# TWO-SIDED ESTIMATES OF FUNCTIONAL TYPE FOR GLOBAL AND LOCAL ERROR CONTROL

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# Error Estimate of Residual Type

These estamates have the following form:

$$\|\nabla(u-u_h)\|_{0,\Omega} \leq$$

$$\leq C_0\Big(\Big(\sum_{i=1}^{N} C_{1,i}^2 \mathrm{diam}^2\,(\triangle_i)\|f+\mathrm{div}\sigma_h\|_{0,\triangle_i}^2\Big)^{1/2}+$$

$$+\Big(\sum_{k=1}^{K}\sum_{l>k,E_{kl}\subset\Omega}^{K} C_{2,kl}^2|E_{kl}|\|j(\sigma_h\cdot\nu_{kl})\|_{0,E_{kl}}^2\Big)^{1/2}\Big) \qquad (\bullet)$$

• In the above, the constant $C_0$ strongly depend on properties of the concrete mesh $\mathcal{T}_h$

• In fact, the most difficult type of error $\varepsilon_3$ is neglected

**Quasi-Uniform Meshes:** In this case all generic constants $C_{1,i}$ are approximately of the same value (and, similarly, the constants $C_{2,kl}$), i.e., they can be replaced by only two constants $c_1$ and $c_2$, respectively. Then RHS of the estimate $(\bullet)$ has a form

$$C\Big(\sum_{i=1}^{N}\eta^2(\triangle_i)\Big)^{1/2}$$

where

$$\eta^2(\triangle_i) = c_1^2\mathrm{diam}^2\,(\triangle_i)\|f+\mathrm{div}\sigma_h\|^2_{0,\triangle_i}+\frac{c_2^2}{2}\sum_{E_{kl}\subset\partial\triangle_i}|E_{kl}|\|j(\sigma_h\cdot\nu_{kl})\|^2_{0,E_{kl}}$$

and the constant $C$ depends on $c_1$, $c_2$, and $C_0$

- However, for strongly nonhomogeneous meshes this estimate cannot be properly computed or, at least, lead to high overestimation of the error

**Arbitrary Meshes:** In the general case, sharper evaluation of the interpolation constants is desired. It leads to considerations of the following two variational problems

$$\inf_{v\in H_0^1(\Omega),v\neq 0}\frac{\|v\|_{1,\Omega_N(\triangle_i)}}{\|v-\pi_h v\|_{0,\triangle_i}}\mathrm{diam}\,(\triangle_i)$$

and

$$\inf_{v\in H_0^1(\Omega),v\neq 0}\frac{\|v\|_{1,\Omega_N(E_{kl})}}{\|v-\pi_h v\|_{0,E_{kl}}}|E_{kl}|^{1/2}$$

- The constants $C_{1,i}$ and $C_{2,kl}$ are very difficult to compute or even to estimate from above, see, e.g., the paper [Carstensen, Funken]
- Moreover, the number of those constants depends on the dimension of the finite-element space $V_h$ and can be very large
- *In fact, the estimation derived is only error indicator*

# Error Estimates Based on Post-Processing

These estimates have the following forms:

$$\|\nabla(u - u_h)\|_{0,\Omega} \approx \|G_h(\nabla u_h) - \nabla u_h\|_{0,\Omega}$$

$$\ell(u - u_h) \approx E_0(u_h, v_\tau) + \int_\Omega (G_h(\nabla u_h) - \nabla u_h) \cdot (G_\tau(\nabla v_\tau) - \nabla v_\tau)\, dx$$

where $G_h$ and $G_\tau$ are suitable gradient averaging procedures

• The above error indicators are quite efficient and reliable in many situations, but there are problems (even simple ones) for which they completely fail

**Test 1:** In this test we demonstrate that the global error indicator completely fails if the problem data is not sufficiently smooth

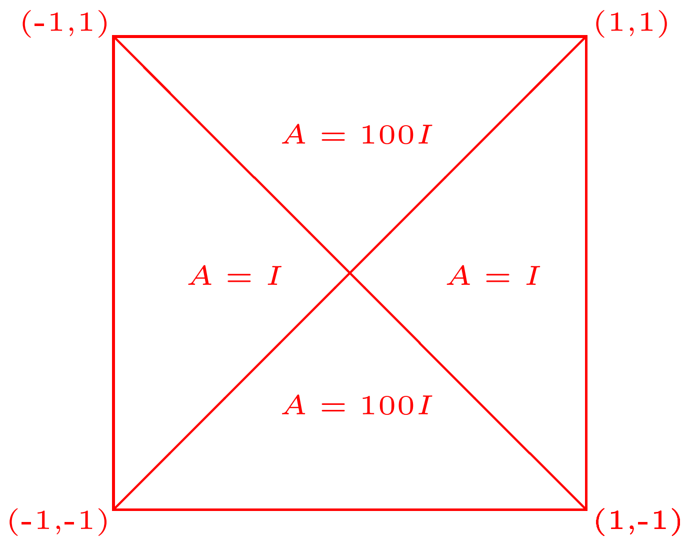



Figure 1: Definition of the coefficient matrix $\mathcal{A}$

For the indicator value we have 17.9628, which is quite different from the error $\|\nabla(u - u_h)\|_\Omega^2 \approx 0.8584$

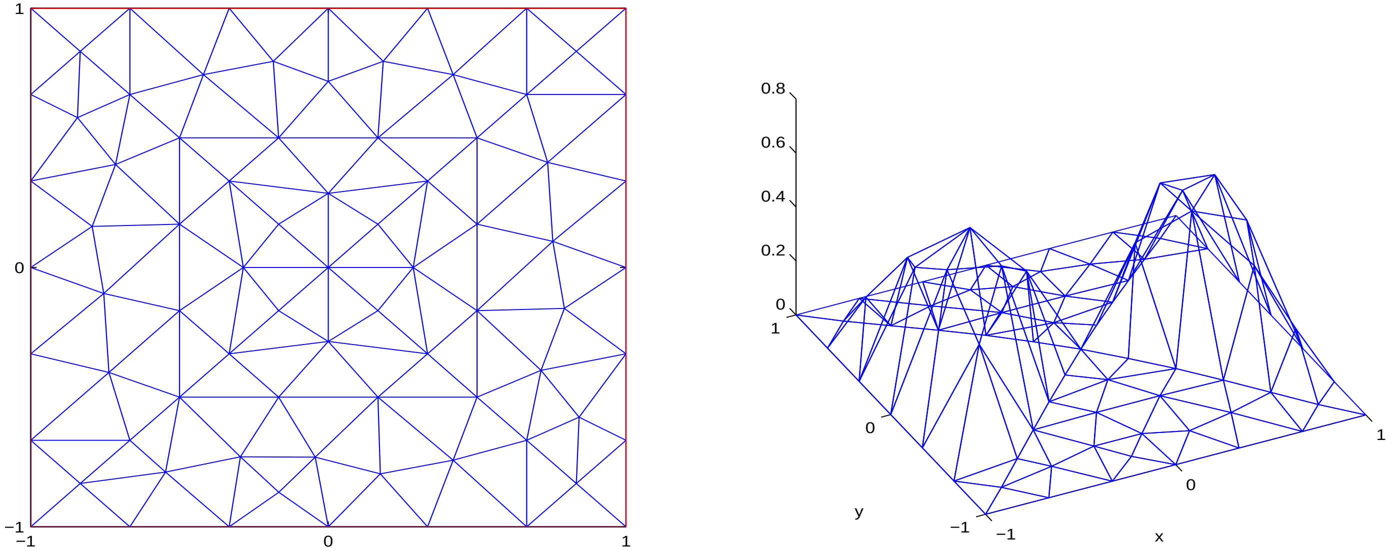
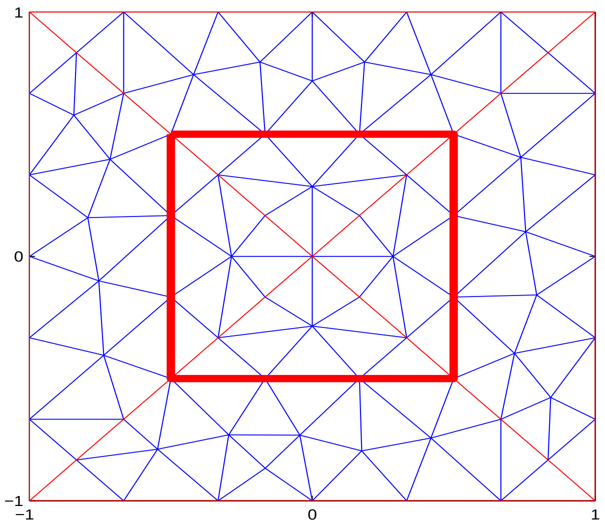
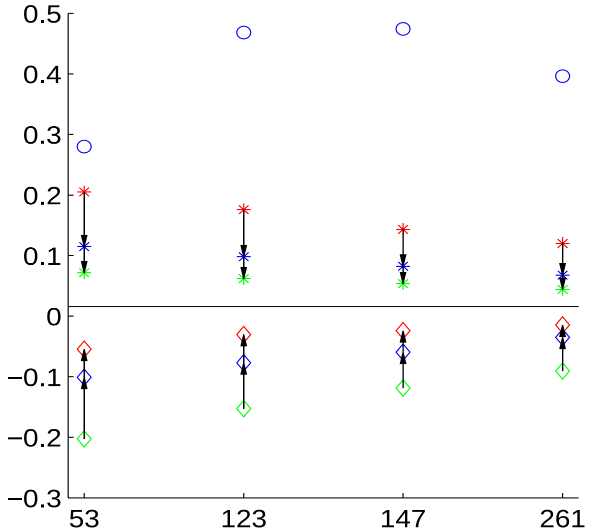



Figure 2: Solution domain $\Omega$ with computational mesh $\mathcal{T}_h$ having 78 nodes, the finite element approximation $u_h$

**Test 2:** To demonstrate that the local indicator fails, we take the same problem as in **Test 1**




We performed error estimation using several diferent meshes for the adjoint problem. We clearly see that the values of idicator (by circles) are essentially bigger than $\ell(u - u_h) = 0.0157$ for all choices of the mesh for the adjoint problem

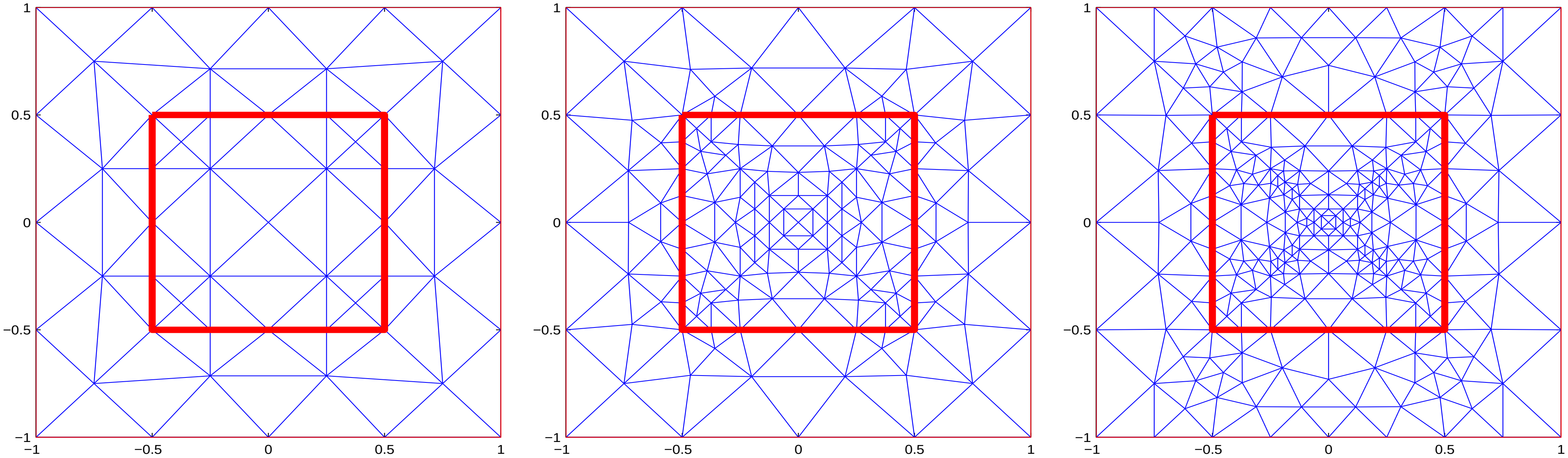



Figure 3: Meshes for adjoint problem with 53, 147, and 261 nodes

# NEW TECHNOLOGIES FOR ERROR CONTROL DURING COMPUTATIONS: THEORETICAL PART

## Main Targets of New Technologies

To create a set of numerical algorithms (and respective computer codes), which is able to give the following information about the accuracy of obtained approximate solutions:

• guaranteed two-sided bounds of the error in the global norm

• guaranteed two-sided estimation of various local errors

• guidelines for efficient mesh adaptation in order to decrease the errors

- All error estimates are required to be absolutely independent of the nature of the method used to obtain approximations, since we never have in practice e.g. true finite element solutions

# Model Elliptic Type BVPs

Find a function $u$ such that

$$-\mathrm{div}(A\nabla u) + cu = f \quad \text{in} \quad \Omega \tag{1}$$

$$u = u_0 \quad \text{on} \quad \Gamma_D \tag{2}$$

$$\nu^T \cdot A\nabla u = g \quad \text{on} \quad \Gamma_N \tag{3}$$

where $\Omega$ is bounded domain in $\mathbf{R}^d$ with Lipschitz boundary
$\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\mathrm{meas}_{\,d-1}\Gamma_D > 0$, $\nu$ is the outward normal to $\partial\Omega$

**Assumptions:** $f \in L_2(\Omega)$, $u_0 \in H^1(\Omega)$, $g \in L_2(\Gamma_N)$, $c \in L_\infty(\Omega)$, matrix of coefficients $A$ is symmetric, with entries $a_{ij} \in L_\infty(\Omega)$, $i, j = 1, \ldots, d$, and is such that

$$C_2|\xi|^2 \geq A(x)\xi \cdot \xi \geq C_1|\xi|^2 \qquad \forall \xi \in \mathbf{R}^d \quad \forall x \in \Omega \tag{4}$$

In addition, let us assume that almost everywhere in $\Omega$

$$c \geq 0 \tag{5}$$

and stand the denotation

$$\Omega^c := \operatorname{supp} c = \{x \in \Omega \mid c(x) > 0\} \tag{6}$$

**Weak formulation:** Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla w \, dx + \int_\Omega cuw \, dx = \int_\Omega fw \, dx + \int_{\Gamma_N} gw \, ds \quad \forall w \in H^1_{\Gamma_D}(\Omega) \tag{7}$$

where

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\} \tag{8}$$

• Existence and uniqueness of the *weak solution* defined by (7) is provided by the well-known Lax-Milgram lemma

We can define *bilinear and linear forms* as follows

$$a(v,w) := \int_\Omega A\nabla v \cdot \nabla w \, dx + \int_\Omega cvw \, dx, \quad F(w) := \int_\Omega fw \, dx + \int_{\Gamma_N} gw \, ds \tag{9}$$

- Then formulation (7) can be written in a more compact form:

Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that $a(u,w) = F(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega) \quad (\dagger)$

- Lax-Milgram lemma is, in fact, applied to such a problem:

Find $u_\star \in H^1_{\Gamma_D}(\Omega)$ such that $a(u_\star,w) = \bar{F}(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega) \quad (\star)$

where $\bar{F}(w) = F(w) - a(u_0,w)$. And then we take $u = u_0 + u_\star$

- Formulations $(\dagger)$ and $(\star)$ are equivalent

# Energy Functional

Having formulation $(\star)$, it is possible to define a convenient *energy functional* $J$ as follows

$$J(w) := \frac{1}{2}a(w,w) - \bar{F}(w) \tag{10}$$

- The term *"energy functional"* comes from the fact that the functional taken in the form (10) often presents the total energy of the system under analysis

- It is well-known that problem $(\star)$ is equivalent to the problem of finding the minimizer of the energy functional over space $H^1_{\Gamma_D}(\Omega)$

# Control of Computational Errors

• Let $\bar{u} = u_0 + \bar{u}_\star$, where $\bar{u}_\star \in H^1_{\Gamma_D}(\Omega)$, be *any function* from the set $u_0 + H^1_{\Gamma_D}(\Omega)$ (e.g., computed by some numerical method) considered as approximation of $u$

• We are interested in *diverse and reliable estimation and control of the deviation (or error)* $e := u - \bar{u} = u_\star - \bar{u}_\star$ during computations

• So far *two ways* of the error control *aimed at two different final goals of the whole computational process* are well studied and widely used in mathematical and engineering communities

• Roughly speaking, those are – *control of the overall accuracy* of obtained approximations and *control of various local errors*

# Overall Error in Energy Norm

First of all, we can easily prove that

$$a(u-\bar{u}, u-\bar{u}) = a(u_\star - \bar{u}_\star, u_\star - \bar{u}_\star) = 2(J(\bar{u}_\star) - J(u_\star)) \qquad (11)$$

which means that the value

$$a(u-\bar{u}, u-\bar{u}) = \int_\Omega A\nabla(u-\bar{u}) \cdot \nabla(u-\bar{u})\, dx + \int_\Omega c(u-\bar{u})^2\, dx \quad (12)$$

is a natural measure for the overall quality of approximation $\bar{u}$

- Since $\sqrt{a(\cdot,\cdot)}$ is called the *energy norm*, the estimation of value (12) is often called the *error estimation in the energy norm*

• In the case $c \equiv 0$, the value (12) reduces to the well-known form

$$\int_{\Omega} A\nabla(u - \bar{u}) \cdot \nabla(u - \bar{u})\, dx \tag{13}$$

which is nowadays widely used as a measure of accuracy of $\bar{u}$

• The error control in the global energy norm (13) has been in a focus of active research work since the works by Babuška and Rheinboldt in 1978

## Control of Local Errors

- Another trend in a posteriori error estimation is based on concept of local error control in addition to classical control in energy norm

- This approach is motivated by practical needs: analysts are often interested not only in the value of the overall error, but also in *controlling errors over certain parts of solution domain*, or relative to some interesting characteristics (*"quantities of interest"*)

- Common way for performing such type control is to introduce *linear functional* $\ell$ associated with subdomain of interest (and/or with a "quantity of interest") and to obtain a computable estimate for $\ell(u - \bar{u})$

• For example, one can be interested in estimation of

$$\ell(e) = \ell(u - \bar{u}) \; = \int\limits_{\Omega} \varphi(u - \bar{u}) \; dx \tag{14}$$

where $\varphi \in L_2(\Omega)$ and $\operatorname{supp} \varphi = \omega \subseteq \Omega$, which provides us with info on error $e$ locally, in subdomain $\omega$

• Estimates for $\ell(u - \bar{u})$ can also be used for estimation of unknown *"quantity of interest"* $\ell(u)$ , since

$$\ell(u) = \ell(\bar{u}) + \ell(u - \bar{u})$$

where $\ell(\bar{u})$ is computable and $\ell(u - \bar{u})$ is estimated

• Certain "quantities of interest" (e.g., in linear elasticity) are often more interesting for the practitioners than the solution itself

# Inequalities and Constants

- In what follows we need Friedrichs' inequality

$$\|w\|_{0,\Omega} \leq C_{\Omega,\Gamma_D} \|\nabla w\|_{0,\Omega} \quad \forall w \in H^1_{\Gamma_D}(\Omega) \tag{15}$$

and the inequality in the trace theorem

$$\|w\|_{0,\partial\Omega} \leq C_{\partial\Omega} \|w\|_{1,\Omega} \quad \forall w \in H^1(\Omega) \tag{16}$$

where $C_{\Omega,\Gamma_D}$ and $C_{\partial\Omega}$ are positive constants depending only on $\Omega$, $\Gamma_D$, and $\partial\Omega$

- $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$ stand for norms in $L_2(\Omega)$ and $H^1(\Omega)$
- $\|\cdot\|_{0,\partial\Omega}$ is the norm in $L_2(\partial\Omega)$

## Two-Sided Estimates of Error in Energy Norm

- Let $\chi_S$ be the characteristic function of set $S$: $\chi_S(x) := 1$ if $x \in S$, and $\chi_S(x) := 0$ if $x \notin S$

- Let $\|y\|_\Omega := \sqrt{\int\limits_\Omega Ay \cdot y\,dx}$ for $y \in L_2(\Omega, \mathbf{R}^d)$

- We shall need two above defined constants $C_{\Omega,\Gamma_D}$, $C_{\partial\Omega}$, and also

$$C_{\Omega,\partial\Omega} := \frac{C_{\partial\Omega}\sqrt{1 + C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}$$

- $H_N(\Omega, \mathrm{div}\,) := \{y \in L_2(\Omega, \mathbf{R}^d)\,|\,\mathrm{div}\,y \in L_2(\Omega),\ \nu^T \cdot y \in L_2(\Gamma_N)\}$

- $\Omega^c := \mathrm{supp}\,c = \{x \in \Omega \mid c(x) > 0\}$

**Upper estimate:** We have the following upper estimate

$$a(u-\bar{u},u-\bar{u}) \leq \| \frac{1}{\sqrt{c}}(f+\operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega^c} +$$

$$+(1+\alpha)\|A^{-1}y^* - \nabla\bar{u}\|^2_{\Omega} + (1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f+\operatorname{div} y^*\|^2_{0,\Omega\setminus\overline{\Omega}^c} \tag{17}$$

$$+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|^2_{0,\Gamma_N}$$

where $\alpha,\beta$ are any positive numbers and $y^*$ is any function from $H_N(\Omega,\operatorname{div})$

**P r o o f :** First of all, we notice that

$$a(u-\bar{u},u-\bar{u}) = \|\nabla(u-\bar{u})\|_{\Omega}^2 + \|\sqrt{c}(u-\bar{u})\|_{0,\Omega^c}^2 \qquad (18)$$

Further, using fact that $u-\bar{u} \in H^1_{\Gamma_D}(\Omega)$, integral identity $(7)$, Green formula, and simple regroupping of terms in below we get

$$a(u-\bar{u},u-\bar{u}) = \int_{\Omega} f(u-\bar{u})dx + \int_{\Gamma_N} g(u-\bar{u})\,ds - \int_{\Omega} A\nabla\bar{u}\cdot\nabla\,(u-\bar{u})\,dx$$

$$-\int_{\Omega} c\bar{u}(u-\bar{u})\,dx = \int_{\Omega}(f-c\bar{u})(u-\bar{u})\;dx + \int_{\Gamma_N} g(u-\bar{u})\,ds$$

$$-\int_{\Omega}(A\nabla\bar{u}-y^*)\cdot\nabla(u-\bar{u})\;dx - \int_{\Omega} y^*\cdot\nabla(u-\bar{u})\;dx = \qquad (19)$$

$$= \int_{\Omega} (f + \operatorname{div} y^* - c\bar{u})(u - \bar{u}) \, dx - \int_{\Omega} A(\nabla \bar{u} - A^{-1} y^*) \cdot \nabla (u - \bar{u}) \, dx$$

$$+ \int_{\Gamma_N} g(u - \bar{u}) \, ds - \int_{\Gamma_N} \nu^T \cdot y^* (u - \bar{u}) \, ds =$$

$$= \int_{\Omega} A(A^{-1} y^* - \nabla \bar{u}) \cdot \nabla (u - \bar{u}) \, dx + \int_{\Omega} (f + \operatorname{div} y^* - c\bar{u})(u - \bar{u}) \, dx$$

$$+ \int_{\Gamma_N} (g - \nu^T \cdot y^*)(u - \bar{u}) \, ds$$

where $y^*$ is any function from the space $H_N(\Omega, \operatorname{div})$

Now, the right-hand side (RHS) of the above equality can be estimated, using the Cauchy-Schwarz inequality, denotation $\Omega^c$ and trace theorem from above as follows

$$\text{RHS} \quad (19) \leq \|A^{-1}y^* - \nabla\bar{u}\|_\Omega \, \|\nabla(u-\bar{u})\|_\Omega + \|g - \nu^T \cdot y^*\|_{0,\Gamma_N} \|u-\bar{u}\|_{0,\Gamma_N}$$

$$+ \int_\Omega (f + \operatorname{div} y^* - c\bar{u})(u - \bar{u}) \; dx \leq$$

$$\leq \|A^{-1}y^* - \nabla\bar{u}\|_\Omega \, \|\nabla(u-\bar{u})\|_\Omega + \|g - \nu^T \cdot y^*\|_{0,\Gamma_N} C_{\partial\Omega} \|u-\bar{u}\|_{1,\Omega} \quad (20)$$

$$+ \int_{\Omega^c} \frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u}) \, \sqrt{c}(u-\bar{u}) \; dx + \int_{\Omega \setminus \overline{\Omega}^c} (f + \operatorname{div} y^* - c\bar{u}) \, (u-\bar{u}) \; dx$$

Further, using the ellipticity condition, Friedrichs' inequality, and the inequality

$$|a\, b| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \tag{21}$$

we observe that

$$\text{RHS} \quad (20) \leq$$

$$\leq \left( \|\!| A^{-1}y^* - \nabla \bar{u}) \|\!|_\Omega + \frac{C_{\partial\Omega}\sqrt{1 + C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}} \| g - \nu^T \cdot y^* \|_{0,\Gamma_N} \right) \|\!| \nabla (u - \bar{u}) \|\!|_\Omega$$

$$+ \frac{1}{2} \| \sqrt{c}(u - \bar{u}) \|^2_{0,\Omega^c} + \frac{1}{2} \| \frac{1}{\sqrt{c}} (f + \operatorname{div} y^* - c\bar{u}) \|^2_{0,\Omega^c}$$

$$+ \int_\Omega \chi_{\Omega \setminus \overline{\Omega}^c} (f + \operatorname{div} y^* - c\bar{u})\, (u - \bar{u})\; dx \leq$$

$$\leq \Big( |||A^{-1}y^* - \nabla \bar{u})|||_\Omega + C_{\Omega,\partial\Omega}\|g - \nu^T \cdot y^*\|_{0,\Gamma_N} \Big) |||\nabla(u - \bar{u})|||_\Omega \quad (22)$$

$$+\frac{1}{2}\|\sqrt{c}(u - \bar{u})\|^2_{0,\Omega^c} + \frac{1}{2}\|\frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega^c}$$

$$+\|\chi_{\Omega\setminus\overline{\Omega}^c}(f + \operatorname{div} y^* - c\bar{u})\|_{0,\Omega}\,\|u - \bar{u}\|_{0,\Omega}$$

where $C_{\Omega,\partial\Omega} := \frac{C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}$

Regrouping terms in RHS of (22) and using again the inequality (21), we get an estimate

$$\text{RHS} \quad (22) \le$$

$$\Big( |\!|\!| A^{-1}y^* - \nabla \bar{u}) |\!|\!|_{\Omega} + C_{\Omega,\partial\Omega} \| g - \nu^T \cdot y^* \|_{0,\Gamma_N} + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}} \| f + \operatorname{div} y^* \|_{0,\Omega\setminus\overline{\Omega}^c} \Big) \times$$

$$\times |\!|\!| \nabla(u - \bar{u}) |\!|\!|_{\Omega} + \frac{1}{2} \| \sqrt{c}(u - \bar{u}) \|^2_{0,\Omega^c} + \frac{1}{2} \| \frac{1}{\sqrt{c}} (f + \operatorname{div} y^* - c\bar{u}) \|^2_{0,\Omega^c} \le$$

$$\frac{1}{2} \Big( |\!|\!| A^{-1}y^* - \nabla \bar{u}) |\!|\!|_{\Omega} + C_{\Omega,\partial\Omega} \| g - \nu^T \cdot y^* \|_{0,\Gamma_N} + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}} \| f + \operatorname{div} y^* \|_{0,\Omega\setminus\overline{\Omega}^c} \Big)^2 \tag{23}$$

$$+ \frac{1}{2} |\!|\!| \nabla(u - \bar{u}) |\!|\!|^2_{\Omega} + \frac{1}{2} \| \sqrt{c}(u - \bar{u}) \|^2_{0,\Omega^c} + \frac{1}{2} \| \frac{1}{\sqrt{c}} (f + \operatorname{div} y^* - c\bar{u}) \|^2_{0,\Omega^c}$$

Using now (18) and the final inequality resulting from (19)–(20) and (22)–(23), multiplying it by two and regroupping, we immediately get for the error in the energy norm that

$$a(u-\bar{u},u-\bar{u}) =$$

$$= \|\nabla(u-\bar{u})\|_{\Omega}^2 + \|\sqrt{c}(u-\bar{u})\|_{0,\Omega^c}^2 \leq \|\frac{1}{\sqrt{c}}(f+\operatorname{div} y^* - c\bar{u})\|_{0,\Omega^c}^2$$

$$+\Big(\|A^{-1}y^*-\nabla\bar{u}\|_{\Omega}+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|f+\operatorname{div} y^*\|_{0,\Omega\setminus\overline{\Omega}^c}+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)^2 \tag{24}$$

Finally, using two times the inequality

$$(a+b)^2 \leq (1+\lambda)a^2 + (1+\frac{1}{\lambda})b^2 \ (\lambda > 0)$$

for terms in round brackets in (24), we get our upper estimate

**Lower estimate:** We have the following lower bound

$$a(u-\bar{u}, u-\bar{u}) \geq 2(J(\bar{u}_\star) - J(w)) \tag{25}$$

where $w$ is any function from $H^1_{\Gamma_D}(\Omega)$ and $J$ is energy functional

**P r o o f :** First, we prove that

$$a(u-\bar{u}, u-\bar{u}) = a(u_\star - \bar{u}_\star, u_\star - \bar{u}_\star) = 2(J(\bar{u}_\star) - J(u_\star)) \tag{26}$$

Indeed, we observe

$$2(J(\bar{u}_\star) - J(u_\star)) = a(\bar{u}_\star, \bar{u}_\star) - 2\bar{F}(\bar{u}_\star) - a(u_\star, u_\star) + 2\bar{F}(u_\star)$$

$$= a(\bar{u}_\star, \bar{u}_\star) - a(u_\star, u_\star) + 2\bar{F}(u_\star - \bar{u}_\star)$$

$$= a(\bar{u}_\star, \bar{u}_\star) - a(u_\star, u_\star) + 2a(u_\star, u_\star - \bar{u}_\star)$$

$$= a(\bar{u}_\star, \bar{u}_\star) + a(u_\star, u_\star) - 2a(u_\star, \bar{u}_\star) = a(u_\star - \bar{u}_\star, u_\star - \bar{u}_\star)$$

Since $u_\star$ minimizes $J$, we have $J(u_\star) \leq J(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$, which proves (25)

# Comments on Estimates

- In order to derive the upper and lower estimates, we did not require the function $\bar{u}$ be a finite element approximation (or computed by some another specific numerical method); it is simply any function from $u_0 + H^1_{\Gamma_D}(\Omega)$
- The upper estimate cannot be improved. Indeed, if one takes $y^* = A\nabla u$, which obviously belongs to $H_N(\Omega, \mathrm{div}\,)$, then the last two terms in the right-hand side of (17) vanish. Further, taking $\alpha \to 0$, we finally observe that the inequality (17) holds as equality. To prove that the lower estimate cannot be improved either, we should, obviously, take $w = u_\star \in H^1_{\Gamma_D}(\Omega)$

- The upper estimate contains only two global constants, $C_{\Omega,\Gamma_D}$ and $C_{\partial\Omega}$, which do not depend on the computational process at all. They have to be computed (or accurately estimated from above) only once when the problem is posed

- In many works, devoted to a posteriori error estimation, one usually takes $c \equiv 0$. In this case $a(u-\bar{u}, u-\bar{u}) = \|\nabla(u-\bar{u})\|_{\Omega}^2$, the set $\Omega^c = \emptyset$, and the upper estimate takes a simpler form

$$a(u-\bar{u}, u-\bar{u}) \leq$$

$$\leq (1+\alpha)\|A^{-1}y^* - \nabla\bar{u}\|_{\Omega}^2 + (1+\frac{1}{\alpha})(1+\beta)\frac{C_{\Omega,\Gamma_D}^2}{C_1}\|f + \operatorname{div} y^*\|_{0,\Omega}^2$$

$$+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C_{\Omega,\partial\Omega}^2\|g - \nu^T \cdot y^*\|_{0,\Gamma_N}^2 \qquad (27)$$

- For the pure Dirichlet boundary condition, the third term in RHS of (27) does not exist, and, since the estimate is valid for any positive $\beta$, we can take it be zero. Then, we get

$$a(u-\bar{u}, u-\bar{u}) \leq$$

$$\leq (1+\alpha)\|A^{-1}y^* - \nabla\bar{u}\|^2_\Omega + (1+\frac{1}{\alpha})\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|f + \operatorname{div} y^*\|^2_{0,\Omega} \quad (28)$$

- The upper estimate (28) was first obtained in [Repin, 1997] using quite complicated tools of the duality theory. Later it was obtained in [Repin, Sauter, Smolianski, 2003] for simple Poisson equation, using an idea of the Helmholz decomposition of $L_2(\Omega, \mathbf{R}^d)$. The estimate (27) is derived in [Repin, Sauter, Smolianski, 2004] using again the duality theory

- Our approach for derivation of the estimates is different from those used in the above mentioned works and is simplier
- More general problems with mixed Dirichlet/Neumann/Robin (D/N/R) boundary conditions are treated in a just presented way in [Korotov, 2006], where (17) follows as a particular case
- In the case of pure Dirichlet (D) condition or Dirichlet/Robin (D/R) mixed boundary condition we have to compute, or estimate from above, only one constant $C_{\Omega,\Gamma_D}$
- If, in addition to D or D/R boundary conditions, we have $c(x) \geq c_0 > 0$ in $\Omega$ we do not need any global constants for our estimation at all

# Two-Sided Estimates for Local Errors

• The second technology is developed for controlling local errors

• To demonstrate the main idea of the approach and avoid unnecessary technical details we shall consider only the simplest problem. More general cases can be treated just in the same fashion

**Model Problem:** Find a function $u$ such that

$$-\mathrm{div}(A\nabla u) = f \quad \text{in} \quad \Omega \tag{29}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{30}$$

• I.e., $c \equiv 0$, $u_0 = 0$, $\Gamma_N = \emptyset$, and $a(u - \bar{u}, u - \bar{u}) = \|\nabla e\|^2_\Omega$ then

**Weak formulation:** Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in H_0^1(\Omega) \tag{31}$$

• We want to estimate the deviation $e = u - \bar{u}$ measured in terms of suitable linear functional $\ell$

$$\text{Example:} \qquad \ell(e) = \int_\Omega \varphi \,(u - \bar{u})\; dx \tag{32}$$

where $\varphi \in L_2(\Omega)$ and $\operatorname{supp} \varphi = \omega \subseteq \Omega$

**Adjoint problem (AP):** Find $v \in H_0^1(\Omega)$ such that

$$\int_\Omega A\nabla v \cdot \nabla w \, dx = \ell(w) \quad \forall w \in H_0^1(\Omega) \tag{33}$$

• Adjoint problem is uniquely solvable

• Usually we only have some approximation $\bar{v}$ (e.g., computed by FEM on *adjoint mesh* $\mathcal{T}_\tau$)

• It can be shown that

$$\ell(u-\bar{u}) = E_0(\bar{u},\bar{v}) + E_1(e,e_\ell) \tag{34}$$

where

$$E_0(\bar{u},\bar{v}) = \int_\Omega f\bar{v}\,dx - \int_\Omega \mathrm{A}\nabla\bar{v}\cdot\nabla\bar{u}\,dx \tag{35}$$

and

$$E_1(e,e_\ell) = \int_\Omega \mathrm{A}\nabla e\cdot\nabla e_\ell\,dx \tag{36}$$

• Denotation $e_\ell := v - \bar{v}$ stands for computational error in **AP**

Proof: From the integral equality for **AP,** we have $(w := u - \bar{u})$

$$\ell(u-\bar{u}) = \int_\Omega \mathrm{A}\nabla v \cdot \nabla(u-\bar{u})\ dx =$$

$$= \int_\Omega \mathrm{A}\nabla(v-\bar{v}) \cdot \nabla(u-\bar{u})\ dx + \int_\Omega \mathrm{A}\nabla\bar{v} \cdot \nabla(u-\bar{u})\ dx =$$

$$= E_1(e, e_\ell) - \int_\Omega \mathrm{A}\nabla\bar{v} \cdot \nabla\bar{u}\ dx + \int_\Omega \mathrm{A}\nabla u \cdot \nabla\bar{v}\ dx =$$

$$= E_1(e, e_\ell) - \int_\Omega \mathrm{A}\nabla\bar{v} \cdot \nabla\bar{u}\ dx + \int_\Omega f\bar{v}\ dx =$$

$$= E_0(\bar{u}, \bar{v}) + E_1(e, e_\ell)$$

• First term $E_0$ is directly computable once $\bar{u}$ and $\bar{v}$ are computed, but term $E_1$ contains unknown gradients $\nabla u$ and $\nabla v$

• In order to estimate $E_1$, we use the following identity

$$2E_1(e, e_\ell) = \|\nabla(\gamma e + \frac{1}{\gamma} e_\ell)\|_\Omega^2 - \gamma^2 \|\nabla e\|_\Omega^2 - \frac{1}{\gamma^2} \|\nabla e_\ell\|_\Omega^2 \qquad (37)$$

valid for any $\gamma > 0$

• The above identity contains errors in energy norm for both, primal and adjoint problems, and we can use the corresponding two-sided estimates

$$M_f^\ominus \leq \|\nabla e\|_\Omega^2 \leq M_f^\oplus \quad \& \quad M_\ell^\ominus \leq \|\nabla e_\ell\|_\Omega^2 \leq M_\ell^\oplus \qquad (38)$$

• For the first term in RHS of (37), we observe that

$$\|\nabla(\gamma e + \frac{1}{\gamma} e_\ell)\|_\Omega^2 = \|\nabla(\gamma u + \frac{1}{\gamma} v) - \nabla(\gamma \bar{u} + \frac{1}{\gamma}\bar{v})\|_\Omega^2 \qquad (39)$$

• Function $\gamma u + \frac{1}{\gamma} v$ can be perceived as the solution of problem: Find $u_\gamma \in H_0^1(\Omega)$ such that

$$\int_\Omega A\nabla u_\gamma \cdot \nabla w \, dx = \gamma \int_\Omega f w \, dx + \frac{1}{\gamma}\ell(w) \quad \forall w \in H_0^1(\Omega) \qquad (40)$$

and function $\gamma\bar{u} + \frac{1}{\gamma}\bar{v}$ can be considered as an approximation of $u_\gamma$

• Then we can again use two-sided estimates

$$M^{\ominus}_{\gamma,f,\ell} \leq \|\nabla(\gamma e + \frac{1}{\gamma} e_\ell)\|_\Omega^2 \leq M^{\oplus}_{\gamma,f,\ell} \qquad (41)$$

- We immediately observe that

$$\frac{1}{2}(M^{\ominus}_{\gamma,f,\ell} - \gamma^2 M^{\oplus}_f - \frac{1}{\gamma^2} M^{\oplus}_{\ell}) \le E_1(e, e_\ell)$$

and

$$E_1(e, e_\ell) \le \frac{1}{2}(M^{\oplus}_{\gamma,f,\ell} - \gamma^2 M^{\ominus}_f - \frac{1}{\gamma^2} M^{\ominus}_{\ell})$$

which together with directly computable term

$$E_0(\bar{u}, \bar{v}) = \int\limits_{\Omega} f\bar{v}\, dx - \int\limits_{\Omega} \mathrm{A}\nabla\bar{v} \cdot \nabla\bar{u}\, dx$$

provide with two-sided estimates for the error $\ell(e) = E_0 + E_1$

# Mesh Adaptivity

- For both types of eror control, the proposed upper and lower estimates have integral form, i.e, each one can be presented as a single integral over the solution domain $\Omega$

- Such a form suggests a standard mesh adaptivity procedure based on refining those elements of the mesh which give a large contribution to the total error

# NEW TECHNOLOGIES:

# PRACTICAL REALIZATION

# AND NUMERICAL TESTS

## On Computation of Global Constants

- Our estimates contains only two unknown constants $C_{\Omega,\Gamma_D}, C_{\partial\Omega}$
- Estimation of those two constants from above is sufficient
- Other estimation techniques involve *many unknown constants:*
  - which are very hard to estimate
  - which have to be always recomputed if the mesh changes
- But $C_{\Omega,\Gamma_D}, C_{\partial\Omega}$ remain the same under any changes of meshes

# Series of Meshes

- We often perform computations on a series of meshes $\mathcal{T}_{h_1}$, $\mathcal{T}_{h_2}$, $\mathcal{T}_{h_3}$, ..., where $h = h_1 > h_2 > h_3 > \dots$

- Thus, we always have several successive approximations $u_{h_1}$, $u_{h_2}$, $u_{h_3}$, ..., where $u_{h_i} = u_0 + \bar{u}_{h_i}$

- We normally try to have $J(\bar{u}_{h_1}) > J(\bar{u}_{h_2}) > J(\bar{u}_{h_3}) > \dots$

# On Computation of Upper Bounds

• For simplicity we consider the minimization issue under condition that approximation is computed by FEM, i.e., $\bar{u} \equiv u_h$

$$M^{\oplus} = M^{\oplus}(u_h, \alpha, \beta, y^*) \tag{42}$$

• Coarse upper bounds can be computed fast using, e.g., $y^* = G_\mu(\nabla u_\mu) \in H_N(\Omega, \mathrm{div})$, $\mu = h_1, h_2, h_3, \ldots,$ and $G_\mu$ is some gradient averaging operator

• Sharp estimates require a real minimization of $M^{\oplus}(u_h, \alpha, \beta, y^*)$ with respect to *"free variables"* $y^*$ and $\alpha$ & $\beta$

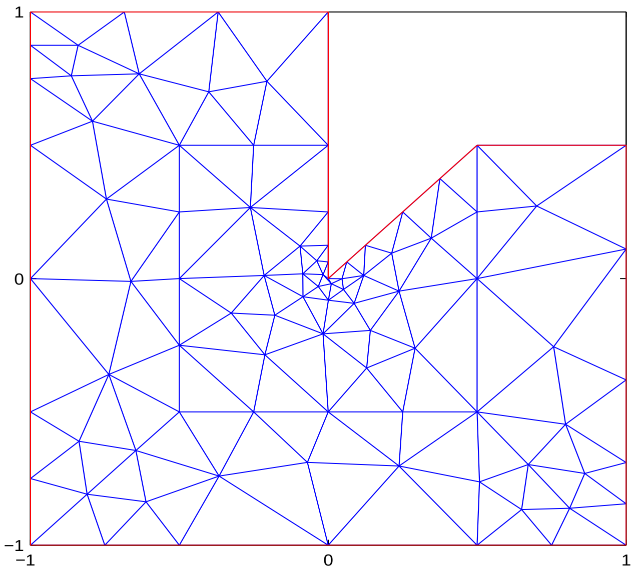
# On Computation of Lower Bounds

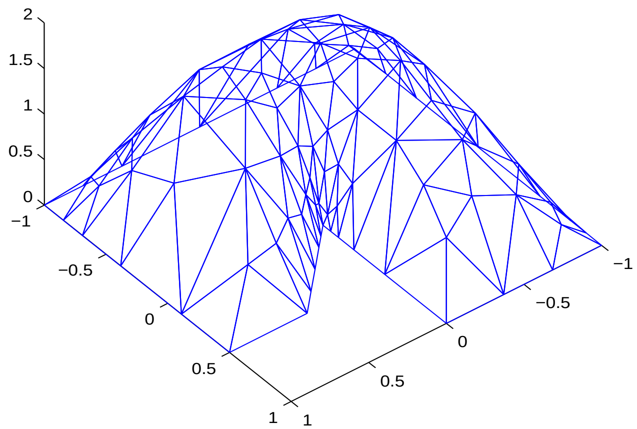
We usually try to have $J(\bar{u}_{h_1}) > J(\bar{u}_{h_2}) > J(\bar{u}_{h_3}) > \ldots,$ which suggests an easy way to construct the lower bounds as follows

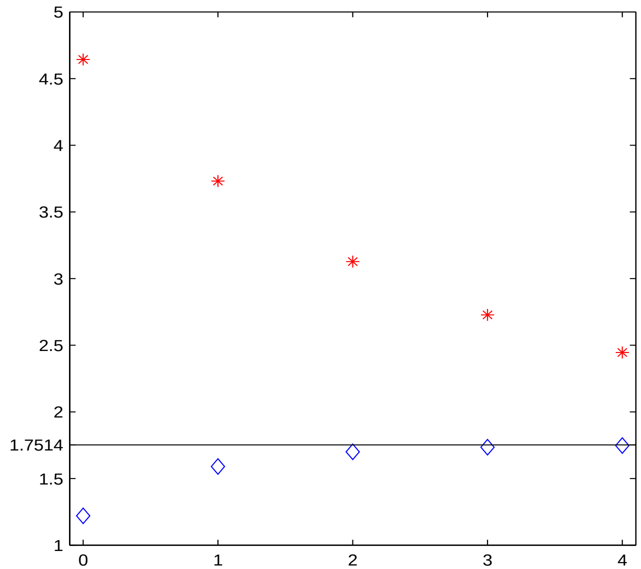
$$a(u - u_h, u - u_h) \geq 2(J(\bar{u}_h) - J(\bar{u}_\mu)) > 0 \qquad (43)$$

where $\mu = h_2, h_3, \ldots$

# Test No. 1

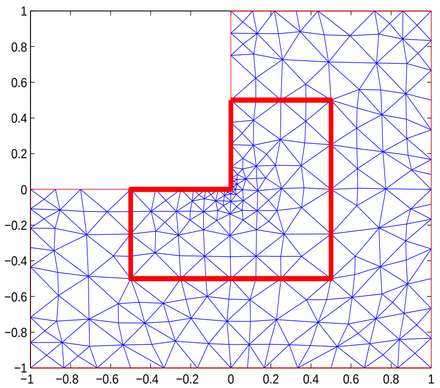


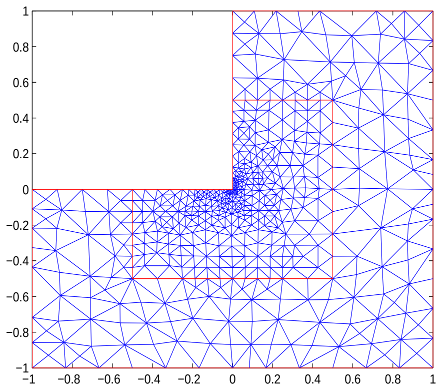



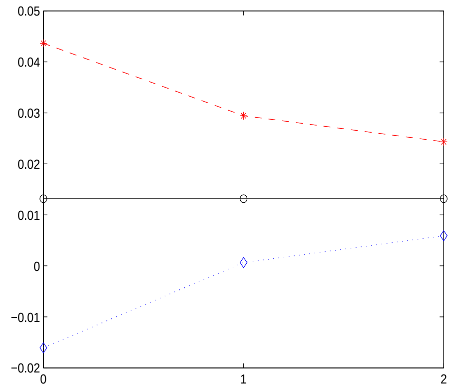




- $A \equiv I$, $f \equiv 10$, $\Gamma_N = \emptyset$, $u_0 \equiv 0$, $C_{\Omega,\Gamma_D} \leq \frac{\sqrt{2}}{\pi}$, $\mathcal{T}_h$ with 92 nodes
- Several successive meshes are used in computations of bounds
- Upper bound decreases from 4.6426 to 2.4443
- Lower bound grows from 1.2191 to 1.7469
- $\|\nabla e\|^2 \approx 1.7514$ (computed using the reference solution)

# Test No. 2



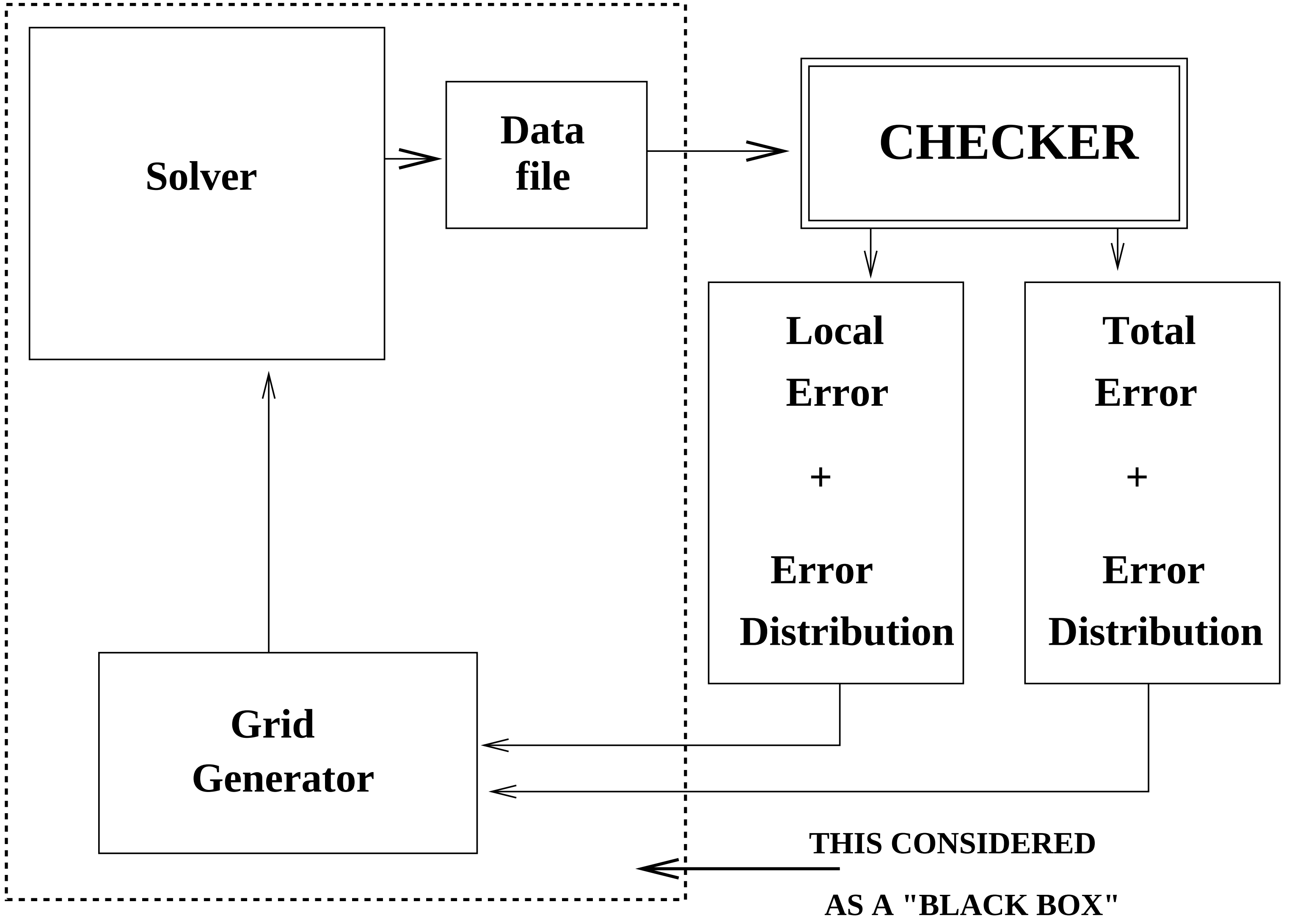




- $A \equiv I$, $f \equiv 10$, $\Gamma_N = \emptyset$, $u_0 \equiv 0$, $\mathcal{T}_h$ with 249 nodes
- $\ell(w) := \int\limits_{\Omega} \varphi w \, dx$, $\operatorname{supp} \varphi = \omega \subset \Omega$, $\varphi \in L_2(\Omega)$, $\mathcal{T}_\tau$ with 541 nodes
- Upper bound decreases from 0.04364 to 0.02433
- Lower bound grows from -0.01606 to 0.00591
- $\ell(e) = \int\limits_{\omega} 1 \cdot (u - u_h) \, dx \approx 0.01317$

# FINAL COMMENTS

- Techniques proposed can be adapted for the other boundary conditions and other linear elliptic problems (e.g., in linear elasticity), also for parabolic problems.
- For both types of control the corresponding upper and lower bounds can be made arbitrary close to the true errors
- Such closeness only depends on computer resources (memory, velocity) of a concrete computer used for calculations
- Technologies can be easily coded and added as **block-checker** to most of existing educational and industrial software products like MATLAB, FEMLAB, ANSYS, etc.

# Compatibility With Existing Codes



## Computational Benefits

By means of the above checkers, we can

- explicitely verify the error for the concrete computed solution and, therefore, see when the desired tolerance is achieved
- verify the exisiting codes and discover their possible drawbacks and bugs
- create optimal (or "quasi-optimal") meshes in the process of the mesh-refinement
- improve the quality of computer–aided modelling
- reduce the time of computer simulation

- One of the main advantages of our technologies is that the error estimators proposed are valid for a wide spectrum of different approximations, i.e., the checking code is absolutely independent of the solver used

- The connection between a particular solver and the checker can be organized via the exchange of the data files

## INPUT DATA:

a) geometry of the body

b) material constants

c) approximate solution obtained on the concrete mesh and the mesh parameters

• The input data are to be delivered as files of real numbers, the structure of such files is to be specified

## OUTPUT DATA:

a) two-sided estimates of errors in terms of special "goal-oriented" quantities suggested by customer

b) two-sided bounds of the error in the global norm

c) distribution of local errors

• The output data can be presented both as tables of numbers and in graphical form