SOURCES OF ERRORS

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One of the most challenging problems in mathematical modeling and numerical analysis, which also has great importance for most of applications in industry is

RELIABLE VERIFICATION OF ACCURACY

OF APPROXIMATE SOLUTIONS OBTAINED

IN COMPUTER SIMULATIONS

• Mathematically, this task is related to the so-called *a posteriori error estimates,* giving computable bounds for errors of various types and detecting zones, where such errors are excessively high and some mesh-refinement algorithm should be used

Modeling Error

- In modeling phase a physical process (or object) of interest is usually described by means of a certain mathematical model. Then, the so-called *modeling error* $\varepsilon_1 = |U - u|$ arises, where U is the value characterizing the process (object) and u is the corresponding value obtained from the mathematical model. This error exists due to, e.g., the fact that usually "second-order" phenomena are neglected, there is also an undeterminacy in the problem data, often dimensional reduction is used to simplify models, etc.

- The symbol $|\cdot|$ denotes a convenient measure of the difference between U and u (e.g., the absolute value of the difference or a norm in a suitable functional space)

Discretization Error

• In most of cases, the "exact" solution u cannot be obtained in analytical (explicit) form due to a high complexity of models constructed for real-life problems. Indeed, adequate models for complicated processes normally involve several differential and integral equations, various algebraic relations, etc. The solution u of such complicated models is often understood in an abstract sense – as an element of a certain functional space

• Qualitative properties of u can be studied by pure mathematical methods, but the quantitative analysis normally requires a replacement of the original problem by a sequence of simpler problems whose solutions can be found relatively easily. Let u_h be solution of approximate problem obtained on mesh with characteristic size h. Then *discretization error* $\varepsilon_2 = |u - u_h|$ appears

Computational Error

- In turn, those approximate problems are themselves solved approximately, using concrete computers and concrete software packages, so the third type error, called the *computational error*, $\varepsilon_3 = |u_h - u_h^\varepsilon|$, appears, where u_h^ε is what we really obtain in computer simulations

- The error ε_3 includes roundoff errors, errors due to forcibly stopped iterative processes and errors caused by bugs in computer codes. It is clear that estimation of ε_3 is an extremely difficult task

- All said above is presented in the following diagram

U **Physical process**

$\Downarrow$

$\boldsymbol{\varepsilon_1}$ $\longrightarrow$ Modeling error

$\Downarrow$

u **Mathematical model** $Au = f$

$\Downarrow$

$\boldsymbol{\varepsilon_2}$ $\longrightarrow$ Discretization error

$\Downarrow$

u_h **Discrete model** $A_h u_h = f_h$

$\Downarrow$

$\boldsymbol{\varepsilon_3}$ $\longrightarrow$ Computational error

$\Downarrow$

u_h^ϵ **Numerical solution** $A_h^\epsilon u_h^\epsilon = f_h^\epsilon$

Two Principal Relations

• **Computations on the base of a reliable (certified) model. Here ε_1 is assumed to be small and computed u_h^ϵ gives a desired information on U**

$$|U - u_h^\epsilon| \leq \quad \varepsilon_1 + \boxed{\varepsilon_2 + \varepsilon_3} \tag{1}$$

• **Verification of a mathematical model. Here physical data U and computed results u_h^ϵ are compared to judge on the quality of mathematical model**

$$|\varepsilon_1| \leq |U - u_h^\epsilon| + \boxed{\varepsilon_2 + \varepsilon_3} \tag{2}$$

Thus, two major problems of mathematical modeling:

- **reliable computer simulation**
- **verification of mathematical models by comparing physical and mathematical experiments**

require efficient methods able to provide

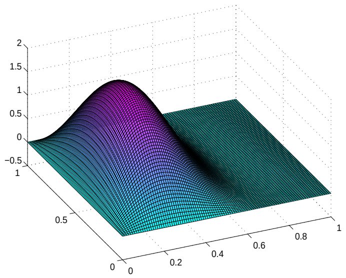
COMPUTABLE AND REALISTIC estimates of $\boxed{\varepsilon_2 + \varepsilon_3}$

No Error Control - No Reliability

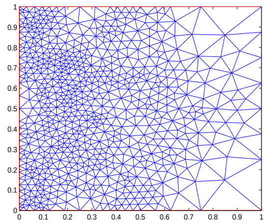
- In order to have really reliable computations we have to do both things: obtain an aproximate solution and also explicitly control the error $\boxed{\varepsilon_2 + \varepsilon_3}$

- In practical computations the second part of the total work is often ignored. Mostly it happenes because the problem is very complicated and computable error estimates are simply unknown. In other cases, it happens because the analysts believe that approximations obtained with help of "sufficiently fine" meshes and "powerful" computers have values ε_2 and ε_3 so small that they should not be taken into an account at all

• Such an approach does not provide with reliable computer simulations. Approximation solutions are quite sensitive to mesh perturbation and restructuring mesh can lead to very different results

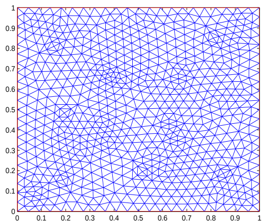


exact solution



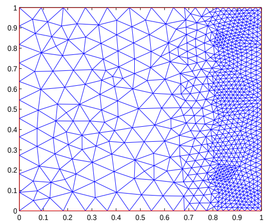
mesh with 741 nodes

error = 0.09212



mesh with 785 nodes

error = 0.11562



mesh with 855 nodes

error = 0.44662

Drawbacks of A Priori Error Estimation

- A priori error estimates cannot guarantee that the error monotonically decreases as $h \to 0$
- The constants in those estimates are often not known at all or highly overestimated

A priori error estimates have only theoretical meaning – they show that the approximation method is correct in principle. However, in practice we are strongly interested in the error for the concrete approximation on the concrete mesh

• Because of these reasons, starting from 70-th a different approach for the error control has been developing. It is called as *a posteriori error control for partial differential equations*