably worse. In this case, the shortest-diagonal refinement seems to have the practical convergence rate of $\mathcal{O}(N^{-1/3})$, whereas the longest-diagonal refinement $\mathcal{O}(N^{-1/10})$, where N is the number of degrees of freedom in the mesh.

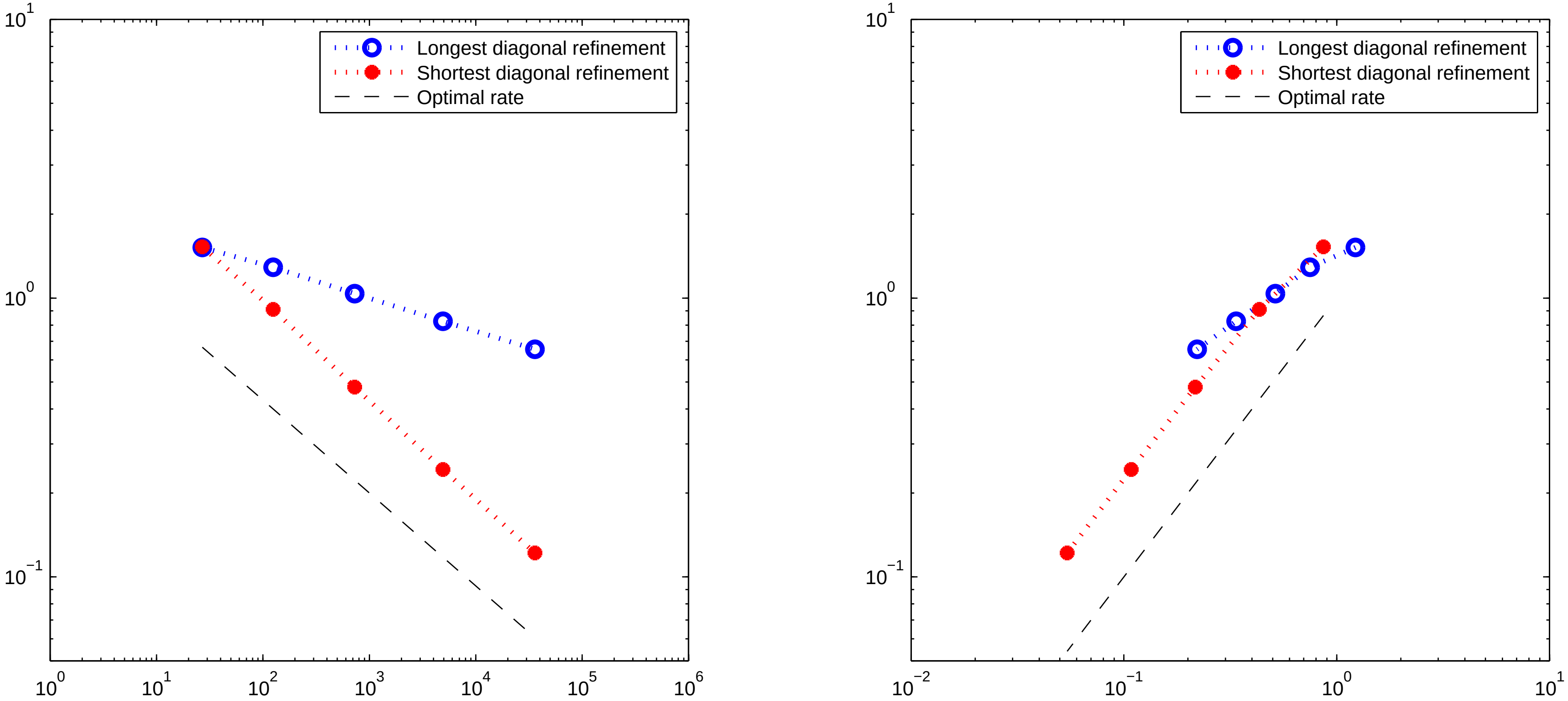


Figure 2: The H^1-norm of the discretization error versus the number of degrees of freedom (left) and the discretization parameter (right).

This example shows that sometimes large dihedral angles make convergence considerably slower. Thus, the maximum angle condition is really essential, even though it is not necessary. The same observation was done on planar triangulations illustrated earlier.

Although the presented examples are rather academic, they show that large angles may produce small finite element discretization error, even though the interpolation error is large.