DISCRETE MAXIMUM PRINCIPLES IN FINITE ELEMENT ANALYSIS

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Model Elliptic Problem

Find a function u such that

$$-\text{div}(\mathcal{A}\nabla u) + cu = f \quad \text{in} \quad \Omega, \tag{1}$$

$$u = g \quad \text{on} \quad \partial\Omega. \tag{2}$$

- $\Omega \subset \mathbf{R}^d$ is a bounded polytope with Lipschitz boundary $\partial\Omega$.

- The diffusive tensor $\mathcal{A}$ is assumed to be a symmetric and uniformly positive definite matrix.

- The reactive coefficient c is assumed to be nonnegative in Ω.

Continuous Maximum Principle

• The classical (smooth) solutions of many PDE problems are known to satisfy various (continuous) maximum principles (or CMPs in short).

For our model elliptic problem standard CMP reads as follows:

$$f \leq 0 \qquad \Longrightarrow \qquad \max_{x \in \overline{\Omega}} u(x) \leq \max\{0\,, \max_{s \in \partial\Omega} g(s)\}. \tag{3}$$

• `CMPs are not only ``pure mathematical'' properties of PDE models, they also ``reflect'' well physical behaviour of phenomena modelled.`

Preliminaries

• Any reasonable discrete analogue of CMP (which may depend, in general, on the nature of numerical technique used) is called the discrete maximum principle (or DMP in short).

• The first DMP and certain conditions providing its validity were (probably ?) formulated by R. Varga for the finite difference method (FDM) in "On a discrete maximum principle", *J. SIAM Numer. Anal.* 3 (1966), 355–359. However, only a special case $f = 0$ was analysed there and CMP considered was of the following form:

$$\max_{x \in \overline{\Omega}} |u(x)| \leq \max_{s \in \partial\Omega} |g(s)|.$$

• Later, in 1970 Ph. Ciarlet presented a more sophisticated form of DMP suitable for both, finite element (FE) and finite difference (FD) types of discretization. He also proposed a practically convenient set of (sufficient) conditions on matrix blocks involved, which provides a validity of his DMP.

• Since that time Ciarlet–conditions became popular in numerical community for proving DMPs for various problems of elliptic type, see e.g. [Křížek, Lin Qun, 1995], [Karátson, Korotov, 2005], and references therein.

• `We shall present in detail the conditions proposed by Ciarlet and discuss some practical problems with them when FEM is used as a main numerical technique.`

Weak Formulation

Find $u \in g + H_0^1(\Omega)$ such that

$$a(u, v) = \mathcal{F}(v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega c u v \, \mathrm{d}x \quad \text{and} \quad \mathcal{F}(v) = \int_\Omega f v \, \mathrm{d}x.$$

Here, the matrix $\mathcal{A} \in [L^\infty(\Omega)]^{d \times d}$, $c \in L^\infty(\Omega)$, $g \in H^1(\Omega)$, and $f \in L^2(\Omega)$.

The existence and uniqueness of the weak solution u is provided by the Lax-Milgram lemma.

FE Discretization: Mesh & Basis Functions

• Let $\mathcal{T}_h$ be a FE mesh of $\overline{\Omega}$ with interior nodes $B_1, \ldots, B_N$ lying in Ω and boundary nodes $B_{N+1}, \ldots, B_{N+N^\partial}$ lying on $\partial\Omega$.

• Let V_h be a finite-dimensional subspace of $H^1(\Omega)$, associated with nodes of $\mathcal{T}_h$, and spanned by basis functions $\phi_1, \phi_2, \ldots, \phi_{N+N^\partial}$ such that

$$\phi_i \geq 0 \text{ in } \overline{\Omega}, \; i = 1, \ldots, N + N^\partial, \quad \text{and} \quad \sum_{i=1}^{N+N^\partial} \phi_i \equiv 1 \text{ in } \overline{\Omega}.$$

• We also assume that $\phi_1, \phi_2, \ldots, \phi_N$ vanish on the boundary $\partial\Omega$, so that they span a finite-dimensional subspace V_h^0 of $H_0^1(\Omega)$.

FE Discretization: Matrix Equation

- Let $g_h = \sum_{i=N+1}^{N+N^\partial} g_i \phi_i \in V_h$ be a suitable approximation of the function g, for example its nodal interpolant.

FE approximation is defined as a function $u_h \in g_h + V_h^0$ such that

$$a(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h^0.$$

In fact, $u_h = \sum_{i=1}^{N+N^\partial} y_i \phi_i$, where $\bar{\mathbf{y}} = [y_1, \dots, y_{N+N^\partial}]^\top$ is the solution of

$$\bar{\mathbf{A}} \bar{\mathbf{y}} = \bar{\mathbf{F}}, \tag{4}$$

with

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^\partial \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^\partial \end{bmatrix}. \tag{5}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^{\partial} \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{A}}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

- $\mathbf{A}$ and $\mathbf{A}^{\partial}$ are matrices of size $N \times N$ and $N \times N^{\partial}$, respectively, $\mathbf{I}$ – unit matrix, and $\mathbf{0}$ – zero matrix. Entries of $\bar{\mathbf{A}}$ are $a_{ij} = a(\phi_j, \phi_i)$, $i = 1, \ldots, N,\ j = 1, \ldots, N + N^{\partial}$.

- The block-vector $\mathbf{F}$ consists of entries $f_i = \mathcal{F}(\phi_i),\ i = 1, \ldots, N$, and the block-vector $\mathbf{F}^{\partial}$ has entries $f_i = g_i,\ i = N + 1, \ldots, N + N^{\partial}$.

- `For the later reference we include the formula for` $\bar{\mathbf{A}}^{-1}$`.`

- $\bar{\mathbf{A}}$ `is nonsingular if and only if` $\mathbf{A}$ `is nonsingular.`

Conditions of Ph. Ciarlet

We distinguish two basically different types of DMPs.

Algebraic DMP: A natural algebraic analogue of CMP is

$$\mathbf{F} \leq \mathbf{0} \qquad \Longrightarrow \qquad \max_{i=1,\ldots,N+N^\partial} y_i \leq \max\{0\,, \max_{j=N+1,\ldots,N+N^\partial} y_j\}.$$

Functional DMP: A natural functional imitation of CMP is

$$f \leq 0 \qquad \Longrightarrow \qquad \max_{\overline{\Omega}} u_h \leq \max\{0\,, \max_{\partial\Omega} u_h\}.$$

• `Often we see that validity of algebraic DMP ‘‘implies’’ validity of (desired) functional DMP,e.g. for linear, multilinear, or prismatic FEs. (However, this may not be true, in general, e.g. for higher-order FEs.)`

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^{\partial} \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{A}}^{\mathbf{-1}} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

• In "Discrete maximum principle for finite-difference operators", *Aequationes Math.* 4 (1970), Ciarlet proved

Theorem 1. *The algebraic DMP is satisfied iff*

(A) $\bar{\mathbf{A}}$ *is monotone (i.e.,* $\bar{\mathbf{A}}$ *nonsingular and* $\bar{\mathbf{A}}^{-1} \geq 0$*)*

(B) $\xi + \mathbf{A}^{-1}\mathbf{A}^{\partial}\xi^{\partial} \geq 0$, *where* ξ *and* ξ^{∂} *are vectors of all ones of sizes* N *and* N^{∂}*, respectively*

• Since conditions (A) and (B) are difficult to use in practice, Ciarlet also proposed a more convenient set of sufficient conditions

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^{\partial} \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{A}}^{-\mathbf{1}} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Theorem 2. *The algebraic DMP is valid provided* $\bar{\mathbf{A}}$ *satisfies*

(a) $a_{ii} > 0, \quad i = 1, \ldots, N$

(b) $a_{ij} \leq 0, \quad i \neq j, \; i = 1, \ldots, N, \; j = 1, \ldots, N + N^{\partial}$

(c) $\displaystyle\sum_{j=1}^{N+N^{\partial}} a_{ij} \geq 0, \quad i = 1, \ldots, N$

(d) $\mathbf{A}$ *is irreducibly diagonally dominant*

Basically, Ciarlet proposed these conditions in order to utilize the results of R. Varga from his book *Matrix Iterative Analysis*, 1962.

Definition 1. A square $n \times n$ matrix $\mathbf{M} = (m_{ij})_{i,j=1}^{n}$ is called irreducibly diagonally dominant if

(i) $\mathbf{M}$ is irreducible, i.e., for any $i \neq j$ there exists a sequence of nonzero entries $\{m_{i,i_1}, m_{i_1,i_2}, \ldots, m_{i_s,j}\}$ of $\mathbf{M}$, where $i, i_1, i_2, \ldots, i_s, j$ are distinct indices,

(ii) $\mathbf{M}$ is diagonally dominant, i.e., $|m_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |m_{ij}|, \; i = 1, ..., n,$

(iii) for at least one index $i_0 \in \{1, ..., n\}$ the inequality in (ii) is strict, i.e.,

$$|m_{i_0,i_0}| > \sum_{\substack{j=1 \\ j \neq i_0}}^{n} |m_{i_0,j}|.$$

Lemma 3. *If* $\mathbf{A} \in \mathbf{R}^{N\times N}$ *is an irreducibly diagonally dominant matrix with strictly positive diagonal and nonpositive off-diagonal entries then* $\mathbf{A}^{-1} > 0$.

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}^{\partial} \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{A}}^{\mathbf{-1}} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{A}^{\partial} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Remark 1. However, in the proof we only need monotonicity of $\mathbf{A}$, i.e. $\mathbf{A}^{-1} \geq 0$, not more. E.g. if $g = 0$ the system reduces to $\mathbf{Ay} = \mathbf{F}$, and CMP is $\quad f \leq 0 \quad \Longrightarrow \quad \max_{x\in\overline{\Omega}} u(x) \leq 0$. To immitate this, it is enough to have $\mathbf{A}^{-1} \geq 0$.

Analysis of Ciarlet–Conditions

- Condition (a) is always satisfied for elliptic problems.

- Row sums in (c) are nonnegative automatically for our problem as

$$\sum_{j=1}^{N+N^\partial} a_{ij} = a\left(\sum_{j=1}^{N+N^\partial} \phi_j, \phi_i\right) = a(1, \phi_i) = \int_\Omega c\phi_i \geq 0, \quad i = 1, \ldots, N. \tag{6}$$

- On the other hand, the irreducibility of $\mathbf{A}$ required in (d) is not always obvious.

Problems with Irreducibility

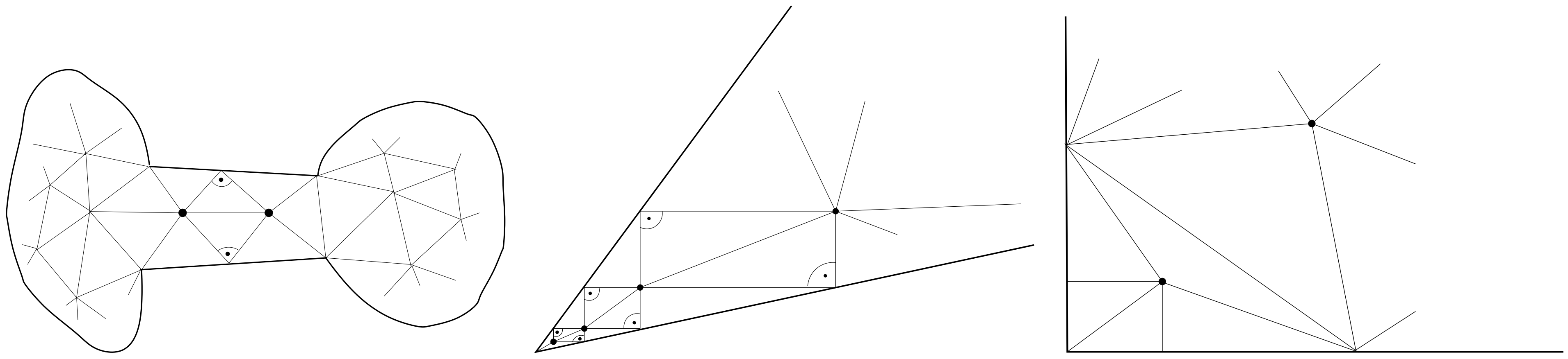


Figure 1: [Hannukainen, Korotov, Vejchodský, 2009]. Meshes leading to reducible matrix $\mathbf{A}$ for the Poisson problem with Dirichlet boundary conditions. Angles marked by dots are right.

Associated Geometric Conditions

- For some types of FEs, $\bar{\mathbf{A}}$ can be computed explicitly, therefore

$$a_{ij} \le 0, \qquad i \ne j,$$

can often be guaranteed a priori.

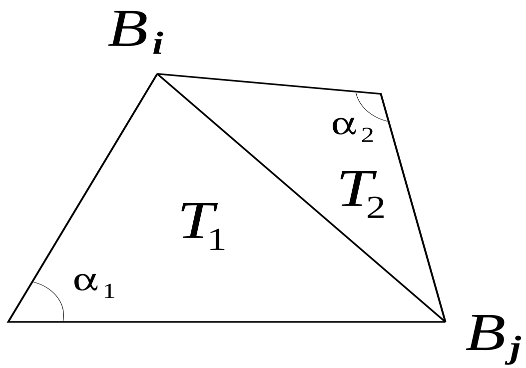
- For example, if

$$\mathcal{A} = I$$

then there are well-known geometrical conditions providing (b).

Simplicial Meshes

- For simplicity we assume that c is constant and $d = 2$
- If nodes B_i and B_j are not connected by an edge then $a_{ij} = 0$



- Otherwise

$$a_{ij} = -\frac{1}{2}\Big(\operatorname{ctg}\alpha_1 + \operatorname{ctg}\alpha_2\Big) + \frac{c}{12}\Big(\operatorname{meas} T_1 + \operatorname{meas} T_2\Big),$$

see [Fujii, 1973], [Ruas Santos, 1982], [Ciarlet-book, 1978].

$$a_{ij} = -\frac{1}{2}\Big(\operatorname{ctg}\alpha_1 + \operatorname{ctg}\alpha_2\Big) + \frac{c}{12}\Big(\operatorname{meas} T_1 + \operatorname{meas} T_2\Big)$$

- In case $c = 0$, we get $a_{ij} \le 0$ if

$$\operatorname{ctg}\alpha_1 + \operatorname{ctg}\alpha_2 \ge 0$$

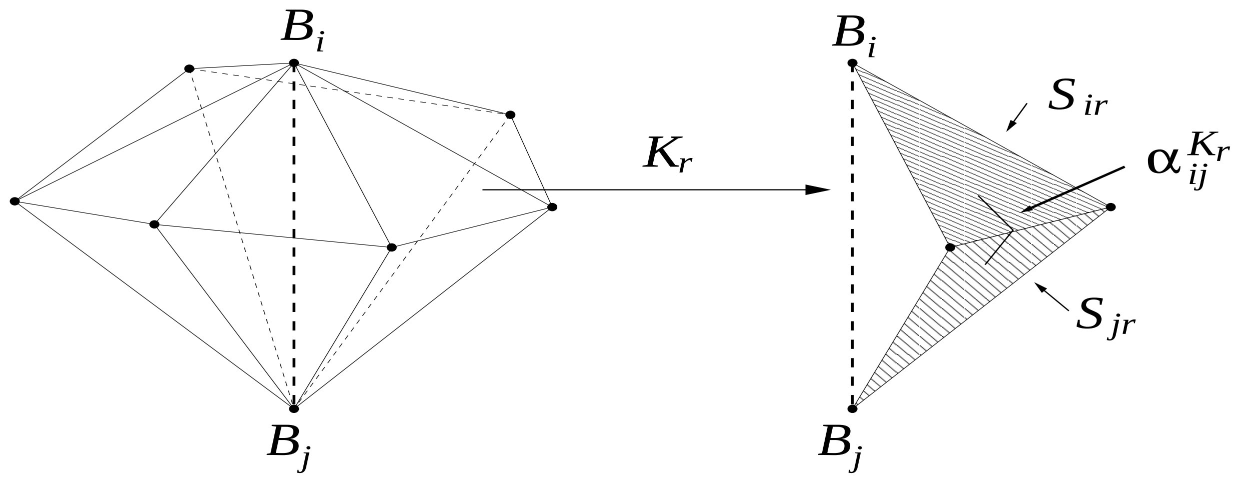
It holds if e.g. $\alpha_k \le \frac{\pi}{2}$. This observation leads to the concept of *nonobtuse triangulations.*

- In case $c > 0$, we get $a_{ij} \le 0$ if

$$\frac{c}{12}\Big(\operatorname{meas} T_1 + \operatorname{meas} T_2\Big) \le \frac{1}{2}\Big(\operatorname{ctg}\alpha_1 + \operatorname{ctg}\alpha_2\Big)$$

It is possible if e.g. $\alpha_k \le \frac{\pi}{2} - \varepsilon$ $(\varepsilon > 0)$ and triangulations are sufficiently fine. This leads to the concept of *acute triangulations.*

- **Similarly for $d \geq 3$:**



$$\int_{K_r} \nabla\phi_j \nabla\phi_i \, dx = -\frac{\mathrm{meas}_{d-1} S_{ir} \cdot \mathrm{meas}_{d-1} S_{jr}}{d^2 \, \mathrm{meas}_d K_r} \cos \alpha_{ij}^{K_r}$$

$$\int_{K_r} \phi_j \phi_i \, dx = \frac{d!}{(d+2)!} \mathrm{meas}_d K_r$$

See [Křížek, Lin Qun, 1995], [Xu, Zikatanov, 1999], [Brandts, Korotov, Křížek, 2007].

Block Meshes

[Karátson, Korotov, Křížek, 2007], [Korotov, Vejchodský, 2010]: the results strongly depend on dimension

Let K be a block of a d-dimensional block mesh with edges of lengths $b_1, b_2, \ldots, b_d$, and let B_i and B_j be its vertices connected by b_1, then

$$a_{ij}^K = \frac{b_1 b_2 \ldots b_d}{3^{d-1}} \Big(\sum_{k=2}^{d} \frac{1}{2b_k^2} - \frac{1}{b_1^2} \Big), \quad i \neq j$$

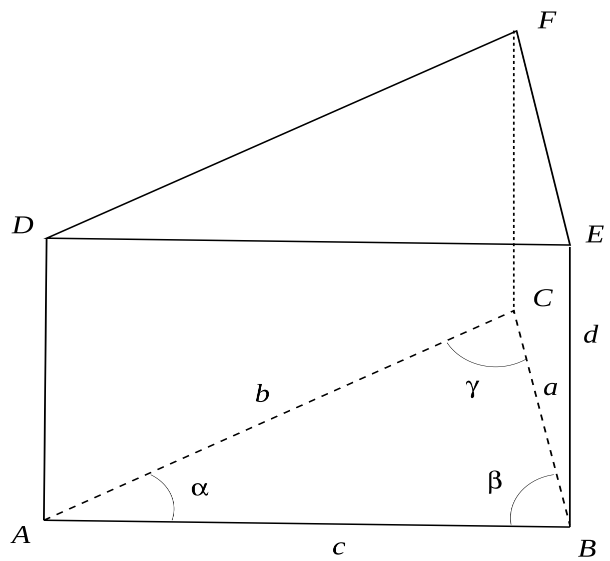
- $d = 2$: all rectangular elements have to be *nonnarrow* for (b), i.e.

$$\sqrt{2}/2 \leq b_1/b_2 \leq \sqrt{2}$$

- $d = 3$: trilinear elements have to be cubes
- $d \geq 4$: condition (b) is not valid even on hyper-cubes

Prismatic Meshes

• [Hannukainen, Korotov, Vejchodský, 2009]: for 3D meshes consiting of right triangular prisms the altitudes of prisms should be limited from both sides by certain quantities dependent on the area and angles of the triangular base (and magnitude of c) to provide (b).



No Irreducibility-Proof is Needed

• A real square matrix $\mathbf{A}$ is an *M-matrix* if all its off-diagonal entries are nonpositive and if it is nonsingular and $\mathbf{A}^{-1} \geq 0$.

• A real square matrix $\mathbf{A}$ is a *Stieltjes matrix* if all its off-diagonal entries are nonpositive and if it is symmetric and positive definite.

Lemma 4. *If* $\mathbf{A}$ *is a Stieltjes matrix then it is also an M-matrix.*

• This lemma, proved in [Varga-book, 1962], enables to eliminate (a), (c), and (d) from the set of Ciarlet–conditions in our case:

Theorem 5. *If FE matrix* $\bar{\mathbf{A}}$, *associated to our model problem, satisfies* $a_{ij} \leq 0, \;\; i \neq j$, *then the algebraic DMP is valid.*

For the proof see [Hannukainen, Korotov, Vejchodský, 2009].

Generation of Acute / Nonobtuse Triangulations

It is not so easy task even in simple cases:

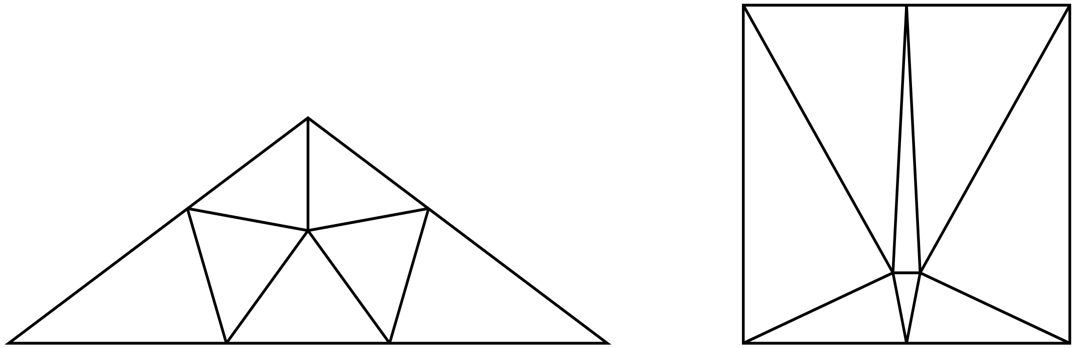


Figure 2: Partition of an obtuse triangle and square into 7 and 8 acute triangles, respectively (cf. [Gardner, 1960]). It was shown in [Lindgren, 1962] and [Manheimer et al., 1960] that these numbers are minimal. Later, [Cassidy, Lord, 1980] proved that for any $n \geq 10$ there exists a triangulation of a square into n acute triangles. They also showed why such a triangulation does not exist for $n = 9$.

Burago and Zalgaller give an algorithm to construct an acute triangulation for an arbitrary polygon. Gerver presents an algorithm that enables us to decompose special polygons into almost equilateral triangles with maximal angle 72°. Maehara proves that every n-gon can be triangulated into $\mathcal{O}(n)$ acute triangles (see also [Yuan]). A short recent survey on acute triangulations was done by Zamfirescu.

Ju.D. Burago and V. A. Zalgaller, Polyhedral embedding of a net, Vestnik Leningrad. Univ., 15 (1960), pp. 66–80 (in Russian).

J. L. Gerver, The dissection of a polygon into nearly equilateral triangles, Geom. Dedicata, 16 (1984), pp. 93–106.

H. Maehara, Acute triangulations of polygons, European J. Combin., 23 (2002), pp. 45–55.

L. Yuan, Acute triangulations of polygons, Discrete Comput. Math., 34 (2005), pp. 697–706.

T. Zamfirescu, Acute triangulations: A short survey, in Proceedings of the VI Annual Conference of the Romanian Society of Mathematical Sciences, Vol. I (Sibiu, 2002), Soc. Stiinte Mat. Romania, Bucharest, 2003, pp. 10–18.

Nonobtuse triangulations of polygons are very well studied; see, for instance:

B. S. Baker, E. Grosse, and C. S. Rafferty, Nonobtuse triangulation of polygons, Discrete Comput. Geom., 3 (1988), pp. 147–168.

S. Korotov and M. Křížek, Acute type refinements of tetrahedral partitions of polyhedral domains, SIAM J. Numer. Anal., 39 (2001), pp. 724–733.

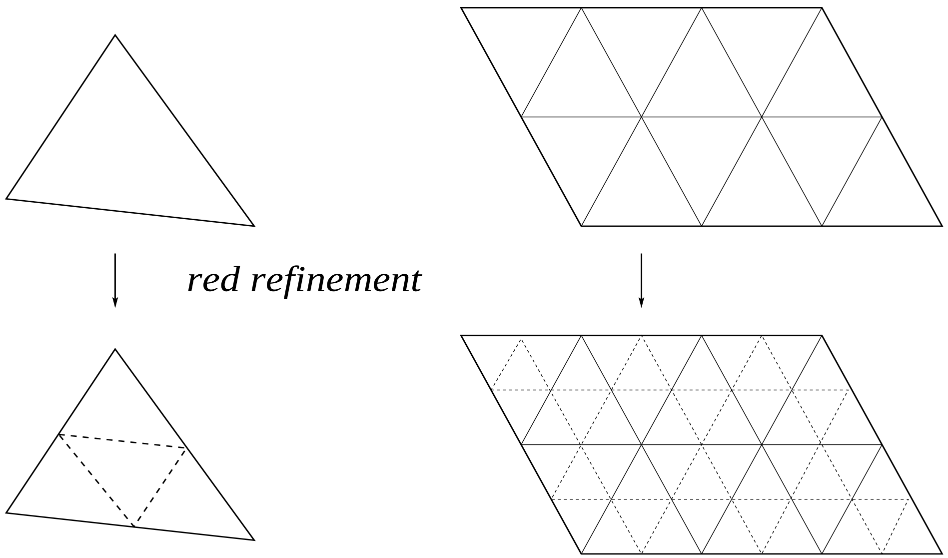
S. Korotov and M. Křížek, Global and local refinement techniques yielding nonobtuse tetrahedral partitions, Comput. Math. Appl., 50 (2005), pp. 1105–1113.

H. Maehara, Acute triangulations of polygons, European J. Combin., 23 (2002), pp. 45–55.

L. Yuan, Acute triangulations of polygons, Discrete Comput. Math., 34 (2005), pp. 697–706.

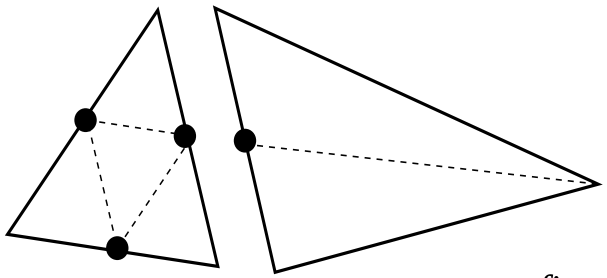
Problems with Mesh Reconstructions

• To improve the accuracy of FE computations we need to refine meshes globally and locally. *How to preserve the desired geometric properties then ?*



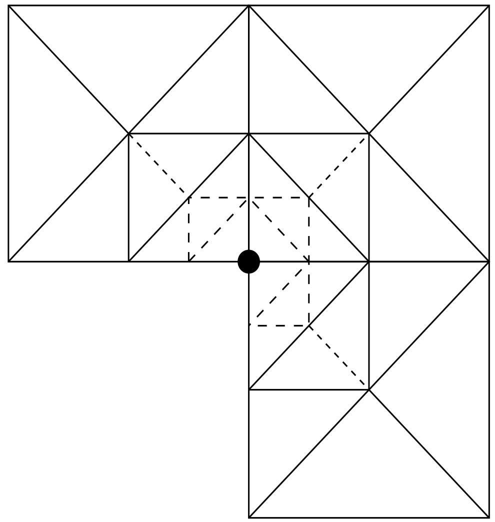
• For global mesh refinements "2D red" technique *always* preserves properties of acuteness and nonobtuseness

• For local mesh refinements "green" post-refining (to preserve conformity) is often needed, but it generally leads to obtuse angles. Only in very special cases we can preserve nonobtuseness for local mesh refinements using both, red and green, refinements:



green refinement

red refinement



red + green producing nonobtuse local refinements

Problems with Tetrahedral Meshes

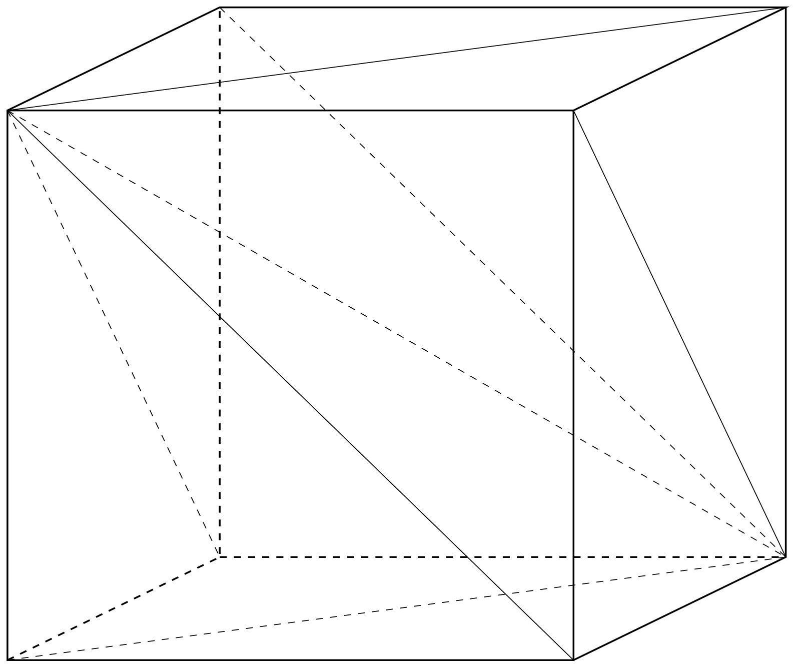
Existence of a face-to-face partition of even a cube into acute tetrahedra was surprisingly (!) an open problem till 2009.

E. Kopczyński, I. Pak, P. Przytycki. Acute triangulations of polyhedra and the Euclidean space. In: Proc. of the Annual Sympos. on Comput. Geom., Snowbird, Utah, 2010, 307–313.

E. VanderZee, A. N. Hirani, V. Zharnitsky, D. Guoy. A dihedral acute triangulation of the cube, Comput. Geom. 43 (2010), 445–452.

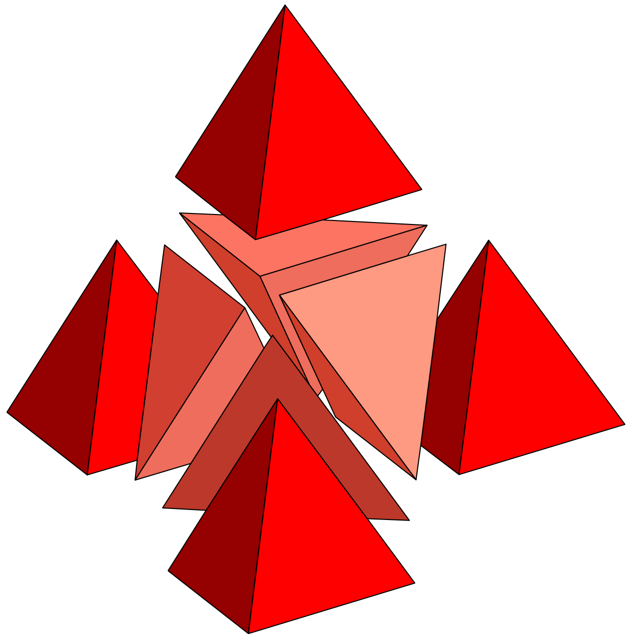
However, tetrahedralizations proposed in the above works involve too many acute tetrahedra, which are, in addition, very densely placed in the interior of the cube, which is not desired in practical calculations (usually we expect singularities, etc at vertices and edges of computational domains).

Partition of a cube in 6 nonobtuse (path) tetrahedra:



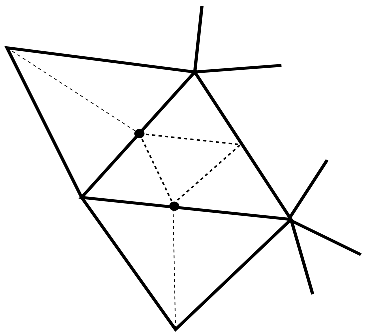
b)

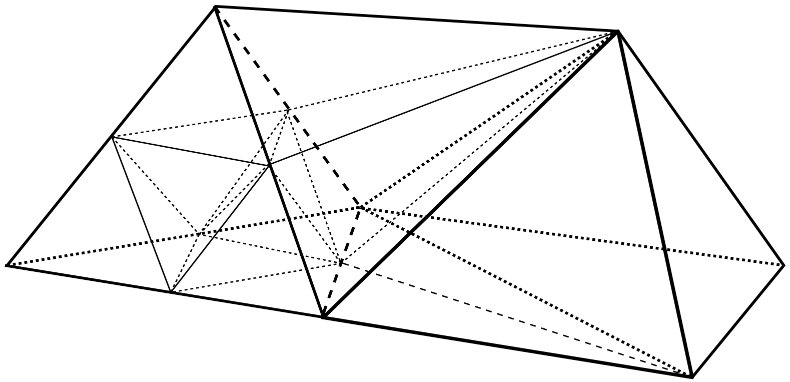
3D red (global) refinement technique does not generally preserve nonobtuseness, and never acuteness.



Bisection technique does not help with these issue either.

The same problems remain with local refinements of unstructured tetrahedral meshes - obtuse angles can be easily produced



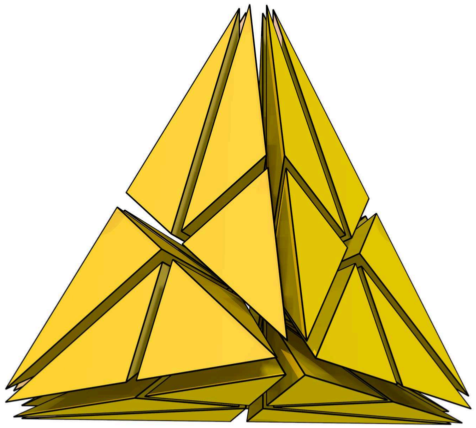


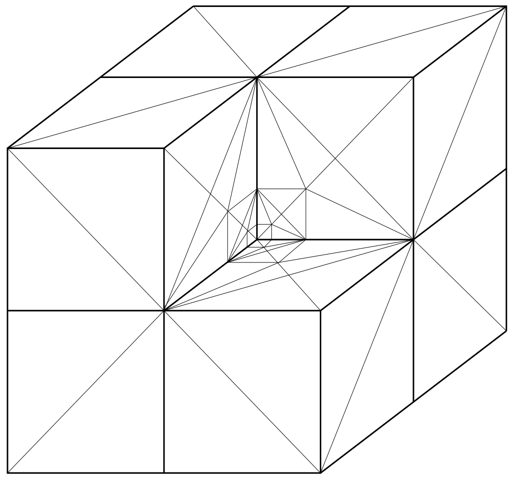
green refinement *red−green refinement*

M. Křížek, T. Strouboulis. How to generate local refinements of unstructured tetrahedral meshes satisfying a regularity ball condition. Numer. Methods Partial Differential Equations 13 (1997), 201–214.

Results for Nonobtuse Tetrahedral Refinements

• In [Korotov, Křížek, SINUM–2001] and [Korotov, Křížek, 2003] we proposed algorithms for global and local (towards a vertex) tetrahedral refinements preserving nonobtuseness





• Very recently new algorithms were developed for nonobtuse local refinements towards edge – [Korotov, Křížek, 2011], and face/interface – [Korotov, Křížek, 2011]

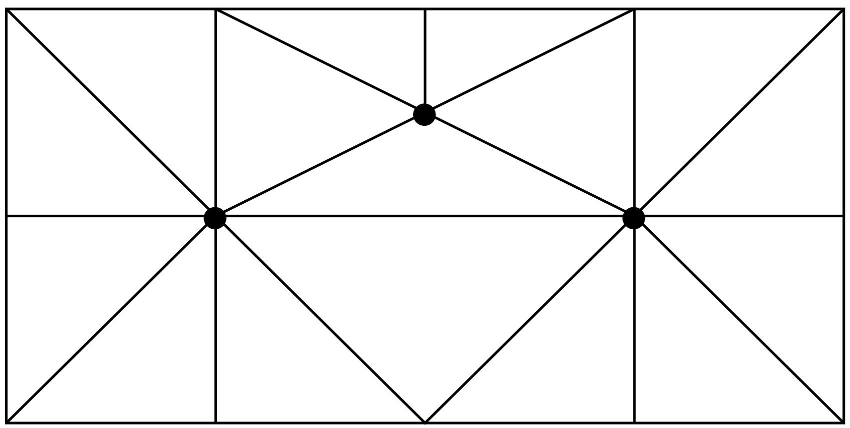
More Problems

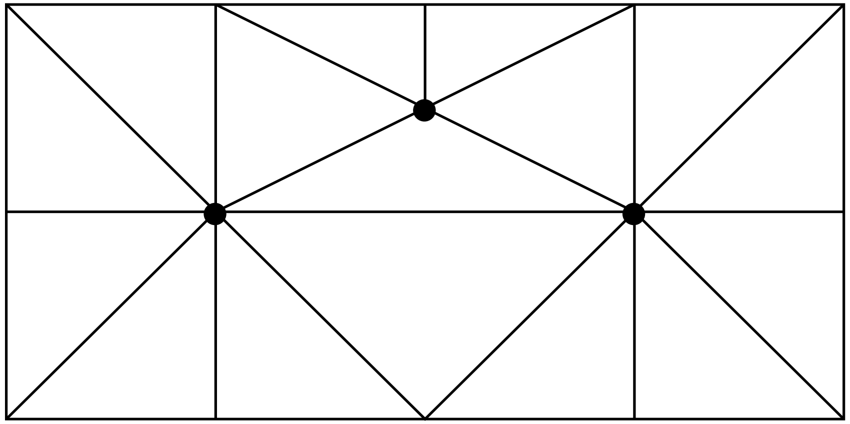
- Local refinements of block and prismatic FE meshes ...
- Full diffusive tensor ...
- Higher-dimensional problems (blocks ?, acute simplices ?) ...

M. Křížek. There is no face-to-face partition of R^5 into acute simplices. Discrete Comput. Geom. 36 (2006), 381–390, Erratum 40 (2010).

Geometrical Conditions are Essential

Just one single obtuse triangle in FE mesh can completely destroy DMP (e.g. for the Poisson equation with zero Dirichlet b.c.), see [Brandts, Korotov, Křížek, Šolc, 2009].





The inverse of **A** equals

$$\begin{pmatrix} \frac{63}{248} & -\frac{1}{248} & \frac{1}{16} \\ -\frac{1}{248} & \frac{63}{248} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{37}{160} \end{pmatrix}.$$

Some idea: Allow some off-diagonal entries of $\bar{\mathbf{A}}$ be positive.

Some results are available in this direction – [Bramble, Hubbard, 1964], [Ruas Santos, 1982], [Korotov, Křížek, Neittaanmäki, 2001], [Bouchon, 2007]. However, again e.g. the nontrivial issue of providing (or proving) irreducibility is involved ...

Some Numerical Tests

- Stieltjes matrices form only a certain subclass of monotone matrices.

- In what follows we test how sharp the conditions based on the Stieltjes matrix concept are, i.e., we try to test how wide is the class of meshes which lead to $\mathbf{A}$ being monotone but not Stieltjes.

- For this purpose we solve the 2D Poisson problem with homogenous Dirichlet boundary conditions on various domains using various triangulations.

- We present three representative tests ...

• In each test we construct a simple triangulation which is characterized by two parameters (angles) $0 < \alpha < \pi$ and $0 < \gamma < \pi$:

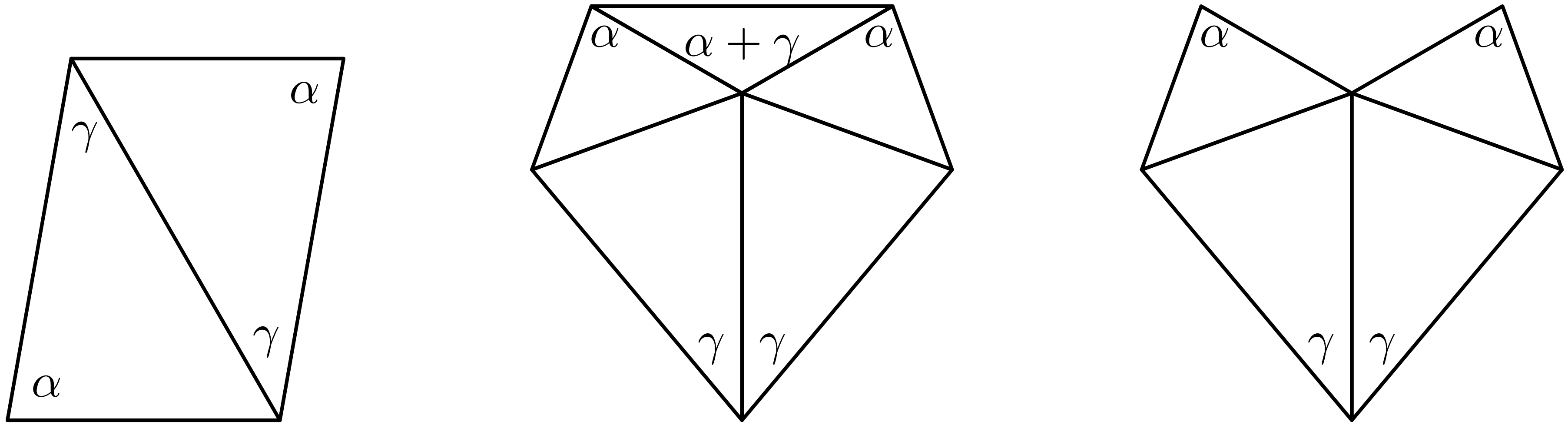


Figure 3: The basic meshes for the three tests. The meshes used for the actual computations are 10-fold refined basic meshes.

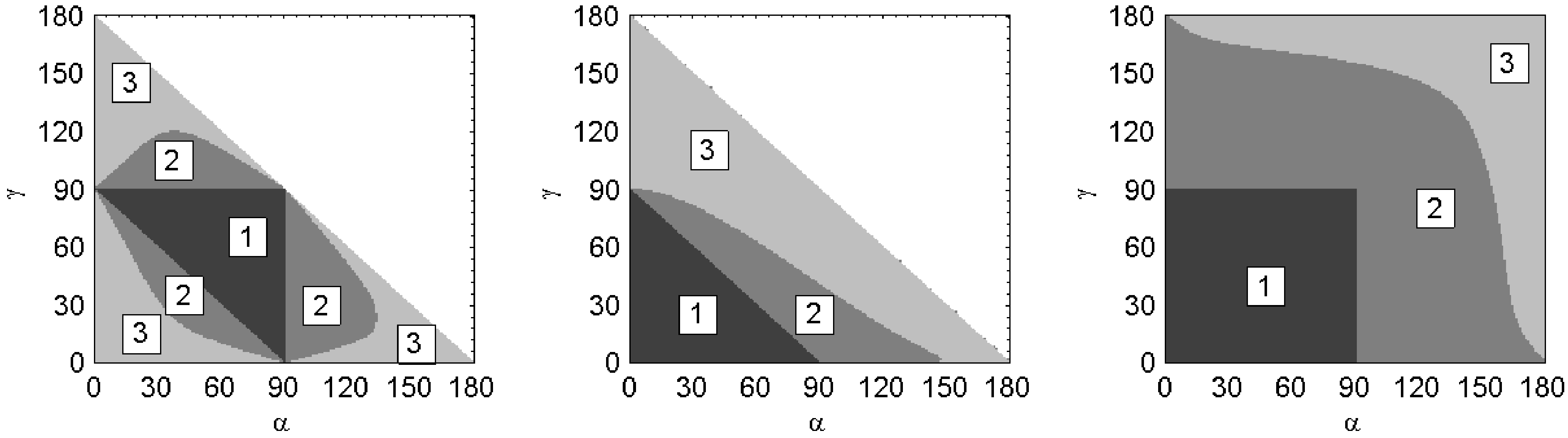
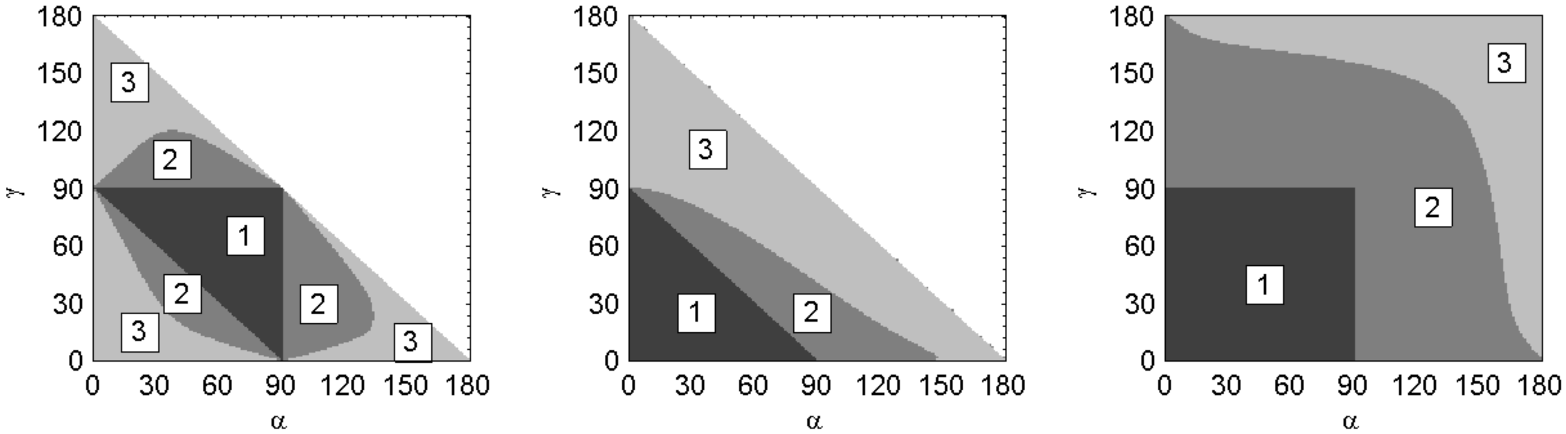


Figure 4: Domain 1: triangulations with non-obtuse maximal angle (i.e. matrix $\mathbf{A}$ is a Stieltjes matrix). Domain 2: triangulations with obtuse maximal angle providing the DMP (i.e. matrix $\mathbf{A}$ is monotone but not Stieltjes). Domain 3: triangulations with obtuse maximal angle, DMP is not valid (matrix $\mathbf{A}$ is not monotone).



• If we compare the sizes of domains 1 with domains 2 we may conclude that the standard sufficient condition (b) is not very sharp. There is a wide space for its generalization.

• However, any generalization have to utilize more general criteria for the monotonicity of a matrix. These criteria are often quite complicated and their application for proving DMPs is not so straightforward.