A POSTERIORI ERROR ESTIMATES BASED ON NEGATIVE NORMS OF RESIDUALS

Sergey Korotov

Basque Center for Applied Mathematics / IKERBASQUE

http://www.bcamath.org & http://www.ikerbasque.net

Model Problem

Find a function u such that

$$-\mathrm{div}\,(\mathcal{A}\,\nabla u) = f \quad \text{in } \Omega \tag{1}$$

$$u = u_0 \quad \text{on } \Gamma_D \tag{2}$$

$$n^T \mathcal{A}\,\nabla u = g \quad \text{on } \Gamma_N \tag{3}$$

• We assume that $a_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $u_0 \in H^1(\Omega)$. Moreover let for a.a. $x \in \Omega$ one has $a_{ij}(x) = a_{ji}(x)$ and there exist constants $C_1, C_2 > 0$ such that

$$C_1\|\xi\|^2 \le \mathcal{A}(x)\xi\cdot\xi \le C_2\|\xi\|^2 \quad \forall \xi \in \mathbf{R}^d$$

Weak Formulation

We introduce the space of test functions

$$V = \{v \in H^1(\Omega) \mid v = 0 \ \text{ on } \ \Gamma_D\}$$

and define the bilinear and linear forms

$$a(v,w) = \int_\Omega \mathcal{A}\nabla w \cdot \nabla v \, dx, \quad F(v) = \int_\Omega f v \, dx + \int_{\Gamma_N} g v \, ds.$$

Definition: Let $\Omega \in \mathcal{L}$ and $\Gamma_D \neq \emptyset$. A function $u \in H^1(\Omega)$ is called a weak (or generalized) solution of the problem (1)–(3) if

$$u - u_0 \in V \qquad \text{and} \qquad a(v,u) = F(v) \quad \forall v \in V$$

Theorem: Let $\Omega \in \mathcal{L}$ and $\Gamma_D \neq \emptyset$. Then there exists exactly one weak solution $u \in H^1(\Omega)$ of the problem (1)–(3)

Equivalent Norms for Measuring Errors

- The solution $u = u_0 + u_\star$, where $u_\star \in V$. Let $\bar{u} = u_0 + \bar{u}_\star$ ($\bar{u}_\star \in V$) be an approximation of u. We want to find (or estimate) the following value $\|u - \bar{u}\|_{1,\Omega}$
- First, for $w \in H^1(\Omega)$ we have by definition

$$\|w\|_{1,\Omega}^2 := \|w\|_{0,\Omega}^2 + \|\nabla w\|_{0,\Omega}^2$$

- Also, the following Friederichs' inequality holds for $w \in V$

$$\|w\|_{0,\Omega} \leq C_{\Omega,\Gamma_D} \|\nabla w\|_{0,\Omega} \qquad (FI)$$

- The function $u_\star$ minimizes the energy functional over V

$$J(w) = \frac{1}{2} a(w, w) - F(w)$$

Now, since $u - \bar{u} \in V$ we observe that

$$\|u - \bar{u}\|_{1,\Omega}^2 \leq \left(1 + C_{\Omega,\Gamma_D}^2\right) \|\nabla(u - \bar{u})\|_{0,\Omega}^2 \leq \frac{1 + C_{\Omega,\Gamma_D}^2}{C_1} a(u - \bar{u}, u - \bar{u})$$

and, also, it holds

$$\frac{1}{C_2} a(u - \bar{u}, u - \bar{u}) \leq \|\nabla(u - \bar{u})\|_{0,\Omega}^2 \leq \|u - \bar{u}\|_{1,\Omega}^2$$

which altogether means that the estimation of the error in the energy norm $\sqrt{a(\cdot,\cdot)}$ is equivalent to the error estimation in the norm $\|\cdot\|_{1,\Omega}$ with known equivalence constants

• Moreover, since $a(u - \bar{u}, u - \bar{u}) = 2(J(\bar{u}_\star) - J(u_\star))$, and J often presents the total energy of the system under analysis, the measurement of the error in terms of the energy norm is physically more natural

• From above we also observe that estimation of $\|u - \bar{u}\|_{1,\Omega}$ is equivalent to estimation of $\|\nabla(u - \bar{u})\|_{0,\Omega}^2$

First Attempt of Error Control

- The first attempt of the error control was probably done in W. Prager, J.L. Synge "Approximation in elasticity based on the concept of functional spaces", Quart. Appl. Math., **5** (1947)

- Their estimate has been obtained using geometrical arguments. In terms of our model elliptic BVP in the simplest setting

$$-\Delta u = f \quad \text{in} \quad \Omega \quad \& \quad u = 0 \quad \text{on} \quad \partial\Omega$$

i.e., $\mathcal{A} \equiv I$, $u_0 \equiv 0$, $\Gamma_N = \emptyset$, it reads as follows

$$\|\nabla(u - \bar{u})\|^2_{0,\Omega} + \|\nabla u - \tau\|^2_{0,\Omega} = \|\nabla \bar{u} - \tau\|^2_{0,\Omega} \qquad (PS)$$

where $\bar{u} \in H^1_0(\Omega)$ is an appoximation and τ satisfies $\operatorname{div} \tau + f = 0$

- RHS of the estimate (PS) is explicitly computable once we know $\bar{u}$ and τ, and we need not know anything about u

The estimate (PS) can be proved as follows. First we notice that $\operatorname{div}(\nabla u - \tau) = 0$ coming from the condition on τ and our equation. Thus, the following orthogonality relation holds

$$0 = \int_{\Omega} (u - \bar{u}) \operatorname{div}(\nabla u - \tau)\, dx = \int_{\Omega} \nabla(u - \bar{u}) \cdot (\nabla u - \tau)\, dx$$

which, obviously, results then into

$$\|\nabla(u - \bar{u})\|_{0,\Omega}^2 + \|\nabla u - \tau\|_{0,\Omega}^2 = \|\nabla(u - \bar{u}) - (\nabla u - \tau)\|_{0,\Omega}^2 = \|\nabla \bar{u} - \tau\|_{0,\Omega}^2$$

The estimate (PS) can be written also as

$$a(u - \bar{u}, u - \bar{u}) \equiv \|\nabla(u - \bar{u})\|_{0,\Omega}^2 \leq \|\nabla \bar{u} - \tau\|_{0,\Omega}^2 \qquad (PS')$$

• The drawback of the above estimates is that we do not know how to construct τ satisfying the differential relation $\operatorname{div} \tau + f = 0$

ERRORS and RESIDUALS

Case of System of Linear Algebraic Equations

- One natural way to check the quality of an approximate solution is to substitute the approximation into the governing equation and check how large the resulting residual is

First we demonstrate how this procedure works for the case of a system of linear algebraic equations in $\mathbf{R}^d$:

$$\mathbf{A}\,\mathbf{u} = \mathbf{f}$$

Let $\bar{\mathbf{u}}$ be any vector from $\mathbf{R}^d$, then we observe

$$\mathbf{A}\,(\mathbf{u} - \bar{\mathbf{u}}) =: \mathbf{A}\,\mathbf{e} = \mathbf{r} := \mathbf{f} - \mathbf{A}\,\bar{\mathbf{u}}$$

which gives the following *residual type estimate*

$$\|\mathbf{e}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{r}\|$$

Let us define quantities

$$\lambda_{min} = \min_{\mathbf{x}\in\mathbf{R}^d,\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\,\mathbf{x}\|}{\|\mathbf{x}\|} \quad \text{and} \quad \lambda_{max} = \max_{\mathbf{x}\in\mathbf{R}^d,\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\,\mathbf{x}\|}{\|\mathbf{x}\|}$$

Since $\mathbf{Ae} = \mathbf{r}$ and $\mathbf{Au} = \mathbf{f}$ we have

$$\lambda_{min} \leq \frac{\|\mathbf{A}\,\mathbf{e}\|}{\|\mathbf{e}\|} = \frac{\|\mathbf{r}\|}{\|\mathbf{e}\|} \leq \lambda_{max} \quad \text{and} \quad \lambda_{min} \leq \frac{\|\mathbf{A}\,\mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\|\mathbf{f}\|}{\|\mathbf{u}\|} \leq \lambda_{max}$$

or, in another form,

$$\lambda_{max}^{-1}\|\mathbf{r}\| \leq \|\mathbf{e}\| \leq \lambda_{min}^{-1}\|\mathbf{r}\| \quad \text{and} \quad \lambda_{max}^{-1}\|\mathbf{f}\| \leq \|\mathbf{u}\| \leq \lambda_{min}^{-1}\|\mathbf{f}\|$$

Thus,

$$\kappa^{-1}(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{f}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \leq \kappa(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{f}\|}$$

where $\kappa(\mathbf{A}) = \frac{\lambda_{max}}{\lambda_{min}}$ is a condition number of $\mathbf{A}$

• It means that the error (relative error) is directly controlled via the value (relative value) of the residual

Case of Boundary Value Problems

• The above agruments can obviously be used for a wider class of linear problems, where now

$$A : V \to U$$

is a coercive linear operator from Banach space V to Banach space U and f is a given element from U

• If A is related to some BVP we must define what the spaces V and U are then, and find a meaningful analogue of the estimate constructed just before for the case of a system of algebraic equations

• Let $A : V \to U$ be a linear elliptic operator. Consider the following BVP

$$Au = f \quad \text{in} \quad \Omega \quad \& \quad u = 0 \quad \text{on} \quad \partial\Omega$$

• Assume that $\bar{u} \in V$ is an approximation of u. We should obviously measure the error in the space V and the residual – in the space U

• That is the principial form of the desired estimate is as follows

$$\|u - \bar{u}\|_V \le C\|f - A\bar{u}\|_U \qquad (DE)$$

where the constant C is independent of $\bar{u}$

• Then the main question is – which spaces V and U do we choose for a concrete BVP

Model Problem: First Trial

- Consider first our model elliptic BVP

$$-\mathrm{div}(\mathcal{A}\nabla u) = f \quad \text{in} \quad \Omega \quad \& \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with standard conditions on matrix of coefficients (see Lecture 1)

- The generalized solution of BVP $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$

- The question now is if it is suitable to take spaces as follows $V = H_0^1(\Omega)$ and $U = L^2(\Omega)$?

Idea to Get Relevant Estimate: Solution $u \in H^1_0(\Omega)$ satisfies the integral identity

$$\int\limits_{\Omega} \mathcal{A}\nabla u \cdot \nabla w \, dx = \int\limits_{\Omega} f w \, dx \quad \forall w \in H^1_0(\Omega) \qquad (II)$$

which implies (taking $w = u$) the so-called *energy estimate*

$$\|\nabla u\|_{0,\Omega} \leq \frac{C_{\Omega,\Gamma_D}}{C_1} \|f\|_{0,\Omega} \qquad (EE)$$

where C_{Ω,Γ_D} is a constant in the Friedrichs' inequality (FI)

- This estimate is a particular case of (DE) with $\bar{u} = 0$. We can try to get similarly an estimate involving also the approximation $\bar{u}$

• Let $\bar{u}$ be such that $\operatorname{div}(\mathcal{A}\nabla\bar{u}) \in L^2(\Omega)$. Then, similarly to (II)

$$\int_\Omega \mathcal{A}\nabla(u-\bar{u})\cdot\nabla w\,dx = \int_\Omega (f + \operatorname{div}(\mathcal{A}\nabla\bar{u}))w\,dx \quad \forall w \in H_0^1(\Omega)$$

which now implies (with $w = u - \bar{u}$) another estimate

$$\|\nabla(u-\bar{u})\|_{0,\Omega} \leq \frac{C_{\Omega,\Gamma_D}}{C_1}\|f + \operatorname{div}(\mathcal{A}\nabla\bar{u})\|_{0,\Omega}$$

whose RHS is, as desired, the L^2-norm of the residual

• However, normally a sequence of approximations u_h converges to the exact solution u only in the norm $\|\cdot\|_{1,\Omega}$, i.e., the sequence $\|f + \operatorname{div}(\mathcal{A}\nabla u_h)\|_{0,\Omega}$ need not tend to zero as $h \to 0$. Thus, very important property of consistency for the above estimate is lost

• It follows from the above analysis that another norm of the residuals should be taken

Sobolev Spaces with Negative Indices

• In order to explain the next approach we need extra mathematical tools related to the idea of the so-called "negative norms". First, we recall several definitions for distributions

Definition: Linear continuous functionals defined on the space $C_0^\infty(\Omega)$ are called *distributions*

Denotation: The space of distrubitions is traditionally denoted by $\mathcal{D}'(\Omega)$. The value of a distribution g on a function $\varphi \in C_0^\infty(\Omega)$ is denoted by $< g, \varphi >$

Definition: We say that distributions g_1 and g_2 are equal in Ω if

$$< g_1, \varphi >=< g_2, \varphi > \quad \forall \varphi \in C_0^\infty(\Omega)$$

Definition: We say that the distribution g is a sum of distributions g_1 and g_2 if

$$< g, \varphi >=< (g_1 + g_2), \varphi > \quad \forall \varphi \in C_0^\infty(\Omega)$$

Definition: If a distribution g can be identified with a locally integrable function $\bar{g}$, then it is called *regular.* In this case, the action of g is given by the Lebesgue integral

$$< g, \varphi >= \int\limits_\Omega \bar{g} \varphi \, dx$$

The other distributions are called *singular*

• Distributions possess a very important property – they have derivatives of any order if the differentiation is understood in a special (generalized) sense

Definition: Let g be a distribution. Its *generalized derivative* $D^m g$ is a linear continuous functional defined for any $\varphi \in C_0^\infty(\Omega)$ as follows

$$< D^m g, \varphi > := (-1)^{|m|} < g, D^m \varphi >$$

• Any function from $L^p(\Omega)$ always defines a certain distribution and, therefore, has generalized derivatives (in the sense of distributions) of any order

Definition: Sobolev space $W_p^{-k}(\Omega)$ is the space of distributions $g \in D'(\Omega)$ such that

$$g = \sum_{|m| \leq k} D^m g_m$$

where $g_m \in L^p(\Omega)$

- Let $g \in L^2(\Omega)$. Then the functional

$$< \frac{\partial g}{\partial x_i}, \varphi >:= -\int\limits_{\Omega} g \frac{\partial \varphi}{\partial x_i}\, dx$$

is l. c. not only for functions from $C_0^\infty(\Omega)$ but also for functions from $H_0^1(\Omega)$. This follows from the density of smooth functions in $H_0^1(\Omega)$ and known theorem on the continuation of l. c. functionals. Hence, the first generalized derivatives of g lie in the space dual to $H_0^1(\Omega)$, usually denoted by $H^{-1}(\Omega)$

- For $g \in H^{-1}(\Omega)$ we can introduce a *"negative norm"*

$$\|g\|_{-1,\Omega} := \sup_{w \in H_0^1(\Omega), w \neq 0} \frac{| < g, w > |}{\|\nabla w\|_{0,\Omega}}$$

- From the definition it follows

$$| < g, w > | \leq \|g\|_{-1,\Omega} \|\nabla w\|_{0,\Omega}$$

Consistent Error Estimate

- Now we can look at the integral identity (II)

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in H_0^1(\Omega) \qquad (II)$$

as equality of two distributions from $H^{-1}(\Omega)$ acting on $w \in H_0^1(\Omega)$

$$< -\text{div}\, \mathcal{A}\nabla u, w > = < f, w >$$

which is correct due to $\mathcal{A}\nabla u \in (L^2(\Omega))^d$ and $f \in L^2(\Omega)$

- Using $< f, w > \le \|f\|_{-1,\Omega} \|\nabla w\|_{0,\Omega}$ we get an estimate

$$\|\nabla u\|_{0,\Omega} \le \frac{1}{C_1} \|f\|_{-1,\Omega}$$

which suggests to use negative norm for the residuals

Let $\bar{u} \in H_0^1(\Omega)$ be an approximation of u. We have

$$\int_\Omega \mathcal{A}\nabla(u-\bar{u})\cdot\nabla w\,dx = \int_\Omega \Big(fw-\mathcal{A}\nabla\bar{u}\cdot\nabla w\Big)\,dx =:< f+\operatorname{div}\mathcal{A}\nabla\bar{u}, w >$$

where $f + \operatorname{div}\mathcal{A}\nabla\bar{u} \in H^{-1}(\Omega)$. Setting $w = u - \bar{u}$, we obtain

$$C_1\|\nabla(u-\bar{u})\|_{0,\Omega} \le \|f + \operatorname{div}\mathcal{A}\nabla\bar{u}\|_{-1,\Omega}$$

Further

$$\|f + \operatorname{div}\mathcal{A}\nabla\bar{u}\|_{-1,\Omega} = \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{|< f + \operatorname{div}\mathcal{A}\nabla\bar{u}, w >|}{\|\nabla w\|_{0,\Omega}} =$$

$$= \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{|\int_\Omega \mathcal{A}\nabla(u-\bar{u})\cdot\nabla w\,dx|}{\|\nabla w\|_{0,\Omega}} \le C_2\|\nabla(u-\bar{u})\|_{0,\Omega}$$

Finally

$$\frac{1}{C_2}\|f+\operatorname{div}\mathcal{A}\nabla\bar{u}\|_{-1,\Omega} \le \|\nabla(u-\bar{u})\|_{0,\Omega} \le \frac{1}{C_1}\|f+\operatorname{div}\mathcal{A}\nabla\bar{u}\|_{-1,\Omega}$$

• Thus, the upper and the lower bounds of the error can be evaluated in terms of the negative norm of the residual

• A posteriori error estimates obtained in the 70–90 -th are based on reasonable evaluation of $\|f+\operatorname{div}\mathcal{A}\nabla\bar{u}\|_{-1,\Omega}$, where $\bar{u}$ is assumed to be the "exact" finite element aproximation, i.e., the error ε_3 is neglected again

• We shall present in detail the so-called *explicit residual method,* which is essentially based two key points – Galerkin orthogonality property and Clement interpolation operator, and clearly demonstrates all main features of the approach of estimating the errors via residuals for linear BVPs

Estimation of Residuals in 1D

First we consider the model problem in 1D

$$-(a(x)u'(x))' = f(x)\,, \quad x \in (0,1) \quad \& \quad u(0) = u(1) = 0$$

where $f \in L^2(0,1)$, $a(x) \in C^1([0,1])$ and $a(x) \geq a_0 > 0$

• Let $\triangle_i = (x_i, x_{i+1})$, $x_0 = 0$, $x_{N+1} = 1$ and $h_i = |x_{i+1} - x_i|$

• We also assume that the approximation $\bar{u} \in H_0^1(0,1)$ and is smooth in each subinterval $\triangle_i$, $i = 1, \ldots, N$

Then we observe

$$\|f+(a\bar{u}')'\|_{-1,(0,1)} = \sup_{w\in H_0^1(0,1),\, w\neq 0} \frac{\left|\int_0^1 (-a\bar{u}'w' + fw)\, dx\right|}{\|w'\|_{0,(0,1)}} =$$

$$= \sup_{w\in H_0^1(0,1),\, w\neq 0} \frac{\left|\sum_{i=0}^{N} \int_{\triangle_i} (-a\bar{u}'w' + fw)\, dx\right|}{\|w'\|_{0,(0,1)}} =$$

$$= \sup_{w\in H_0^1(0,1),\, w\neq 0} \frac{\left|\sum_{i=0}^{N} \int_{\triangle_i} r_i(\bar{u})w\, dx + \sum_{i=0}^{N} a(x_i)w(x_i)j(\bar{u}'(x_i))\right|}{\|w'\|_{0,(0,1)}} =$$

where $j(\bar{u}'(x_i)) = \bar{u}'(x_i+0) - \bar{u}'(x_i-0)$ is the jump of gradient of the approximation $\bar{u}$ at the node x_i and $r_i(\bar{u}) = (a\bar{u}')' + f$ is the residual in $\triangle_i$

• It is most probably impossible to get an upper bound for the presented supremum if $\bar{u}$ is an arbitrary function

• However, when $\bar{u} = u_h$, i.e., if it is assumed that approximation is obtained with a help of FEM, it turns to be possible to get the desired upper estimate in a relatively simple way

So, let $V_h \subset H_0^1(0,1)$ be a finite-dimensional subspace and

$$0 = \int_0^1 a u_h' w_h' \, dx - \int_0^1 f w_h \, dx \quad \forall w_h \in V_h$$

This means that we can add RHS of the above identity with any w_h to the numerator in the definition of the residual, which gives the following

$$\|f + (au_h')'\|_{-1,(0,1)} =$$

$$= \sup_{w \in H_0^1(0,1),\, w \neq 0} \frac{\left| \int_0^1 (-au_h'(w - \pi_{\mathbf{h}}\mathbf{w})' + f(w - \pi_{\mathbf{h}}\mathbf{w}))\, dx \right|}{\|w'\|_{0,(0,1)}}$$

where $\pi_h : H_0^1(0,1) \to V_h$ is the interpolation operator defined by conditions $\pi_h w \in V_h$, $\pi_h w(0) = \pi_h w(1) = 0$ and

$$\pi_h w(x_i) = w(x_i), \;\; i = 1, \ldots, N$$

Hence, we can get

$$\|f + (au_h')'\|_{-1,(0,1)} = \sup_{w \in H_0^1(0,1),\, w \neq 0}$$

$$\frac{\left| \sum_{i=0}^{N} \int_{\triangle_i} r_i(u_h)(w - \pi_h w)\, dx + \sum_{i=0}^{N} a(x_i)(w(x_i) - \pi_h w(x_i)) j(u_h'(x_i)) \right|}{\|w'\|_{0,(0,1)}}$$

- Since $w(x_i) - \pi_h w(x_i) = 0$, the second term in the numerator vanishes
- We also have

$$\sum_{i=0}^{N} \int_{\triangle_i} r_i(u_h)(w - \pi_h w)\, dx \leq \sum_{i=0}^{N} \|r_i(u_h)\|_{0,\triangle_i} \|w - \pi_h w\|_{0,\triangle_i}$$

Since for $w \in H_0^1(0,1)$

$$\|w - \pi_h w\|_{0,\triangle_i} \le c_i \|w'\|_{0,\triangle_i}$$

we obtain for the numerator of the above expression that

$$\sum_{i=0}^{N} \int_{\triangle_i} r_i(u_h)(w - \pi_h w)\, dx \le \sum_{i=0}^{N} \|r_i(u_h)\|_{0,\triangle_i} \|w - \pi_h w\|_{0,\triangle_i} \le$$

$$\le \sum_{i=0}^{N} c_i \|r_i(u_h)\|_{0,\triangle_i} \|w'\|_{0,\triangle_i} \le \left(\sum_{i=0}^{N} c_i^2 \|r_i(u_h)\|_{0,\triangle_i}^2 \right)^{1/2} \|w'\|_{0,(0,1)}$$

which implies

$$\|f + (a u_h')'\|_{-1,(0,1)} \le \left(\sum_{i=0}^{N} c_i^2 \|r_i(u_h)\|_{0,\triangle_i}^2 \right)^{1/2}$$

• This bound presents a sum of local residuals $r_i(u_h)$ with weights given by the *interpolation constants* c_i

Interpolation Constants

- There is still an open problem - how to compute the interpolation constants
- It turned out that for the above considered 1D cade this task can be relatively easy solved

Let γ_i be a constant such that

$$\inf_{w \in H^1(\triangle_i), w \neq 0} \frac{\|w'\|^2_{0,\triangle_i}}{\|w - \pi_h w\|^2_{0,\triangle_i}} \geq \gamma_i$$

Then, obviously, we can take $c_i = \gamma_i^{-1/2}$

Note that

$$\int_{x_i}^{x_{i+1}} |w'|^2\,dx = \int_{x_i}^{x_{i+1}} |(w-\pi_h w)' + (\pi_h w)'|^2\,dx$$

where $(\pi_h w)'$ is constant on (x_i, x_{i+1}). Hence

$$\int_{x_i}^{x_{i+1}} (w-\pi_h w)'(\pi_h w)'\,dx = 0$$

and

$$\int_{x_i}^{x_{i+1}} |w'|^2\,dx = \int_{x_i}^{x_{i+1}} |(w-\pi_h w)'|^2\,dx + \int_{x_i}^{x_{i+1}} |(\pi_h w)'|^2\,dx \geq \int_{x_i}^{x_{i+1}} |(w-\pi_h w)'|^2\,dx$$

Thus, we have

$$\inf_{w\in H^1(\triangle_i),w\neq 0}\frac{\|w'\|^2_{0,\triangle_i}}{\|w-\pi_h w\|^2_{0,\triangle_i}} \geq \inf_{w\in H^1(\triangle_i),w\neq 0}\frac{\int\limits_{x_i}^{x_{i+1}} |(w-\pi_h w)'|^2\,dx}{\int\limits_{x_i}^{x_{i+1}} |w-\pi_h w|^2\,dx} \geq$$

$$\geq \inf_{\theta\in H^1(\triangle_i),\theta\neq 0}\frac{\int\limits_{x_i}^{x_{i+1}} |\theta'|^2\,dx}{\int\limits_{x_i}^{x_{i+1}} |\theta|^2\,dx} = \inf_{\theta\in H^1(0,h_i),\theta\neq 0}\frac{\int\limits_{0}^{h_i} |\theta'|^2\,dx}{\int\limits_{0}^{x_i} |\theta|^2\,dx} = \frac{\pi^2}{h_i^2}$$

• The last equality comes from the fact that the infimum is attained on the eigenfunction $\sin\frac{\pi}{h_i}x$ of the problem $\theta''+\lambda\theta=0$ in $(0,h_i)$

• As a result, we take $\gamma_i=\frac{\pi^2}{h_i^2}$ and $C_i=\frac{h_i}{\pi}$

Estimation of Residuals in 2D

Now we consider the model problem in 2D

$$-\mathrm{div}(\mathcal{A}\nabla u) = f \quad \text{in} \quad \Omega \quad \& \quad u = 0 \quad \text{on} \quad \partial\Omega$$

Let $V_h \subset H_0^1(\Omega)$ be FE-space generated by linear elements and triangulation $\mathcal{T}_h$ and let u_h be the corresponding FE solution. i.e.,

$$\int_\Omega \mathcal{A}\nabla u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx \quad \forall w_h \in V_h$$

In this case the negative norm of the residual is

$$\|f + \mathrm{div}\, \mathcal{A}\nabla u_h\|_{-1,\Omega} = \sup_{w \in H_0^1(\Omega), w \neq 0} \frac{|\int_\Omega \Big(f w - \mathcal{A}\nabla u_h \cdot \nabla w \Big)\, dx|}{\|\nabla w\|_{0,\Omega}}$$

- Let $\pi_h : H_0^1(\Omega) \to V_h$ be a continuous interpolation operator
- Then for the finite element approximation u_h we have

$$\|f + \operatorname{div} \mathcal{A}\nabla u_h\|_{-1,\Omega} =$$

$$= \sup_{w \in H_0^1(\Omega), w \neq 0} \frac{\left| \int\limits_{\Omega} \Big(f(w - \pi_{\mathbf{h}}\mathbf{w}) - \mathcal{A}\nabla u_h \cdot \nabla(w - \pi_{\mathbf{h}}\mathbf{w}) \Big)\, dx \right|}{\|\nabla w\|_{0,\Omega}}$$

- One of the most known interpolation operators π_h with desired properties has been suggested in Ph. Clément "Approximations by finite element functions using local regularization", RAIRO Anal. Numér., **9** (1975)
- It is often called Clement's interpolation operator and widely used in a posteriori error estimation analysis

Clement's Interpolation Operator

- Let x_s denote a node of the triangulation $\mathcal{T}_h$, $v \in H_0^1(\Omega)$
- Let Ω_s denote the union of elements in $\mathcal{T}_h$ having x_s as a vertex

For any interior x_s we find a polynomial $p_s(x) \in P^1(\Omega_s)$ such that

$$\int\limits_{\Omega_s} (v - p_s) q \, dx = 0 \qquad \forall q \in P^1(\Omega_s)$$

Now we uniquely define linear and continuous mapping

$$\pi_h : H_0^1(\Omega) \to V_h$$

by setting

$$\pi_h v(x_s) = p_s(x_s) \quad \text{if } x_s \in \Omega \qquad \& \qquad \pi_h v(x_s) = 0 \quad \text{if } x_s \in \partial\Omega$$

Interpolation Estimates

• Let the triangular elements of the triangulation $\mathcal{T}_h$ be denoted by the symbols $\triangle_i$ and let E_{kl} denote the common edge of $\triangle_k$ and $\triangle_l$

The following relation hold

$$\|v - \pi_h v\|_{0,\triangle_i} \leq C_{1,i} \mathrm{diam}\,(\triangle_i) \|v\|_{1,\Omega_N(\triangle_i)}$$

$$\|v - \pi_h v\|_{0,E_{kl}} \leq C_{2,kl} |E_{kl}|^{1/2} \|v\|_{1,\Omega_N(E_{kl})}$$

where $\Omega_N(\triangle_i)$ denotes the union of all triangles having at least one common node with the element $\triangle_i$ and $\Omega_N(E_{kl})$ stands for the union of all triangles having at least a common node with E_{kl}

• For more details see book [Verfürth, 1996] and references therein

Let $\sigma_h := \mathcal{A}\nabla u_h$, then we have

$$\|f + \operatorname{div} \mathcal{A}\nabla u_h\|_{-1,\Omega} =$$

$$= \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{\left|\int\limits_{\Omega} \Big(f(w-\pi_h w) - \sigma_h \cdot \nabla(w-\pi_h w)\Big)\, dx\right|}{\|\nabla w\|_{0,\Omega}}$$

Let ν_{kl} be the unit normal to the edge E_{kl}, then we observe

$$\int\limits_{\triangle_i} \sigma_h \cdot \nabla(w-\pi_h w)\, dx = \sum_{E_{kl}\subset\partial\triangle_i} \int\limits_{E_{kl}} \sigma_h \cdot \nu_{kl}\,(w-\pi_h w)\, ds -$$

$$- \int\limits_{\triangle_i} \mathrm{div}\sigma_h\,(w-\pi_h w)\, dx$$

Further

$$\|f + \operatorname{div} \mathcal{A}\nabla u_h\|_{-1,\Omega} \leq$$

$$\leq \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{|\sum_{i=1}^{N} \int_{\triangle_i} (f + \operatorname{div}\sigma_h)(w - \pi_h w)\, dx|}{\|\nabla w\|_{0,\Omega}} +$$

$$+ \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{|\sum_{k=1}^{K} \sum_{l>k}^{K} \int_{E_{kl}\subset\Omega} j(\sigma_h \cdot \nu_{kl})(w - \pi_h w)\, ds|}{\|\nabla w\|_{0,\Omega}} = \mathbf{I_1} + \mathbf{I_2}$$

where N is the number of elements and K is the number of nodes in the triangulation $\mathcal{T}_h$, the symbol j stands for the jump function

- In the above we consider only edges $E_{kl} \subset \Omega$ since $w - \pi_h w = 0$ on the boundary $\partial\Omega$ by definition

Consider now the term $\mathbf{I_1}$. First, we notice that

$$\int_{\triangle_i} (f + \mathrm{div}\sigma_h)(w - \pi_h w)\, dx \le \|f + \mathrm{div}\sigma_h\|_{0,\triangle_i} \, \|w - \pi_h w\|_{0,\triangle_i} \le$$

$$\le C_{1,i} \|f + \mathrm{div}\sigma_h\|_{0,\triangle_i} \, \mathrm{diam}\,(\triangle_i) \|w\|_{1,\Omega_N(\triangle_i)}$$

Then we get

$$\sum_{i=1}^{N} \int_{\triangle_i} (f + \mathrm{div}\sigma_h)(w - \pi_h w)\, dx \le$$

$$\le d_1 \Big(\sum_{i=1}^{N} C_{1,i}^2 \mathrm{diam}^2\,(\triangle_i) \|f + \mathrm{div}\sigma_h\|_{0,\triangle_i}^2 \Big)^{1/2} \|w\|_{1,\Omega}$$

- d_1 depends on the maximal number of elements in $\Omega_N(\triangle_i)$

Consider now the term $\mathbf{I_2}$. Using interpolation estimates we get

$$\sum_{k=1}^{K} \sum_{l>k}^{K} \int_{E_{kl} \subset \Omega} j(\sigma_h \cdot \nu_{kl})(w - \pi_h w)\, ds \leq$$

$$\leq \sum_{k=1}^{K} \sum_{l>k, E_{kl} \subset \Omega}^{K} \|j(\sigma_h \cdot \nu_{kl})\|_{0,E_{kl}} C_{2,kl} |E_{kl}|^{1/2} \|w\|_{1,\Omega_N(E_{kl})} \leq$$

$$\leq d_2 \Big(\sum_{k=1}^{K} \sum_{l>k, E_{kl} \subset \Omega}^{K} C_{2,kl}^2 |E_{kl}| \|j(\sigma_h \cdot \nu_{kl})\|_{0,E_{kl}}^2 \Big)^{1/2} \|w\|_{1,\Omega}$$

- d_2 depends on the maximal number of elements in $\Omega_N(E_{kl})$

Error Estimate of Residual Type

Finally, we obtain

$$\|f + \operatorname{div} \mathcal{A}\nabla u_h\|_{-1,\Omega} \leq$$

$$\leq C_0\Big(\Big(\sum_{i=1}^{N} C_{1,i}^2 \text{diam}^2\,(\triangle_i)\|f + \text{div}\sigma_h\|_{0,\triangle_i}^2\Big)^{1/2} +$$

$$+\Big(\sum_{k=1}^{K}\sum_{l>k, E_{kl}\subset\Omega}^{K} C_{2,kl}^2 |E_{kl}| \|j(\sigma_h \cdot \nu_{kl})\|_{0,E_{kl}}^2\Big)^{1/2}\Big) \qquad (\bullet)$$

- In the above, the constant C_0 depends on d_1 and d_2

Final Comments

Quasi-Uniform Meshes: In this case all generic constants $C_{1,i}$ are approximately of the same value (and, similarly, the constants $C_{2,kl}$), i.e., they can be replaced by only two constants c_1 and c_2, respectively. Then RHS of the estimate $(\bullet)$ has a form

$$C\Big(\sum_{i=1}^{N}\eta^2(\triangle_i)\Big)^{1/2}$$

where

$$\eta^2(\triangle_i) = c_1^2 \mathrm{diam}^2\,(\triangle_i)\|f+\mathrm{div}\sigma_h\|^2_{0,\triangle_i}+\frac{c_2^2}{2}\sum_{E_{kl}\subset\partial\triangle_i}|E_{kl}|\|j(\sigma_h\cdot\nu_{kl})\|^2_{0,E_{kl}}$$

and the constant C depends on c_1, c_2, and C_0

- However, for strongly nonhomogeneous meshes this estimate can lead to high overestimation of the error

Arbitrary Meshes: In the general case, sharper evaluation of the interpolation constants is desired. It leads to considerations of the following two variational problems

$$\inf_{v\in H_0^1(\Omega),v\neq 0}\frac{\|v\|_{1,\Omega_N(\triangle_i)}}{\|v-\pi_h v\|_{0,\triangle_i}}\mathrm{diam}\,(\triangle_i)$$

and

$$\inf_{v\in H_0^1(\Omega),v\neq 0}\frac{\|v\|_{1,\Omega_N(E_{kl})}}{\|v-\pi_h v\|_{0,E_{kl}}}|E_{kl}|^{1/2}$$

- The constants $C_{1,i}$ and $C_{2,kl}$ are very difficult to compute or even to estimate from above, see, e.g., the paper [Carstensen, Funken]
- Moreover, the number of those constants depends on the dimension of the finite-element space V_h and can be very large
- *In fact, the estimation derived is only* <u>*error indicator*</u>

References

[1] I. Babuška and W. C. Rheinbold. Error estimates for adaptive finite element computations, SIAM J. Numer. Anal., **15**, 736–754 (1978).

[2] I. Babuška and W. C. Rheinbold. A posteriori error estimates for the finite element method, Internat. J. Numer. Methods Engrg., **12**, 1597–1615 (1978).

[3] C. Carstensen and S. A. Funken. Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods, East-West J. Numer. Math. **8**, 153–175 (2000).

[4] R. Verfürth. A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, 1996.