Basis of Mathematical Methods in Fluid Mechanics

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These notes correspond to a course taught in BCAM in november 2011 with some complements. Most of them also follow a DEA course taught in University of Versailles St Quentin some time before. They are very classical basis for equations of fluid mechanics (for viscous fluids) and for the mathematical analysis of these equations and they intend to be essentially self contained. The first chapter can be found in the introductions of many books related to fluid mechanics, for example [1] and the next chapters follow closely notes written by L. Tartar in [3] which is difficult to find. But of course, most of the results and the proofs can be found (sometimes presented differently) in classical books where Navier-Stokes equations are studied like [4] or [5] and the references therein.

Chapter 1

Equations of viscous fluid flows

1.1 Introduction

The physical domain is determined by

- An open subset Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$ (say in general $\mathbb{R}^N$).
- $x \in \Omega$ is the sapce variable.
- $t \in (0, T)$ with $T > 0$ is the time variable.

The so-called "state variables" or quantities which determine the flow are

- The fluid velocity u.
- The pressure p.
- The density of the fluid ρ.
- Sometimes other quantitites like the temperature θ, ...

Eulerian description.
$u(x,t)$ is the velocity of the particle of fluid which is at position x at time t. Therefore, if $t' \neq t$, $u(x,t')$ is the velocity of a different particle. The observer is located at position x and looks at particles passing at this position. Here we will focus essentially on this Eulerian description.
Lagrangian description.
Let us call here the velocity $v(x,t)$ in order to avoid confusions. If we call $(x(t))_{t \geq t_0}$ the trajectory of a particle emanating from point x_0 at time t_0 (in fact this trajectory is an unknown of the problem). Then $v(x_0,t)$ $(t \geq t_0)$ is the velocity, at time t of the particle which was at position x_0 at time t_0 and therefore, the velocity at time

t of the particle which is at position $x(t)$. The observer is here transpoted by the flow and follows the particle.
Particle derivative.
Let $(t, x(t))$ the trajectory of a fluid particle starting from (t_0, x_0). We have with the Eulerian velocity,

$$\frac{dx}{dt}(t) = u(x(t), t),$$
$$x(t_0) = x_0.$$

Let φ be a function of x and t. The particle derivative of φ is

$$\frac{d}{dt}\varphi(x_0, t_0) = \frac{d}{dt}(\varphi(x(t), t))_{/t=t_0}$$
$$= \frac{\partial \varphi}{\partial t}(x_0, t_0) + \sum_{i=1}^{N} u_i(x_0, t_0)\frac{\partial \varphi}{\partial x_i}(x_0, t_0)$$
$$= \Big(\frac{\partial \varphi}{\partial t} + \sum_{i=1}^{N} u_i \frac{\partial \varphi}{\partial x_i}\Big)_{/(x_0, t_0)}$$
$$= \Big(\frac{\partial \varphi}{\partial t} + u.\nabla\varphi\Big)_{/(x_0, t_0)}.$$

1.2 Conservation laws

Let $V(t)$ be a volume of fluid which, when t varies, contains the same particles (which have been moving). That is to say $V(t)$ is a volume transported by the velocity field $u(x, t)$ or for $t' \geq t$

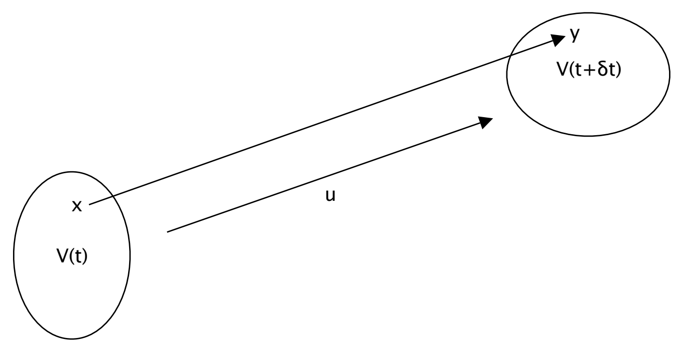
$$V(t') = \{x(t'), \quad \frac{dx}{ds}(s) = u(x(s), s),\ t \leq s \leq t', \quad x(t) \in V(t)\}.$$

We say that a quantity φ, which is a function of x and t is **conserved** if

$$\forall V(t), \quad \frac{d}{dt}\int_{V(t)} \varphi(x, t)dx = 0.$$

Therefore for every $V(t)$ and δt (small) we have

$$\int_{V(t+\delta t)} \varphi(y, t + \delta t)dy = \int_{V(t)} \varphi(x, t)dx.$$



We can write

$$V(t+\delta t) = \{y(\delta t), \quad \frac{dy}{dt}(s) = u(y(s), s), \quad , y(0) = x, \quad x \in V(t)\}.$$

Let us expand every quantity at the first order in δt. We have

$$y(x, \delta t) = x + \delta t.u(x,t) + l.o.t.$$

so that

$$dy_i = dx_i + \delta t \sum_{j=1}^{N} \frac{\partial u_i}{\partial x_j}(x,t) dx_j + l.o.t.$$

Therefore

$$dy = dy_1 \wedge dy_2 \wedge \cdots \wedge dy_N = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_N$$

$$+\delta t\Big(\sum_{j=1}^{N}\frac{\partial u_1}{\partial x_j}dx_j\wedge dx_2\wedge\cdots\wedge dx_N\Big)+\cdots$$

$$+\delta t\Big(dx_1\wedge dx_2\wedge\cdots\wedge\sum_{j=1}^{N}\frac{\partial u_N}{\partial x_j}\Big)+l.o.t.$$

so that

$$dy=(1+\delta t(\operatorname{div}u)+l.o.t.)dx.$$

Now we have

$$\varphi(y(x,\delta t),t+\delta t)=\varphi(x+\delta t u(x,t)+l.o.t.,t+\delta t)$$
$$=\varphi(x,t)+\delta t\frac{\partial\varphi}{\partial t}(x,t)+\delta t\sum_{j=1}^{N}u_j(x,t)\frac{\partial\varphi}{\partial x_j}(x,t)+l.o.t.$$

Therefore

$$\int_{V(t+\delta t)}\varphi(y,t+\delta t)dy$$
$$=\int_{V(t)}\Big(\varphi(x,t)+\delta t(\frac{\partial\varphi}{\partial t}(x,t)+u(x,t).\nabla\varphi(x,t)+o(\delta t))(1+\delta t\operatorname{div}u(x,t)+o(\delta t))dx$$
$$=\int_{V(t)}\varphi(x,t)dx+\delta t\int_{V(t)}\Big(\frac{\partial\varphi}{\partial t}(x,t)+u(x,t).\nabla\varphi(x,t)+\varphi(x,t)\operatorname{div}u(x,t)\Big)dx+o(\delta t).$$

As we have

$$\int_{V(t+\delta t)}\varphi(y,t+\delta t)dy=\int_{V(t)}\varphi(x,t)dx,$$

we obtain

$$\forall V(t),\quad \forall\delta t>0,\quad \int_{V(t)}\Big(\frac{\partial\varphi}{\partial t}(x,t)+u(x,t).\nabla\varphi(x,t)+\varphi(x,t)\operatorname{div}u(x,t)\Big)dx=o(\delta t),$$

and therefore

$$\forall V(t),\quad \int_{V(t)}\Big(\frac{\partial\varphi}{\partial t}(x,t)+u(x,t).\nabla\varphi(x,t)+\varphi(x,t)\operatorname{div}u(x,t)\Big)dx=0,$$

that is to say the equation

$$\frac{\partial\varphi}{\partial t}(x,t)+u(x,t).\nabla\varphi(x,t)+\varphi(x,t)\operatorname{div}u(x,t)=0$$

or

$$\frac{\partial\varphi}{\partial t}(x,t)+\operatorname{div}\left(u(x,t).\varphi(x,t)\right)=0.$$

Another way to obtain this equation is the following. Let $S(t)$ be the surface limiting the volume $V(t)$ and let ν be the outward pointing unit normal vector to this surface. We first have to show that

$$\frac{d}{dt}\int_{V(t)} \varphi(x,t)dx = \int_{V(t)} \frac{\partial\varphi}{\partial t}(x,t)dx + \int_{S(t)} \varphi(x,t)u(x,t)\nu(x)dS.$$

Then applying Stokes-Ostrogradskii formula we have

$$\int_{S(t)} \varphi(x,t)u(x,t)\nu(x)dS = \int_{V(t)} \mathrm{div}\,(u(x,t).\varphi(x,t))dx,$$

which shows that

$$\frac{d}{dt}\int_{V(t)} \varphi(x,t)dx = \int_{V(t)} \Big(\frac{\partial\varphi}{\partial t}(x,t) + \mathrm{div}\,(u(x,t).\varphi(x,t))\Big)dx.$$

1.2.1 Conservation of mass

Here we take

$$\varphi(x,t) = \rho(x,t)$$

where ρ is the density of the fluid. We obtain the equation of conservation of mass which can take different forms.

$$\frac{\partial\rho}{\partial t}(x,t) + \mathrm{div}\,(u(x,t)\rho(x,t)) = 0 \text{ in } \Omega\times(0,T), \tag{1.2.1}$$

or

$$\frac{\partial\rho}{\partial t}(x,t) + u(x,t).\nabla\rho(x,t) + \rho(x,t)\mathrm{div}\,u(x,t) = 0 \text{ in } \Omega\times(0,T), \tag{1.2.2}$$

or even using the particle derivative

$$\frac{d}{dt}\rho(x,t) + \rho(x,t)\mathrm{div}\,u(x,t) = 0 \text{ in } \Omega\times(0,T). \tag{1.2.3}$$

1.2.2 Conservation of volume. Incompressibility

A fluid is incompressible if the volume occupied by a group of fluid particles remains constant during the flow. Therefore we here take

$$\varphi(x,t) = 1$$

which gives the incompressibility condition

$$\operatorname{div} u(x,t) = 0 \text{ in } \Omega \times (0,T). \tag{1.2.4}$$

In that case, the conservation of mass becomes

$$\frac{\partial \rho}{\partial t}(x,t) + u(x,t).\nabla\rho(x,t) = 0 \text{ in } \Omega \times (0,T), \tag{1.2.5}$$

$$\rho(x,0) = \rho_0(x) \text{ in } \Omega. \tag{1.2.6}$$

When the inital density satisfies $\rho_0(x) = \rho_0 = \text{Cst}$ (independent of x), this implies

$$\rho(x,t) = \rho_0.$$

Also if $\rho(x,t) = \rho_0$ equation for conservation of mass implies that $\operatorname{div} u(x,t) = 0$. In fact we see that

$$\operatorname{div} u = 0 \Rightarrow \frac{d}{dt}\rho = 0 \Rightarrow \rho(x(t),t) = \text{Cst}$$

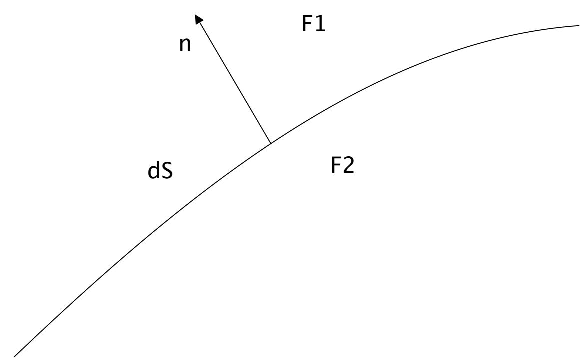
and

$$\rho(x(t),t) = \text{Cst} \Rightarrow \frac{d}{dt}\rho = 0 \Rightarrow \operatorname{div} u = 0.$$

Therefore the fluid is incompressible ($\operatorname{div} u = 0$) if and only if the density of each element stays constant during the flow ($\frac{d}{dt}\rho = 0$).

1.2.3 Stress tensor. Conservation of momentum

Let dS be a surface element in the fluid which separates the fluid in two parts $F1$ and $F2$ and let n be the unit normal vector to dS pointing towards the exterior of $F2$.



The force exerted by $F1$ on $F2$ per surface unit is called **stress**.
For a fluid at rest ($u = 0$), this force is normal to dS, so it is characterized by a scalar quantity at each point. This quantity is the **hydrostatic pressure**.
For a moving fluid, it appears tangential stresses : friction between the fluid layers gliding along each other, due to the **viscosity** of the fluid.
One can show (or it is commonly admitted) that there exists a tensor $\sigma = (\sigma_{ij})$ (represented by a $(2,2)$ or $(3,3)$ or (N,N) matrix called the **stress tensor** such that

- σ is symmetric : $\sigma_{ij} = \sigma_{ji}$. This comes from an equilibrium equation.
- The force exerted by $F1$ on $F2$ is given by

$$\sigma.n = (\sum_{j=1}^{N} \sigma_{ij} n_j)_{i=1,\cdots,N} = (\sigma_{ij} n_j)_{i=1,\cdots,N}$$

  with the convention of summation for repeated indices.

Tensor of viscosity stresses.
Among the stresses, it is convenient to separate those which do not depend of the fluid deformation, that is to say which exist when the fluid is at rest, and those which are due to the fluid deformation. We set

$$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij},$$

where p is the pressure, δ_{ij} is the Kronecker symbol and (σ'_{ij}) is the tensor of **viscosity stresses**. (Sign $-$ in front of the pressure is just a choice indicating that usually, a fluid at rest is compressed).
(σ'_{ij}) is independent of translations and local rotations, and therefore independent of u itself and of $\omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})$ ($\omega = \mathrm{curl} u$ is the vorticity of the fluid).
Therefore, σ'_{ij} only depends on the symmetric part of the tensor of velocity gradients $e = (e_{ij})$ where

$$e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

Newtonian fluids.
For Newtonian fluids (only case which will be considered here), the relation between (σ'_{ij}) and (e_{ij}) is linear and (after some remarks on isotropy etc) the relation can be written

$$\sigma'_{ij} = \eta . \underbrace{(2e_{ij} - \frac{2}{3}\delta_{ij}e_{ll})}_{\text{deformation without volume changes}} + \zeta . \underbrace{(\delta_{ij}e_{ll})}_{\text{isotropic dilation}},$$

where η is the **shear viscosity** and ζ is the **volumic velocity**.
For an incompressible fluid, we have $\mathrm{div}\, u = 0$ so that $e_{ll} = 0$ and then

$$\sigma'_{ij} = 2\eta e_{ij}.$$

Conservation of momentum.
Let $V(t)$ be any volume transported by the flow asociated with the velocity u and limited by a surface $S(t)$. In the presence of external forces represented by f and the action of the exterior of $V(t)$ exerted on the surface $S(t)$, the fundamental law of dynamics takes the (vectorial) form

$$\frac{d}{dt}\int_{V(t)} \rho u dx = \int_{V(t)} \rho f dx + \int_{S(t)} \sigma . n dS.$$

Here f is the volumic density of external forces per unit of mass. For example f can represent

- Gravity forces.

- An electrostatic force for a fluid with electrical charges.
- A Coriolis force for a fluid in a rotating reference frame.
- A magnetic force for a fluid containing magnetic particles (ferrofluid).
- ....

For each component $i = 1, \cdots, N$ we have

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \rho u_i dx &= \int_{V(t)} \Big(\frac{\partial}{\partial t}(\rho u_i) + \mathrm{div}\,(\rho u_i u)\Big)dx \\ &= \int_{V(t)} u_i\Big(\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u)\Big)dx + \int_{V(t)} \rho\Big(\frac{\partial u_i}{\partial t} + u.\nabla u_i\Big)dx.\end{aligned}$$

From the equation expressing conservation of mass we have

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = 0,$$

so that

$$\frac{d}{dt}\int_{V(t)} \rho u_i dx = \int_{V(t)} \rho\Big(\frac{\partial u_i}{\partial t} + u.\nabla u_i\Big)dx = \int_{V(t)} \rho \frac{du_i}{dt} dx.$$

On the other hand, from Stokes-Ostrogradskii formula we have

$$\int_{S(t)} (\sigma.n)_i dS = \int_{V(t)} \sum_{j=1}^{N} \sigma_{ij} n_j dS = \int_{V(t)} \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} dx.$$

Therefore we obtain for every volume $V(t)$

$$\int_{V(t)} \rho\Big(\frac{\partial u_i}{\partial t} + u.\nabla u_i\Big)dx = \int_{V(t)} \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} dx + \int_{V(t)} \rho f_i dx.$$

This gives us the equation for conservation of momentum

$$\rho\Big(\frac{\partial u_i}{\partial t} + u.\nabla u_i\Big) = \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i, \tag{1.2.7}$$

which can also be written as

$$\rho\Big(\frac{\partial u_i}{\partial t} + u.\nabla u_i\Big) = -\frac{\partial p}{\partial x_i} + \sum_{j=1}^{N} \frac{\partial \sigma'_{ij}}{\partial x_j} + \rho f_i, \tag{1.2.8}$$

or in vectorial form

$$\rho\Big(\frac{\partial u}{\partial t} + (u.\nabla)u\Big) = -\nabla p + \mathrm{div}\,\sigma' + \rho f. \tag{1.2.9}$$

1.3 Basic equations : Navier-Stokes, Euler, Stokes,....

If we neglect the spatial variations of viscosities $\frac{\partial \eta}{\partial x_j}$ and $\frac{\partial \zeta}{\partial x_j}$ we obtain :
• For a Newtonian viscous compressible fluid,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \tag{1.3.10}$$

$$\rho\Big(\frac{\partial u}{\partial t} + (u.\nabla)u\Big) = -\nabla p + \eta \Delta u + (\zeta + \frac{\eta}{3})\nabla(\operatorname{div} u) + \rho f. \tag{1.3.11}$$

In order to complete this system we need to give the pressure law, or the energy equation.
• For a Newtonian viscous incompressible fluid we obtain the Navier-Stokes equations

$$\operatorname{div} u = 0, \tag{1.3.12}$$

$$\rho\Big(\frac{\partial u}{\partial t} + (u.\nabla)u\Big) = -\nabla p + \eta \Delta u + \rho f. \tag{1.3.13}$$

• For a perfect (inviscid) incompressible fluid we obtain the Euler equations

$$\operatorname{div} u = 0, \tag{1.3.14}$$

$$\rho\Big(\frac{\partial u}{\partial t} + (u.\nabla)u\Big) = -\nabla p + \rho f. \tag{1.3.15}$$

• For a steady fluid ($u = 0$) we obtain

$$\rho f = \nabla p, \tag{1.3.16}$$

which is the fundamental principle of hydrostatics.
A-dimensional form of Navier-Stokes equations.
Let L and U be the respective reference scales of length and velocity of the flow. We write

$$x' = \frac{x}{L}, \quad u' = \frac{u}{U}, \quad t' = \frac{t}{L/U}, \quad p' = \frac{p - p_0}{\frac{1}{2}\rho U^2},$$

where p_0 is the hydrostatic pressure (in absence of flow). We obtain

$$\frac{\partial u'}{\partial t'} + (u'.\nabla')u' = -\nabla' p' + \frac{\eta}{\rho U L}\Delta' u' + f'. \tag{1.3.17}$$

Setting

$$\nu = \frac{\eta}{\rho} \quad \frac{1}{\mathrm{Re}} = \frac{\nu}{UL},$$

this defines the Reynolds number Re.

Stokes equations.

For small Reynolds numbers, or large viscosity, for laminar flows, we can neglect the convective terms $(u.\nabla)u$ to obtain the Stokes equations

$$\operatorname{div} u = 0, \tag{1.3.18}$$

$$\frac{\partial u}{\partial t} = -\nabla p + \nu \Delta u + f. \tag{1.3.19}$$

Boundary conditions.

• Boundary conditions on the surface of a solid body.

No penetration of the fluid

$$u_{\text{sol}}.n = u_{\text{fluid}}.n.$$

For a viscous fluid : no-slip boundary condition

$$u_{\text{fluid}} = u_{\text{sol}}.$$

• Boundary condition at the (fixed) interface of two fluids.

Continuity of velocities

$$u^1 = u^2.$$

Equilibrium between the stresses in each of the fluids and the stresses localized on the interface (n is the normal vector and τ are tangent vectors)

$$(\sigma^1.n)\tau = (\sigma^2.n)\tau \quad \text{(equality of tangential stresses)},$$

$$(\sigma^1.n)n - (\sigma^2.n)n = \gamma(\frac{1}{R} + \frac{1}{R'}),$$

where γ is the surface tension coefficient between fluid 1 and fluid 2 and R and R' are the principal curvature radii of the interface.

Initial conditions.

They must describe the flow at initial time ($t = 0$) by the datas

$$u_{/t=0} = u_0, \quad \rho_{/t=0} = \rho_0, \cdots$$

Remark 1.3.1 • *If we want to consider an interaction between a fluid and a structure, the coupling must occur in the boundary conditions.*

• *If we want to consider the thermal effects, we must have to add an equation for the energy and the coupling will appear through ρ or σ in the fluid equation.*

• *We can also consider coupling with other phenomena like transport of a specy (salinity in an ocean, physico-chemical elements of a fluid, ...). The coupling in the fluid equation will then have to be defined in a consistent way.*

Remark 1.3.2 *We can consider the stationnary problems corresponding to the previous equations. This does not say that the fluid is at rest, but says only that the flow does not vary with time. Therefore it corresponds to cancel all terms containing* $\frac{\partial}{\partial t}$.

Chapter 2

Stokes equations. Mathematical formulation

2.1 Stationnary Stokes equations

Let Ω be a connected open subset of $\mathbb{R}^N$ that we will suppose to be bounded and regular and let Γ be its boundary. The Stokes equation with no-slip boundary condition can be written in the following (vectorial) form

$$-\nu\Delta u = -\nabla p + f \quad \text{in} \quad \Omega, \tag{2.1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{2.1.2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{2.1.3}$$

where $\nu > 0$ is the given viscosity and f is the given external force and u and p are the velocity and the pressure.
For each component u_i, $i = 1, \cdots, N$ we have

$$-\nu\Delta u_i = -\frac{\partial p}{\partial x_i} + f_i.$$

Let $w = (w_1, \cdots, w_N)$ be a vector function which is "regular" and which is zero on the boundary, with $\operatorname{div} w = 0$. If we multiply the equation for u_i by w_i, integrate on Ω and sum up for $i = 1$ to N we obtain

$$-\nu\sum_{i=1}^{N}\int_\Omega \Delta u_i w_i dx = -\sum_{i=1}^{N}\int_\Omega \frac{\partial p}{\partial x_i} w_i dx + \sum_{i=1}^{N}\int_\Omega f_i w_i dx.$$

As $w_{/\Gamma} = 0$, by integration by parts we have

$$-\sum_{i=1}^{N}\int_\Omega \frac{\partial p}{\partial x_i} w_i dx = \int_\Omega p(\operatorname{div} w) dx = 0,$$

and

$$-\nu \sum_{i=1}^{N} \int_{\Omega} \Delta u_i w_i dx = \nu \sum_{i=1}^{N} \int_{\Omega} \nabla u_i \nabla w_i dx.$$

Therefore u must satisfy for all w regular, vanishing on Γ such that $\operatorname{div} w = 0$,

$$\nu \sum_{i=1}^{N} \int_{\Omega} \nabla u_i \nabla w_i dx = \sum_{i=1}^{N} \int_{\Omega} f_i w_i dx.$$

This suggests a variational formulation of Stokes system.

Precise mathematical formulation.

Let us define the space

$$V = \{w \in H_0^1(\Omega)^N, \quad \operatorname{div} w = 0\} \tag{2.1.4}$$

and the bilinear form defined for all $v \in V$ and $w \in V$ by

$$a(v, w) = \nu \sum_{i=1}^{N} \int_{\Omega} \nabla v_i \nabla w_i dx. \tag{2.1.5}$$

If $f = (f_1, \cdots, f_N) \in L^2(\Omega)^N$ (for example but we could also take $f \in H^{-1}(\Omega)^N$), we look for u such that

$$\begin{aligned} & a(u, w) = \sum_{i=1}^{N} \int_{\Omega} f_i w_i dx \quad (\text{or} \quad \sum_{i=1}^{N} < f_i, w_i >), \quad \forall w \in V, \\ & u \in V. \end{aligned} \tag{2.1.6}$$

Here $< .,. >$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We then have

Theorem 2.1.1 *For every $f \in H^{-1}(\Omega)^N$, there exists a unique solution $u \in V$ to the problem*

$$\begin{aligned} & a(u, w) = \sum_{i=1}^{N} < f_i, w_i >, \quad \forall w \in V, \\ & u \in V. \end{aligned} \tag{2.1.7}$$

The mapping $f \to u$ is linear continuous from $H^{-1}(\Omega)^N$ to V and moreover u is also solution to the minimization problem

$$\begin{aligned} & J(u) = \min_{w \in V} J(w), \\ & u \in V, \end{aligned} \tag{2.1.8}$$

where

$$J(w) = \frac{1}{2}a(w,w) - \sum_{i=1}^{N} < f_i, w_i > . \tag{2.1.9}$$

Proof.
First of all, as Ω is bounded, from Poincaré inequality, we can equip $H_0^1(\Omega)$ with the norm $v \to (\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}}$ and the associated scalar product and it is then a Hilbert space. Then $H_0^1(\Omega)^N$ is also a Hilbert space for the norm (and the corresponding scalar product)

$$w = (w_1, \cdots, w_N) \to (\sum_{i=1}^{N} \int_\Omega |\nabla w_i|^2 dx)^{\frac{1}{2}}.$$

Now the mapping $w \to \operatorname{div} w$ is linear continuous from $H_0^1(\Omega)^N$ to $L^2(\Omega)$, and the space V is the kernel of this mapping. Therefore, V is a closed subspace of $H_0^1(\Omega)^N$ and therefore, it is a Hilbert space for the norm (and the scalar product) induced by the one in $H_0^1(\Omega)^N$.
It is now immediate to see that

$$(v, w) \in V \times V \to a(v, w)$$

is a continuous bilinear form on $V \times V$ which is obviously coercive when $\nu > 0$.
On the other hand, when $f \in H^{-1}(\Omega)^N$, the mapping $w \to \sum_{i=1}^{N} < f_i, w_i >$ is linear continuous from V to $\mathbb{R}$.
We can then apply Lax-Milgram Theorem which shows the first part of Theorem 2.1.1.
That u is the solution to the minimization problem for functional J comes from the fact that the bilinear form $a(.,.)$ is symmetric.
Interpretation. Relation with Stokes problem.

What is the relation with Stokes problem and in particular what about the pressure p which seems to have disappeared?
Let us write $\mathcal{V}$ the space

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega)^N, \quad \operatorname{div} \varphi = 0\},$$

where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$. Of course $\mathcal{V} \subset V$ and it can be shown that $\mathcal{V}$ is dense in V.
Using distribution theory we have for $\varphi \in \mathcal{V}$

$$a(u, \varphi) = \nu \sum_{i=1}^{N} \int_\Omega \nabla u_i \nabla \varphi_i dx = \nu \sum_{i=1}^{N} < \nabla u_i, \nabla \varphi_i >_{\mathcal{D}', \mathcal{D}} = -\nu \sum_{i=1}^{N} < \Delta u_i, \varphi_i >_{\mathcal{D}', \mathcal{D}} .$$

Therefore we obtain

$$\sum_{i=1}^{N} < -\nu\Delta u_i - f_i, \varphi_i >_{\mathcal{D}',\mathcal{D}} = 0 \quad \forall \varphi \in \mathcal{V}.$$

The distribution $T = (T_1, \cdots, T_N) \in \mathcal{D}'(\Omega)^N$ with $T_i = -\nu\Delta u_i - f_i$ is such that

$$\forall \varphi \in \mathcal{V}, \quad < T, \varphi >_{\mathcal{D}',\mathcal{D}} = 0.$$

Lemma 2.1.2 *(de Rham's Lemma) Let $T \in \mathcal{D}'(\Omega)^N$ such that*

$$\forall \varphi \in \mathcal{V}, \quad < T, \varphi >_{\mathcal{D}',\mathcal{D}} = 0.$$

Then there exists a distribution $p \in \mathcal{D}'(\Omega)$ such that

$$T = -\nabla p.$$

In fact it is easy to see that

$$\forall p \in \mathcal{D}'(\Omega), \quad \forall \varphi \in \mathcal{V}, \quad < -\nabla p, \varphi >_{\mathcal{D}',\mathcal{D}} = < p, \operatorname{div} \varphi >_{\mathcal{D}',\mathcal{D}} = 0.$$

de Rham's Lemma, which is very difficult, shows that the only distributions which vanish on $\mathcal{V}$ are of the form $T = -\nabla p$.
So here we see that there exists $p \in \mathcal{D}'(\Omega)$ such that for $i = 1, \cdots, N$

$$\begin{aligned} &-\nu\Delta u_i = f_i - \frac{\partial p}{\partial x_i} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ &\operatorname{div} u = 0, \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

In fact we can see that for every $i = 1, \cdots, N$

$$\frac{\partial p}{\partial x_i} = f_i + \nu\Delta u_i \in H^{-1}(\Omega).$$

Therefore $p \in \mathcal{D}'(\Omega)$ and $\nabla p \in H^{-1}(\Omega)$ and this implies that $p \in L^2(\Omega)$. Of course p is defined up to the addition of a constant. We then obtain

Theorem 2.1.3 *Let $f \in H^{-1}(\Omega)^N$. Then there exists a unique $u \in V$ and a unique $p \in L^2(\Omega)_{/\mathbb{R}}$ (p is unique up to the addition of a constant) such that*

$$-\nu\Delta u_i = f_i - \frac{\partial p}{\partial x_i} \quad \textit{in} \quad H^{-1}(\Omega)^N, \tag{2.1.10}$$

$$\operatorname{div} u = 0, \quad \textit{in} \quad \Omega, \tag{2.1.11}$$

$$u = 0 \quad \textit{on} \quad \Gamma. \tag{2.1.12}$$

((u, p) is solution to Stokes problem).

We can also obtain a regularity result.

Theorem 2.1.4 *Let $f \in L^2(\Omega)^N$. The solution (u,p) of Stokes problem satisfies*

$$u \in H^2(\Omega)^N \cap V, \quad p \in H^1(\Omega)_{/\mathbb{R}}$$

and we have

$$||u||_{H^2(\Omega)^N} + |p|_{L^2(\Omega)_{/\mathbb{R}}} \leq C|f|_{L^2(\Omega)^N}.$$

2.2 Complements of functional analysis and applications

This section will provide some complementary results of functional analysis which enable to prove most of the properties which have been accepted so far without proof. This section follows very closely lecture notes given by L. Tartar on the subject, see [3].

First of all we start with some abstract results.

Let F, G, H be three Hilbert spaces and let A be a continuous linear operator from H to F. Then

$$\mathrm{Ker}A = \{h \in H, \quad Ah = 0\}$$

is a closed linear subspace of H. Let us define

$$N = \mathrm{Ker}A^{\perp} = \{h \in H, \quad \forall l \in \mathrm{Ker}A, \quad (h,l)_H = 0\}.$$

Then N is also a closed linear subspace of H.

If $h \in H$, we can consider $h_1 = \mathrm{Proj}_{\mathrm{Ker}A} h$ and $h_2 = h - h_1$. From the definition of projection we have

$$\forall l \in \mathrm{Ker}A, \quad (h - h_1, l)_H = 0$$

which implies

$$h_2 = h - h_1 \in N.$$

Therefore for every $h \in H$ we have the decomposition

$$h = h_1 + h_2, \quad h_1 \in \mathrm{Ker}A, \quad h_2 \in N.$$

Moreover, if $h \in \mathrm{Ker}A \cap N$, then it is clear that $h = 0$. This says that

$$H = \mathrm{Ker}A \oplus N$$

and $\mathrm{Ker}A$ and N are closed, so that they are topological supplements. The above decomposition is then unique and moreover there exist two constants $C_1 > 0$ and $c_2 > 0$ such that

$$\forall h \in H, \quad |h_1|_H \leq C_1|h|_H, \quad |h_2|_H \leq C_2|h|_H$$

which says that the projections are continuous. (It is clear here in the case of Hilbert spaces that the constants C_1 and C_2 are less or equal to 1.)
Now we will assume that there exists a **compact** linear operator B from H to G such that

(2.2.13) $$\exists C > 0, \quad \forall h \in H, \quad |h|_H \leq C(|Ah|_F + |Bh|_G).$$

Lemma 2.2.1 *Under the above hypotheses*

(2.2.14) $$\exists C_N, \quad \forall h \in N, \quad |h|_H \leq C_N |Ah|_F.$$

Moreover, the image of A denoted ImA *is closed in F.*

Proof.
Let us suppose that (2.2.14) is not true. Then

$$\forall n \in \mathbb{N}, \quad \exists h_n \in N, \quad |h_n|_H \geq n|Ah_n|_F.$$

Let us define

$$\tilde{h}_n = \frac{h_n}{|h_n|_H} \in N.$$

Then $|\tilde{h}_n|_H = 1$ and $|A\tilde{h}_n|_F \leq \frac{1}{n}$ (therefore $|A\tilde{h}_n|_F \to 0$ in F).
As $|\tilde{h}_n|_H = 1$ we can extract a subsequence, still denoted $(\tilde{h}_n)$ such that

$$\tilde{h}_n \rightharpoonup h_0 \quad \text{in} \quad H \text{ weakly}.$$

As N is a closed subspace (then closed for the weak topology) and $\tilde{h}_n \in N$, $h_0 \in N$. As A is linear continuous from H to F, it is continuous for the weak topologies and therefore,

$$A\tilde{h}_n \rightharpoonup Ah_0 \quad \text{in} \quad F \text{ weakly}.$$

But we already know that $A\tilde{h}_n \to 0$ in F strongly. Therefore $Ah_0 = 0$ and $h_0 \in N$. This says that $h_0 \in \mathrm{Ker}A \cap N$ so that $h_0 = 0$.
Now we have

$$\tilde{h}_n \rightharpoonup 0 \quad \text{in} \quad H \text{ weakly} \quad \text{and} \quad A\tilde{h}_n \to 0 \quad \text{in} \quad F \text{ strongly}.$$

As B is compact we have

$$B\tilde{h}_n \to B0 = 0 \quad \text{in} \quad G \text{ strongly}$$

so that $|B\tilde{h}_n|_G \to 0$.
But from (2.2.13) we have

$$|\tilde{h}_n|_H \leq C(|A\tilde{h}_n|_F + |B\tilde{h}_n|_G) \to 0.$$

This gives a contradiction with the fact that $|\tilde{h}_n|_H = 1$. Therefore (2.2.14) is true.
Let us now show that ImA is closed in F.
Let us take a sequence (y_n) such that

$$y_n \in \mathrm{Im}A, \quad y_n \to y \quad \text{in} \quad F.$$

As $y_n \in \mathrm{Im}A$, there exists $\bar{h}_n \in H$ such that $A\bar{h}_n = y_n$. We have the decompsition

$$\bar{h}_n = h_n + \hat{h}_n, \quad h_n \in N, \quad \hat{h}_n \in \mathrm{Ker}A.$$

Then $Ah_n = y_n$ and $h_n \in N$.
As $y_n \to y$ in F, (y_n) is a Cauchy sequence in F. and

$$A(h_m - h_p) = y_m - y_p, \quad (h_m - h_p) \in N.$$

From (2.2.14) we see that

$$|h_m - h_p|_H \leq C_N |y_m - y_p|_F$$

and therefore, (h_n) is a Cauchy sequence in H which is complete. This shows that there exists $h \in H$ such that $h_n \to h$ in H and as $A$is continuous,

$$Ah_n \to Ah \quad \text{in} \quad F.$$

But we know that $Ah_n = y_n$ converges to y in F. Then there exists $h \in H$ such that $y = Ah$ and $y \in \mathrm{Im}A$ so that ImA is closed.
This finishes the proof of Lemma 2.2.1.
We are going to show that hese abstract results can be applied to the case

$$H = L^2(\Omega), \quad F = H^{-1}(\Omega^N, \quad G = H^{-1}(\Omega)$$

and

$$A = \nabla \quad \text{(Gradient operator)}$$

in order to show that ∇ has a closed image in $H^{-1}(\Omega)^N$.
Let us define the space

(2.2.15) $$X(\Omega) = \{g \in H^{-1}(\Omega), \quad \nabla g \in H^{-1}(\Omega^N\}.$$

Lemma 2.2.2
(2.2.16) $$X(\mathbb{R}^N) = L^2(\mathbb{R}^N).$$

Proof.
It is clear that $L^2(\mathbb{R}^N) \subset X(\mathbb{R}^N)$. Let us show that if $g \in X(\mathbb{R}^N)$, then $g \in L^2(\mathbb{R}^N)$ by using the Fourier transform.
We know that

$$g \in L^2(\mathbb{R}^N) \iff \hat{g} \in L^2(\mathbb{R}^N) \text{ and that } g \in H^{-1}(\mathbb{R}^N) \iff (1+|\xi|^2)^{-\frac{1}{2}}\hat{g} \in L^2(\mathbb{R}^N).$$

Now

$$\nabla g \in H^{-1}(\mathbb{R}^N)^N \iff (1+|\xi|^2)|^{\frac{1}{2}}|\xi|\hat{g}| \in L^2(\mathbb{R}^N)$$

so that

$$g \in X(\mathbb{R}^N) \iff (1+|\xi|^2)(1+|\xi|^2)|\hat{g}|^2 \in L^1(\mathbb{R}^N) \iff \hat{g} \in L^2(\mathbb{R}^N).$$

Lemma 2.2.3
(2.2.17)
$$X(\mathbb{R}^N_+) = L^2(\mathbb{R}^N_+).$$

Proof.
It is clear that $L^2(\mathbb{R}^N_+) \subset X(\mathbb{R}^N_+)$. Let us show that $X(\mathbb{R}^N_+) \subset L^2(\mathbb{R}^N_+)$. To this end, we are going to exhibit a continuous extension from $X(\mathbb{R}^N_+)$ to $X(\mathbb{R}^N)$.

Let us first define a restriction operator from $H^1(\mathbb{R}^N)$ to $H^1_0(\mathbb{R}^N_+)$. For $u \in \mathcal{D}(\mathbb{R}^N) = C_0^\infty(\mathbb{R}^N)$ we set

$$Qu(x_1,\cdots,x_N) = \begin{cases} 0, & \text{if } \ x_N < 0, \\ u(x_1,\cdots,x_N) + \sum_{j=1}^2 a_j u(x_1,\cdots,x_{N-1},-jx_N) & \text{if } \ x_N > 0. \end{cases}$$

We impose that

$$1 + \sum_{j=1}^{2} a_j = 0$$

which implies

$$Qu(x_1,\cdots,x_{N-1},0) = 0.$$

It is clear that for $i = 1,\cdots,N-1$ we have

$$Q(\frac{\partial u}{\partial x_i}) = \frac{\partial Qu}{\partial x_i}.$$

For $i = N$ we have

$$Q(\frac{\partial u}{\partial x_N}) = \frac{\partial}{\partial x_N} Ru$$

where

$$Ru(x_1,\cdots,x_N) = \begin{cases} 0, & \text{if } \ x_N < 0, \\ u(x_1,\cdots,x_N) + \sum_{j=1}^2 \frac{a_j}{-j} u(x_1,\cdots,x_{N-1},-jx_N) & \text{if } \ x_N > 0. \end{cases}$$

We also want R to be continuous from $H^1(\mathbb{R}^N)$ to $H_0^1(\mathbb{R}_+^N)$ which imposes

$$1+\sum_{j=1}^{2}\frac{a_j}{-j}=0.$$

Thus we choose the coefficients a_j such that

$$1+\sum_{j=1}^{2}a_j=0, \quad 1+\sum_{j=1}^{2}\frac{a_j}{-j}=0$$

and they are well defined by these relations. Now we extend Q and R by continuity to $H^1(\mathbb{R}^N)$ and we define

$$P={}^tQ.$$

As Q is linear continuous from $H^1(\mathbb{R}^N)$ to $H_0^1(\mathbb{R}_+^N)$, P is linear continuous from $H^{-1}(\mathbb{R}_+^N)$ to $H^{-1}(\mathbb{R}^N)$. Moreover we have for $i=1,\cdots,N-1$,

$$\forall\varphi\in H^1(\mathbb{R}^N), \quad <\frac{\partial}{\partial x_i}Pu,\varphi>=-<Pu,\frac{\partial\varphi}{\partial x_i}>=-<u,Q(\frac{\partial\varphi}{\partial x_i})>$$
$$=-<u,\frac{\partial}{\partial x_i}Q\varphi>=<\frac{\partial u}{\partial x_i},Q\varphi>=<P(\frac{\partial u}{\partial x_i}),\varphi>,$$

so that

$$\frac{\partial}{\partial x_i}Pu=P(\frac{\partial u}{\partial x_i}).$$

So, for $i=1,\cdots,N-1$, as $\frac{\partial u}{\partial x_i}\in H^{-1}(\mathbb{R}_+^N)$ we have $P(\frac{\partial u}{\partial x_i})=\frac{\partial}{\partial x_i}Pu\in H^{-1}(\mathbb{R}^N)$. Now for $i=N$ we have in the same way

$$\frac{\partial}{\partial x_N}Pu={}^tR\frac{\partial u}{\partial x_N}$$

and ${}^tR$ is linear continuous from $H^{-1}(\mathbb{R}_+^N)$ to $H^{-1}(\mathbb{R}^N)$.
Therefore, if $u\in X(\mathbb{R}_+^N)$, then $Pu\in X(\mathbb{R}^N)$ and P is a continuous linear operator. It remains to prove that P is an extension.
For $u\in H_0^1(\mathbb{R}_+^N)$ let us define by Eu the extension by 0 outside $\mathbb{R}_+^N$. It is well known that $u\to Eu$ is linear continuous from $H_0^1(\mathbb{R}_+^N)$ to $H^1(\mathbb{R}^N)$. We denote by Π the transposition of this operator E which is the restriction to $\mathbb{R}_+^N$ and which is a linear continuous operator from $H^{-1}(\mathbb{R}^N)$ to $H^{-1}(\mathbb{R}_+^N)$. We have to show that $\Pi.P=Id$.
But $\Pi.P={}^tE.{}^tQ={}^t(Q.E)$ and it is clear that $Q.E=Id$ so that $\Pi.P=Id$.
If $u\in X(\mathbb{R}_+^N)$, $Pu\in X(\mathbb{R}^N)=L^2(\mathbb{R}^N)$ from Lemma 2.2.2. Then $\Pi.Pu=u\in L^2(\mathbb{R}_+^N)$ and this finishes the proof of Lemma 2.2.3.

Lemma 2.2.4 *If Ω is a bounded regular open set in $\mathbb{R}^N$, then*

$$X(\Omega) = L^2(\Omega). \tag{2.2.18}$$

Proof.
Let (θ_i) be a partition of unity so that

$$\theta_i \in C^\infty(\overline{\Omega}), \quad 0 \leq \theta_i \leq 1, \quad \sum_{i=1}^{I} \theta_i = 1.$$

We write

$$u = \sum_{i=1}^{I} \theta_i u.$$

In the case $\theta_i \in C_0^\infty(\Omega)$, $\theta_i u$ can be extended by zero to have $\theta_i u \in X(\mathbb{R}^N)$.
When $\theta_i \in C^\infty(\overline{\Omega})$, we can find a C^2 diffeomorphism η such that $\theta_i u \circ \eta^{-1} \in X(\mathbb{R}^N)$.
Then $\theta_i u \circ \eta^{-1} \in L^2(\mathbb{R}^N)$ which implies $\theta_i u \in L^2(\Omega)$.
Now we have, algebraically, $X(\Omega) = L^2(\Omega)$.
Let us take on $X(\Omega)$ the norm defined by

$$\forall g \in X(\Omega), \quad |g|^2_{X(\Omega)} = |\nabla g|^2_{H^{-1}(\Omega)^N} + |g|^2_{H^{-1}(\Omega)}.$$

Equipped with this norm it is clear that $X(\Omega)$ is a Hilbert space.

Lemma 2.2.5 *There exists a constant $C > 0$ such that*

$$\forall g \in L^2(\Omega), \quad |g|^2_{L^2(\Omega)} \leq C |g|^2_{X(\Omega)}. \tag{2.2.19}$$

Proof.
When $g \in L^2(\Omega)$ we have

$$|g|^2_{H^{-1}(\Omega)} \leq C |g|^2_{L^2(\Omega)}$$

and

$$|\nabla g|^2_{H^{-1}(\Omega)^N} \leq C |g|^2_{L^2(\Omega)}.$$

Let us consider the identity map from $L^2(\Omega)$ to $X(\Omega)$, i.e.

$$\text{Id} \ : \Big(L^2(\Omega), |\cdot|_{L^2(\Omega)}\Big) \to \Big(X(\Omega), |\cdot|_{X(\Omega)}\Big).$$

From Lemma 2.2.4 this is a one-to-one mapping which is continuous between two Hilbert spaces. From Banach Theorem, it is bi-continuous (the inverse is continuous) which says that there exists a constant $C > 0$ such that

$$\forall g \in L^2(\Omega), \quad |g|^2_{L^2(\Omega)} \leq C |g|^2_{X(\Omega)}.$$

This gives Lemma 2.2.5.
Now, as Ω is bounded, $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and then $L^2(\Omega)$ is also compactly embedded in $H^{-1}(\Omega)$.
Let us set

$$H = L^2(\Omega), \quad F = H^{-1}(\Omega)^N, \quad G = H^{-1}(\Omega), \quad A = \nabla, \quad B = \mathrm{Id}.$$

All hypotheses of Lemma 2.2.1 are fulfilled. We then have the following results.

Lemma 2.2.6 *The operator $\nabla\ : L^2(\Omega) \to H^{-1}(\Omega)^N$ has a closed image.*

If we define

$$\mathrm{Ker}\nabla = \{g \in L^2(\Omega), \quad \nabla g = 0\}$$

as Ω is a connected set, we see that the elements of $\mathrm{Ker}\nabla$ are constants in Ω. Therefore we have

$$\mathrm{Ker}\nabla^{\perp} = \{g \in L^2(\Omega), \quad \int_\Omega g dx = 0\}.$$

Lemma 2.2.7 *If (p_n) is a sequence in $L^2(\Omega)$ such that $\int_\Omega p_n dx = 0$ and $|\nabla p_n|_{H^{-1}(\Omega)^N}$ is bounded, then $|p_n|_{L^2(\Omega)}$ is bounded.*

Proof.
We have $p_n \in \mathrm{Ker}\nabla^{\perp}$. Then from Lemma 2.2.1, there exists a constant $C > 0$ independent of n such that

$$|p_n|_{L^2(\Omega)} \leq C|\nabla p_n|_{H^{-1}(\Omega)^N}$$

and this implies the lemma.

Lemma 2.2.8 *Let $f \in H^{-1}(\Omega)^N$ such that*

$$\forall w \in V, \quad < f, w >= 0.$$

Then there exists $p \in L^2(\Omega)_{/\mathbb{R}}$ such that $f = -\nabla p$.
Moreover, there exists a constant $C > 0$ such that

$$|p|_{L^2(\Omega)_{/\mathbb{R}}} \leq C|f|_{H^{-1}(\Omega)^N}.$$

Proof.
Let us write

$$Y = \{\nabla p, \quad p \in L^2(\Omega)\}.$$

Then Y is a closed subspace of $H^{-1}(\Omega^N$. Let us show that

$$\{w \in H_0^1(\Omega)^N, \quad < y, w >= 0, \ \forall y \in Y\} = V.$$

Actually, if $w \in H_0^1(\Omega)^N$ and $< \nabla p, w >= 0$ $\forall p \in L^2(\Omega)$ we have $< w, \nabla\varphi >= 0$ $\forall \varphi \in C_0^\infty(\Omega)$ so that $< \operatorname{div} w, \varphi >= 0$ $\forall \varphi \in C_0^\infty(\Omega)$ and therefore $\operatorname{div} w = 0$ so that $w \in V$.
On the other hand we have

$$\forall p \in L^2(\Omega), \quad \forall w \in V, \quad < \nabla p, w >= 0.$$

Therefore $V = Y^\perp$ which implies

$$V^\perp = (Y^\perp)^\perp = \overline{Y} = Y.$$

Then if $f \in H^{-1}(\Omega)^N$ is such that $\forall w \in V$ $< f, w >= 0$ then $f \in V^\perp = Y$ and this proves Lemma 2.2.8.
Lemma 2.2.8 enables us to give a completely correct interpretation of Stokes problem in Theorem 2.1.3. Indeed, if $g_i = -\nu\Delta u_i - f_i$ and $g = (g_1, \cdots, g_N)$ we have from the variational formulation

$$g \in H^{-1}(\Omega)^N \quad \text{and} \quad \forall w \in V, \quad < g, w >= 0.$$

Then there exists $p \in L^2(\Omega)_{/\mathbb{R}}$ such that $g = -\nabla p$, which gives the correct interpretation of Stokes problem.
This result can also be proved by another method.
For $\epsilon > 0$ let us consider the problem

$$\begin{aligned} &a(u_\epsilon, w) + \frac{1}{\epsilon}\int_\Omega \operatorname{div} u_\epsilon \operatorname{div} w dx = \sum_{i=1}^N < f_i, w_i >, \ \forall w \in H_0^1(\Omega)^N, \\ &u_\epsilon \in H_0^1(\Omega)^N. \end{aligned}$$

This problem has a unique solution u_ϵ which satisfies

$$\begin{aligned} &-\nu\Delta u_\epsilon - \nabla(\frac{1}{\epsilon}\operatorname{div} u_\epsilon) = f \\ &u_\epsilon \in H_0^1(\Omega)^N. \end{aligned}$$

If we write

$$p_\epsilon = -\frac{1}{\epsilon}\operatorname{div} u_\epsilon$$

then

$$p_\epsilon \in L^2(\Omega), \quad \int_\Omega p_\epsilon dx = 0.$$

On the other hand we have

$$a(u_\epsilon, u_\epsilon) + \frac{1}{\epsilon}\int_\Omega |\operatorname{div} u_\epsilon|^2 dx = \sum_{i=1}^N < f_i, u_{\epsilon,i} >$$

so that there exists a constant M independent of ϵ such that

$$||u_\epsilon||_{H_0^1(\Omega)^N} \le M, \quad |\frac{1}{\sqrt{\epsilon}}\mathrm{div}\, u_\epsilon|_{L^2(\Omega)} \le M.$$

Then, after extraction of a subsequence, we have

$$u_\epsilon \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega)^N \text{ weakly,}$$

$$\mathrm{div}\, u_\epsilon \to 0 \quad \text{in} \quad L^2(\Omega) \text{ strongly.}$$

Then $\mathrm{div}\, u = 0$ so that $u \in V$ and for every $w \in V$ $(\mathrm{div}\, w = 0)$ we have

$$a(u_\epsilon, w) = \sum_{i=1}^N < f_i, w_i >$$

so that

$$a(u, w) = \sum_{i=1}^N < f_i, w_i >$$

and u is solution of the variational form of Stokes problem.
Now p_ϵ satisfies

$$\nabla p_\epsilon = f + \nu \Delta u_\epsilon, \quad \int_\Omega p_\epsilon dx = 0.$$

Then p_ϵ is bounded in $H^{-1}(\Omega)^N$ and satisfies $\int_\Omega p_\epsilon dx = 0$ so that p_ϵ is bounded in $L^2(\Omega)$. After extraction of a subsequence,

$$p_\epsilon \rightharpoonup p \quad \text{in} \quad L^2(\Omega) \text{ weakly}$$

and we have

$$\forall w \in H_0^1(\Omega)^N, \quad a(u, w) = \sum_{i=1}^N < f_i, w_i > + \int_\Omega p \mathrm{div}\, w dx$$

so that

$$-\nu \Delta u = f - \nabla p \quad \text{in} \quad H^{-1}(\Omega)^N.$$

Let us now go back to the abstract formulation. We know that $\mathrm{Im}A$ is closed in F.
Then writing

$$L = \{l \in F, \quad \forall u \in H, \quad (Au, l)_F = 0\}$$

we have

$$F = \mathrm{Im}A \oplus L.$$

Here, $F = H^{-1}(\Omega)^N = W'$ where $W = H_0^1(\Omega)^N$ so that $F' = W$.

Lemma 2.2.9

$$W = (\mathrm{Im}A)^{\perp} \oplus L^{\perp}.$$

Proof.
If $u \in (\mathrm{Im}A)^{\perp} \cap L^{\perp}$, then

$$\forall g \in H,\ < u, Ag >= 0, \quad \forall l \in L,\ < u, l >= 0.$$

As $\forall h \in W'$ we have the decomposition $h = Ag + l$ with $g \in H$ and $l \in L$, we have

$$\forall h \in W', \quad < u, h >= 0.$$

Let now $u \in W$. Let us show that there exists $v \in W$ such that

$$\forall h \in W', \quad < v, h >=< u, Ag > .$$

The mapping $h \in W' \to < u, Ag >$ is linear. Moreover

$$| < u, Ag > | \leq ||u||_W ||Ag||_{W'} \leq C||u||_W ||h||_{W'}$$

and so the mapping is continuous. Therefore, there exists $v \in W$ such that

$$\forall h \in W', \quad < v, h >=< u, Ag > .$$

Now for every $l \in L$ we can write $l = A0 + l$ and then

$$\forall l \in L, \quad < v, l >=< u, A0 >= 0$$

and $v \in L^{\perp}$.
This implies

$$\forall g \in H, \quad < u - v, Ag >= 0$$

which says that $u - v \in (\mathrm{Im}A)^{\perp}$.

We then have

$$u = v + w, \quad v \in L^{\perp}, \quad w \in \mathrm{Im}A^{\perp}.$$

Lemma 2.2.10 *If* ${}^tA : W \to H$ *is defined by*

$$\forall u \in W, \quad \forall g \in H, \quad ({}^tAu, g)_H =< u, Ag >$$

then we have

$$\mathrm{Ker}{}^tA = \mathrm{Im}A^{\perp}, \quad \text{and} \quad \overline{\mathrm{Im}{}^tA} = \mathrm{Ker}A^{\perp}.$$

Proof.
If $u \in \mathrm{Ker}^t A$, we have $\forall g \in H,\ < u, Ag >= 0$ and then $u \in \mathrm{Im}A^\perp$.
If $u \in \mathrm{Im}A^\perp$, we have $\forall g \in H,\ < u, Ag >= 0$ and therefore $({}^tAu, g)_H = 0$ so that ${}^tAu = 0$ and $u \in \mathrm{Ker}^t A$.
Now if $g \in \mathrm{Im}^t A$, there exists $u \in W$ such that $g =^t\ Au$. We have for $f \in\ \mathrm{Ker}A$, $(g, f)_H = ({}^tAu, f)_H =< u, Af >= 0$, so that $g \in \mathrm{Ker}A^\perp$. This shows that $\mathrm{Im}^t A \subset \mathrm{Ker}A^\perp$ and as $\mathrm{Ker}A^\perp$ is closed, we have

$$\overline{\mathrm{Im}^t A} \subset \mathrm{Ker}A^\perp.$$

Let us suppose that $\overline{\mathrm{Im}^t A} \neq \mathrm{Ker}A^\perp$. Then there exists $g \in H' = H$ and there exists $f_0 \in \mathrm{Ker}A^\perp$ such that

$$\forall f \in \mathrm{Im}^t A,\ (g, f)_H = 0, \quad \text{and} \quad (g, f_0)_H = 1.$$

Then for every $u \in W$, $(g,^t Au)_H = 0$ so that $< u, Ag >= 0$ and therefore $Ag = 0$ in W'. Therefore, $g \in \mathrm{Ker}A$ which implies $(g, f_0)_H = 0$. This gives a contradiction. Therefore we have

$$\overline{\mathrm{Im}^t A} = \mathrm{Ker}A^\perp.$$

Lemma 2.2.11 $\mathrm{Im}^t A$ *is closed in* H*.*

Proof.
Let (g_n) be a sequence of elements of $\mathrm{Im}^t A$ such that $g_n \to g$ in H. We have $g_n =^t A\tilde{u}_n$ with $\tilde{u}_n \in W$. Moreover we know from the previous lemma that $g_n \in \mathrm{Ker}A^\perp$. From Lemma 2.2.9, we can decompose $\tilde{u}_n$ as $\tilde{u}_n = u_n + v_n$ with $u_n \in L^\perp$ and $v_n \in \mathrm{Im}A^\perp = \mathrm{Ker}^t A$. Then ${}^tA\tilde{u}_n =^t Au_n$ so that $g_n =^t Au_n$ with $u_n \in L^\perp$.
If $h \in W'$ we have $h = Ag + l$ with $g \in \mathrm{Ker}A^\perp = N$ and $l \in L$. (the fact that we can take $g \in \mathrm{Ker}A^\perp$ is immediate since $H = \mathrm{Ker}A \oplus \mathrm{Ker}A^\perp$.) Then

$$\begin{aligned}&< u_m - u_p, h >=< u_m - u_p, Ag >=< u_m - u_p, l >=< u_m - u_p, Ag >\\ &= ({}^tA(u_m - u_p), g)_H = (g_m - g_p, g)_H.\end{aligned}$$

Then for every $h \in W'$

$$\begin{aligned}&| < u_m - u_p, h > | \leq |g_m - g_p|_H |g|_H\\ &\leq |g_m - g_p|_H C_N ||Ag||_{W'}\\ &\leq C.C_N |g_m - g_p|_H ||h||_{W'}.\end{aligned}$$

This implies

$$||u_m - u_p||_W \leq C.C_N |g_m - g_p|_H.$$

Therefore, $(u_n$ is a Cauchy sequence in W which converges to some $u \in W$. Now ${}^tAu_n = g_n \to^t Au$ in H and therefore $g =^t Au$ so that $g \in \mathrm{Im}^t A$ and $\mathrm{Im}^t A$ is closed in H.

Lemma 2.2.12 *For every $f \in \mathrm{Ker} A^{\perp}$, there exists $u \in L^{\perp}$ such that*

$$ {}^{t}Au = f \quad \textit{and} \quad ||u||_W \leq C|f|_H. $$

Proof.
We have $W = \mathrm{Im}A^{\perp} \oplus L^{\perp} = \mathrm{Ker}^{t}A \oplus L^{\perp}$. Now it is immediate to see that ${}^{t}A : L^{\perp} \to H$ is injective. But as $\mathrm{Im}^{t}A = \mathrm{Ker}A^{\perp}$ we see that the mapping ${}^{t}A : L^{\perp} \to \mathrm{Ker}A^{\perp}$ is one-to-one (bijective). On the other hand we have $|{}^{t}Au|_H \leq C||u||_W$ which says that ${}^{t}A$ is continuous.
Now $l^{\perp}$ and $\mathrm{Ker}A^{\perp}$ are Banach spaces as they are closed subspaces of Banach spaces. Then from Banach Theorem, ${}^{t}A : L^{\perp} \to \mathrm{Ker}A^{\perp}$ has a continuous inverse which shows the lemma.
As an application we have the following result concenring operator div .

Theorem 2.2.13 *There exists a constant $C > 0$ such that for every $g \in L^2_0(\Omega) = \{g \in L^2(\Omega),\ \int_{\Omega} g dx = 0\}$, there exists $u \in H^1_0(\Omega)^N$ such that*

$$ \mathrm{div}\, u = g $$

and

$$ ||u||_{H^1_0(\Omega)^N} \leq C|g|_{L^2(\Omega)}. $$

Proof.
Take

$$ A = \nabla : L^2(\Omega) = H \to H^{-1}(\Omega)^N = W'. $$

Then $\mathrm{Ker}A = \{g \in L^2(\Omega),\ g = \mathrm{Cst}\}$ and $\mathrm{Ker}A^{\perp} = L^2_0(\Omega)$.
If $g \in L^2_0(\Omega)$, there exists $u \in L^{\perp} \subset W$ such that ${}^{t}Au = -g$ and $||u||_W \leq C|g|_H$.
Then $u \in H^1_0(\Omega)^N$ and for every $h \in H$ we have

$$ (-g, h)_H = ({}^{t}Au, h)_H =< u, Ah >=< u, \nabla h > . $$

Taking $h \in C^{\infty}_0(\Omega)$ we see that

$$ ({}^{t}Au, h)_H =< u, \nabla h >_{\mathcal{D}',\mathcal{D}}= - < \mathrm{div}\, u, h >_{\mathcal{D}',\mathcal{D}}=< -g, h >_{\mathcal{D}',\mathcal{D}} . $$

Then we have

$$ \mathrm{div}\, u = g, \quad u \in H^1_0(\Omega)^N, \quad \text{and} \quad ||u||_{H^1_0(\Omega)^N} \leq C|g|_{L^2(\Omega)}. $$

Lemma 2.2.14 *If $f \in H^{-1}(\Omega)^N$ and $< f, \varphi >= 0\ \forall \varphi \in \mathcal{V}$, then there exists $p \in L^2(\Omega)$ such that $f = \nabla p$.*

Proof.
We recall that $\mathcal{V} = \{\varphi \in C_0^\infty(\Omega)^N = \mathcal{D}(\Omega)^N,\ \operatorname{div}\varphi = 0\}$.
Let us consider an increasing sequence of open sets (Ω_n) such that $\overline{\Omega}_n \subset \Omega$ and $\cup_n \Omega_n = \Omega$. If $u \in H_0^1(\Omega_n)^N$ and $\operatorname{div} u = 0$, we can extend u by 0 outside Ω_n. If (ρ_ϵ) is a regularizing sequence (mollifiers), for ϵ small enough we have $\rho_\epsilon * u \in C_0^\infty(\Omega)^N$ and $\operatorname{div}(\rho_\epsilon * u) = 0$. Then $< f, u >= \lim_{\epsilon\to 0} < f, \rho_\epsilon * u >= 0$. Therefore

$$\forall u \in H_0^1(\Omega_n)^N, \quad \operatorname{div} u = 0, \qquad < f_{/\Omega_n}, u >= 0.$$

Then there exists $p_n \in L^2(\Omega_n)$ such that $f_{/\Omega_n} = \nabla p_n$.
But $p_{n+1} - p_n$ is constant on Ω_n and we can assume that this constant is 0. Then $f = \nabla p$ where $p \in L^2_{\text{loc}}(\Omega)$.
Now if Ω is starshaped (for example with respect to 0), for $0 < \theta < 1$ we have $\theta\overline{\Omega} \subset \Omega$. If $u \in H_0^1(\Omega)^N,\ \operatorname{div} u = 0$, then u_θ defined by $u_\theta(x) = u(\frac{x}{\theta})$ has a compact support in Ω and $\operatorname{div} u_\theta = 0$. Then we can approximate u_θ by $\rho_\epsilon * u_\theta \in C_0^\infty(\Omega)^N$ with $\operatorname{div}(\rho_\epsilon * u_\theta) = 0$ so that

$$< f, u >= \lim_{\theta\to 1}\lim_{\epsilon\to 0} < f, \rho_\epsilon * u_\theta >= 0.$$

Then, in this case, there exists $p \in L^2(\Omega)$ such that $f = \nabla p$.
In the general case, every point of Γ has a connected neighborhood ω which is regular and strictly starshaped. Then there exists $q \in L^2(\omega)$ such that $f_{/\omega} = \nabla q$.
But $p_{/\omega} - q$ is constant on ω. Therefore $p_{/\omega} \in L^2(\omega)$ and therefore, $p \in L^2(\Omega)$.

Theorem 2.2.15 *The space of regular functions $\mathcal{V}$ (which has been recalled above) is dense in V.*

Proof.
Let $f \in H^{-1}(\Omega)^N$ such that for every $\varphi \in \mathcal{V}$ we have $< f, \varphi >= 0$. Then there exists $p \in L^2(\Omega)$ such that $f = \nabla p$. This implies that for every $w \in V$ we have $< f, w >= 0$. Therefore $\overline{\mathcal{V}} \supset V$, but it is clear that $\overline{\mathcal{V}} \subset V$ and the theorem follows.

Theorem 2.2.16 *The closure of $\mathcal{V}$ in $L^2(\Omega)^N$ is the space*

$$H = \{v \in L^2(\Omega)^N,\ \operatorname{div} v = 0,\ v.n = 0 \ on \Gamma\}.$$

Proof.
Let us define

$$X = \{v \in L^2(\Omega)^N,\ \operatorname{div} v \in L^2(\Omega)\}.$$

Lemma 2.2.17 *The space $C^\infty(\overline{\Omega})^N$ is dense in X and the mapping $\varphi \to \varphi.n$ defined on $C^\infty(\overline{\Omega})^N$ can be extended to a linear continuous map from X to $H^{-\frac{1}{2}}(\Gamma)$.*

Proof.
Let $v \in X$. For every $\psi \in H^{\frac{1}{2}}(\Gamma)$ and $\varphi \in H^1(\Omega)$ such that $\varphi_{/\Gamma} = \psi$ (we take a continuous extension), we define

$$\mathcal{L}(\psi) = \int_\Omega \operatorname{div} v \varphi dx + \int_\Omega v.\nabla\varphi dx.$$

If $\varphi \in H_0^1(\Omega)$, then the right hand side is 0 so that $\mathcal{L}$ depends only on ψ. Moreover we have

$$|\mathcal{L}(\psi)| \leq ||v||_X ||\varphi||_{H^1(\Omega)} \leq C||v||_X ||\psi||_{H^{\frac{1}{2}}(\Gamma)}.$$

Therefore, there exists an element of $H^{\frac{1}{2}}(\Gamma)$ which corresponds to $v.n$ for regular functions v such that

$$\forall \psi \in H^{\frac{1}{2}}(\Gamma), \quad \mathcal{L}(\psi) =< v.n, \psi >_\Gamma .$$

The continuity follows immediately.
Back to the proof of Theorem 2.2.16.
Of course, for elements of $\mathcal{V}$ this normal trace is 0 and therefore the closure of $\mathcal{V}$ is contained in H. Now if $f \in L^2(\Omega)^N$ and $< f, v >= 0$ for every $v \in \mathcal{V}$. Then we know that there exists $p \in L^2(\Omega)$ such that $f = \nabla p$. Then $p \in H^1(\Omega)$. Therefore, for every $v \in H$, we have

$$< f, v >=< \nabla p, v >= \int_\Omega p \operatorname{div} v dx+ < v.n, p >_\Gamma = 0.$$

Then $< f, v >= 0$ for every $v \in H$ and this shows that the closure of $\mathcal{V}$ contains H and the proof of Theorem 2.2.16 is complete.

2.3 Evolution Stokes equations

We consider the following evolution problem : we look for (u, p) (u will be the velocity and p the pressure) such that

$$\frac{\partial u}{\partial t} - \nu \Delta u = f - \nabla p \quad \text{in} \quad \Omega \times (0, T), \tag{2.3.20}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{2.3.21}$$

$$u = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0, T), \tag{2.3.22}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{2.3.23}$$

When Ω is a bounded open set we can use a Fourier method which will be presented below.

2.3.1 Special basis

First of all we define the space H as the closure in $L^2(\Omega)^N$ of $\mathcal{V}$ which is shown to be

(2.3.24) $$H = \{w \in L^2(\Omega)^N, \quad \operatorname{div} w = 0, \quad w.n = 0\}$$

where n is the unit outward pointing normal vector on the boundary Γ.
We go back to the stationnary Stokes problem which can be written in the form, for $f \in H$ (we denote by $(.,.)$ the scalar product in H which is the L^2 scalar product)

$$\begin{aligned} &a(u, w) = (f, w) \quad \forall w \in V, \\ &u \in V. \end{aligned}$$

As the injection $V \subset H$ is compact continuous it is immediate to see that the mapping $T : f \in H \to u \in V \to u \in H$ is linear continuous and compact. Moreover if we write $Tf = u$ and $Tg = v$, we have as $a(.,.)$ is symmetric

$$(Tf, g) = (u, g) = (g, u) = a(v, u) = a(u, v) = (f, v) = (f, Tg),$$

so that T is selfadjoint.
Therefore, (see for example [2]), there exists a decreasing sequence $(\mu_n)_n$ of strictly positive real numbers with $\mu_n \to 0$ when $n \to +\infty$ and a sequence $(w_n)_n$ of elements of V which form an orthonormal basis in H such that

$$Tw_n = \mu_n w_n.$$

Setting now $\lambda_n = \frac{1}{\mu_n}$ we have

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \quad , \quad \lambda_n \to +\infty$$

and

$$\begin{aligned} &a(w_n, w) = \lambda_n (w_n, w) \quad \forall w \in V, \\ &w_n \in V, \\ &(w_m, w_p) = \delta_{mp}. \end{aligned}$$

If we equip V with the (equivalent) scalar product $a(v, w)$, we can see that $(\frac{w_n}{\sqrt{\lambda_n}})_n$ form an orthonormal basis in V.
Let $v \in H$. Then we have

$$v = \sum_{n=1}^{+\infty} (v, w_n) w_n \quad \text{with} \quad \sum_{n=1}^{+\infty} |(v, w_n)|^2 = |v|_H^2 < +\infty.$$

If $v \in V$ we have

$$v = \sum_{n=1}^{+\infty} (v, w_n) w_n \quad \text{with} \quad \sum_{n=1}^{+\infty} \lambda_n |(v, w_n)|^2 = ||v||_V^2 < +\infty.$$

If $v \in V'$ we have

$$v = \sum_{n=1}^{+\infty} < v, w_n > w_n \quad \text{with} \quad \sum_{n=1}^{+\infty} \frac{|< v, w_n >|^2}{\lambda_n} = ||v||_{V'}^2 < +\infty.$$

2.3.2 Existence and uniqueness result

First of all, at least formally, if we multiply the Stokes evolution system by a function $w \in V$, as for the stationnary case, the pressure p disappear and we can write the problem as

$$\text{(2.3.25)} \qquad \frac{d}{dt}(u(t), w) + a(u(t), w) =< f(t), w > \quad \text{on} \quad (0, T), \quad \forall w \in V,$$

$$\text{(2.3.26)} \qquad u(0) = u_0.$$

We obtain the following result

Theorem 2.3.1 *Let us assume that* $u_0 \in H$ *and* $f \in L^2(0, T; H^{-1}(\Omega)^N)$. *Then there exists a unique solution* $u \in C([0, T]; H) \cap L^2(0, T; V)$ *with* $\frac{du}{dt} \in L^2(0, T; V')$ *of problem (2.3.25),(2.3.26). Moreover, there exists* $p \in \mathcal{D}'(]0, T[; L^2(\Omega))$ *(in fact* $p \in H^{-1}(0, T; L^2(\Omega))$*) such that* (u, p) *satisfies the Stokes evolution equation*

$$\text{(2.3.27)} \qquad \frac{\partial u}{\partial t} - \nu \Delta u = f - \nabla p \quad in \quad \Omega \times (0, T),$$

$$\text{(2.3.28)} \qquad \operatorname{div} u = 0 \quad in \quad \Omega \times (0, T),$$

$$\text{(2.3.29)} \qquad u = 0 \quad on \quad \Sigma = \Gamma \times (0, T),$$

$$\text{(2.3.30)} \qquad u(0) = u_0 \quad in \quad \Omega.$$

Proof.
we have

$$u_0 = \sum_{n=1}^{+\infty} (u_0, w_n) w_n \quad \text{with} \quad \sum_{n=1}^{+\infty} |(u_0, w_n)|^2 = |u_0|_H^2 < +\infty$$

and

$$f(t) = \sum_{n=1}^{+\infty} < f(t), w_n > w_n \quad \text{with} \quad \int_0^T \Big(\sum_{n=1}^{+\infty} \frac{|< f(t), w_n >|^2}{\lambda_n} \Big) dt = ||f||_{L^2(0,T;V')}^2 < +\infty.$$

Let us write

$$u_0^M = \sum_{n=1}^{M} (u_0, w_n) w_n$$

and

$$f^M(t) = \sum_{n=1}^{M} < f(t), w_n > w_n.$$

We know that u_0^M converges to u_0 in H and f^M converges to f in $L^2(0,T;V')$ when $M \to +\infty$. First of all, calling $V^M = span\{w_1, \cdots, w_M\}$, we look for

$$u^M(t) = \sum_{n=1}^{M} u_n(t) w_n$$

such that

$$\begin{aligned}
&\frac{d}{dt}(u^M(t), w) + a(u^M(t), w) =< f^M(t), w > \quad \text{on} \quad (0,T), \quad \forall w \in V^M, \\
&u^M(0) = u_0^M.
\end{aligned}$$

This gives us for every $n = 1, \cdots, M$

$$\begin{aligned}
&\frac{d}{dt} u_n(t) + \lambda_n u_n(t) =< f(t), w_n >, \\
&u_n(0) = (u_0, w_n),
\end{aligned}$$

and we even have an explicit formula for u_n

$$u_n(t) = (u_0, w_n) e^{-\lambda_n t} + \int_0^t < f_n(s), w_n > e^{-\lambda_n (t-s)} ds.$$

Let us show that (u^M) is a Cauchy sequence in $C([0,T;H)$ and in $L^2(0,T;V)$. For $M \geq P+1$ we have

$$(u^M - u^P)(t) = \sum_{n=P+1}^{M} u_n(t) w_n,$$

$$|(u^M - u^P)(t)|_H^2 = \sum_{n=P+1}^{M} |u_n(t)|^2,$$

$$||u^M - u^P||^2_{L^2(0,T;V)} = \int_0^T \sum_{n=P+1}^{M} \lambda_n |u_n(t)|^2 dt.$$

From the equation giving u_n we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + \lambda_n|u_n(t)|^2 =< f(t), w_n > u_n(t) \leq \frac{|< f(t), w_n >|}{\sqrt{\lambda_n}}\sqrt{\lambda_n}|u_n(t)|.$$

For $t \in [0, T]$ we now integrate on $(0, t)$ to get

$$|u_n(t)|^2 + \int_0^t \lambda_n|u_n(s)|^2 ds \leq |(u_0, w_n)|^2 + \int_0^t \frac{|< f(s), w_n >|^2}{\lambda_n} ds.$$

Summing up from $n = P + 1$ to M we immediately obtain

$$||u^M - u^P||^2_{C([0,T;H)} + ||u^M - u^P||^2_{L^2(0,T;V)} \leq 2\Big(|u_0^M - u_0^P|^2_H + ||f^M - f^P||^2_{L^2(0,T;V')}\Big).$$

Therefore, (u^M) is a Cauchy sequence in $C([0, T]; H) \cap L^2(0, T; V)$ and consequently converges to a function u in these spaces which are complete. We can also write

$$u(t) = \sum_{n=1}^{+\infty} u_n(t) w_n.$$

The regularity of $\frac{du}{dt}$ comes immediately from the equation for u_n for example.
Taking test functions w in $\cup_M V^M$ first then in its closure which is V it is now easy to see that u satisfies

$$\frac{d}{dt}(u(t), w) + a(u(t), w) =< f(t), w > \quad \text{on} \quad (0, T), \quad \forall w \in V,$$
$$u(0) = u_0.$$

Uniqueness comes immediately by taking in the above equation $w = w_n$ and seeing that $(u(t), w_n)$ satisfies the same equation as $u_n(t)$.
Interpretaton of this equation and the relation with Stokes problem are more difficult. We need to consider

$$U(t) = \int_0^t u(s)ds \quad \text{and} \quad F(t) = \int_0^t f(s)ds,$$

and to integrate our equation in time, which gives

$$(u(t) - u_0, w) + a(U(t), w) =< F(t), w >, \quad \forall w \in V.$$

This is a stationnary Stokes problem at time t. Therefore, there exists $P(t) \in L^2(\Omega)$ such that

$$-\nu\Delta U(t) + u(t) - u_0 = F(t) - \nabla P(t),$$
$$\operatorname{div} U(t) = 0.$$

As $F(\cdot) - u(\cdot) + u_0$ is continuous in time, we clearly have $P \in C([0,T]; L^2(\Omega)_{/\mathbb{R}})$. Now taking the time derivative we obtain with $p = \frac{d}{dt}P$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \nu \Delta u = f - \nabla p \quad \text{in} \quad \Omega \times (0,T), \\
&\operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0,T), \\
&u = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T), \\
&u(0) = u_0 \quad \text{in} \quad \Omega,
\end{aligned}
$$

and $p \in H^{-1}(0,T; L^2(\Omega)_{/\mathbb{R}})$.
Regularity result.
We also have the regularity result.

Theorem 2.3.2 *If $u_0 \in V$ and $f \in L^2(0,T;H)$, then the solution u of the corresponding Stokes equation satisfies*

$$u \in C([0,T],V) \cap L^2(0,T;H^2(\Omega)^N), \ \frac{\partial u}{\partial t} \in L^2(0,T;H), \tag{2.3.31}$$

$$p \in L^2(0,T;H^1(\Omega)_{/\mathbb{R}}). \tag{2.3.32}$$

Proof.
Multiplying the equation by $\frac{\partial u}{\partial t}$ we have

$$\int_\Omega |\frac{\partial u}{\partial t}(t)|^2 dx + \frac{\nu}{2}\frac{d}{dt}\int_\Omega |\nabla u(t)|^2 dx = \int_\Omega f(t)\frac{\partial u}{\partial t}(t) dx \le |f(t)|_H |\frac{\partial u}{\partial t}(t)|_H.$$

This gives an estimate on $\frac{\partial u}{\partial t}$ in $L^2(0,T;H)$ and of u in $L^\infty(0,T,V)$. Then from the equation, for almost every fixed t we can consider it as a stationnary Stokes equation with right hand side in $L^2(0,T;H)$, which gives (from the regularity result for the stationnary problem) $v \in L^2(0,T,H^2(\Omega)^N)$ and $p \in L^2(0,T;H^1(\Omega)_{/\mathbb{R}})$.

Chapter 3

Navier-Stokes equations. The evolution case.

3.1 Notations and preliminaries

All along this chapter we will assume that Ω is bounded and regular, except if it is specially mentionned. We will also take the density $\rho = 1$ in order to simplify the presentation. The Navier-Stokes equations then take the form

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u = f - \nabla p, \quad \text{in} \quad \Omega \times (0,T), \tag{3.1.1}$$

$$\operatorname{div} u = 0, \quad \text{in} \quad \Omega \times (0,T), \tag{3.1.2}$$

$$u = 0, \quad \text{on} \quad \Gamma \times (0,T), \tag{3.1.3}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{3.1.4}$$

We introduce the notations

$$a(v,w) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial v_i}{\partial x_j} . \frac{\partial w_i}{\partial x_j} dx = \int_\Omega \nabla v . \nabla w dx, \quad \forall v, w \in V, \tag{3.1.5}$$

$$b(u,v,w) = \sum_{i,j=1}^{N} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx, \quad \forall u, v, w \in V. \tag{3.1.6}$$

Lemma 3.1.1 *For $N \leq 4$, $b(.,.,.)$ is a trilinear continuous form on $H^1(\Omega)^N \times H^1(\Omega)^N \times H^1(\Omega)^N$. Moreover we have*

$$\forall u \in V,\ \forall v, w \in H^1(\Omega)^N, \quad b(u,v,w) + b(u,w,v) = 0. \tag{3.1.7}$$

In particular we have

$$\forall u \in V,\ \forall v \in H^1(\Omega)^N, \quad b(u,v,v) = 0. \tag{3.1.8}$$

Proof.
For $N \leq 4$ we have $H^1(\Omega) \subset L^4(\Omega)$ with continuous injection. Therefore, if $u, v, w \in V$ we have

$$u_j \in L^4(\Omega, \quad \frac{\partial v_i}{\partial x_j} \in L^2(\Omega), \quad w_i \in L^4(\Omega),$$

and the term

$$\int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx$$

makes perfect sense with

$$|\int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx| \leq C|u_j|_{L^4(\Omega)}||v_i||_{H^1(\Omega)}|w_i|_{L^4(\Omega)} \leq C||u_j||_{H^1(\Omega)}||v_i||_{H^1(\Omega)}||w_i||_{H^1(\Omega)}.$$

Therefore, $b(.,.,.)$ is well defined, is trilinear and

$$|b(u,v,w)| \leq C||u||_{H^1(\Omega)^N}||v||_{H^1(\Omega)^N}||w||_{H^1(\Omega)^N},$$

which says that it is continuous.
Now we have

$$b(u,v,w) + b(u,w,v) = \sum_{i,j=1}^N \int_\Omega u_j (\frac{\partial v_i}{\partial x_j} w_i + \frac{\partial w_i}{\partial x_j} v_i) dx$$

$$= \sum_{i,j=1}^N \int_\Omega u_j \frac{\partial}{\partial x_j}(v_i w_i) dx = -\sum_{i,j=1}^N \int_\Omega (\frac{\partial u_j}{\partial x_j}) v_i w_i dx = 0$$

$$= -\sum_{i=1}^N \int_\Omega (\operatorname{div} u) v_i w_i dx = 0.$$

We now give a key lemma in dimension $N = 2$ which is no longer valid in higher dimension and which makes the crucial difference between the study of Navier-Stokes equations in dimension $N = 2$ and in dimension $N = 3$.

Lemma 3.1.2 *In dimension $N = 2$, there exists a constant $C > 0$ such that*

$$\forall \varphi \in H^1(\mathbb{R}^2), \quad |\varphi|_{L^4(\mathbb{R}^2)} \leq C||\varphi||^{\frac{1}{2}}_{H^1(\mathbb{R}^2)}|\varphi|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}. \tag{3.1.9}$$

In particular we have

$$\forall \varphi \in H^1_0(\Omega), \quad |\varphi|_{L^4(\Omega)} \leq C||\varphi||^{\frac{1}{2}}_{H^1_0(\Omega)}|\varphi|^{\frac{1}{2}}_{L^2(\Omega)}. \tag{3.1.10}$$

Proof.
Of course it is sufficient to prove the above inequalities for $\varphi \in C_0^\infty(\mathbb{R}^2)$. We can write

$$\varphi^2(x_1,x_2) = \int_{-\infty}^{x_2} \frac{\partial}{\partial x_2}\varphi^2(x_1,y)dy = 2\int_{-\infty}^{x_2} \varphi(x_1,y)\frac{\partial \varphi}{\partial x_2}(x_1,y)dy$$

and then

$$\begin{aligned}|\varphi(x_1,x_2)|^2 &\le 2\Big(\int_{-\infty}^{x_2} \varphi(x_1,y)^2dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{x_2} (\frac{\partial \varphi}{\partial x_2}(x_1,y))^2dy\Big)^{\frac{1}{2}}\\ &\le 2\Big(\int_{-\infty}^{+\infty} \varphi(x_1,y)^2dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} (\frac{\partial \varphi}{\partial x_2}(x_1,y))^2dy\Big)^{\frac{1}{2}}.\end{aligned}$$

In the same way we can write (exchanging the roles of x_1 and x_2)

$$|\varphi(x_1,x_2)|^2 \le 2\Big(\int_{-\infty}^{+\infty} \varphi(y,x_2)^2dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} (\frac{\partial \varphi}{\partial x_1}(y,x_2))^2dy\Big)^{\frac{1}{2}}.$$

Multiplying these two inequalities we obtain

$$\begin{aligned}|\varphi(x_1,x_2)|^4 \le 4&\Big(\int_{-\infty}^{+\infty} \varphi(x_1,y)^2dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} (\frac{\partial \varphi}{\partial x_2}(x_1,y))^2dy\Big)^{\frac{1}{2}}\\ &.\Big(\int_{-\infty}^{+\infty} \varphi(y,x_2)^2dy\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty} (\frac{\partial \varphi}{\partial x_1}(y,x_2))^2dy\Big)^{\frac{1}{2}}\\ &\le \lambda(x_1).\mu(x_2).\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^2} |\varphi(x_1,x_2)|^4dx_1dx_2 \le 4\Big(\int_{-\infty}^{+\infty} \lambda(x_1)dx_1\Big)\Big(\int_{-\infty}^{+\infty} \mu(x_2)dx_2\Big).$$

But we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \lambda(x_1)dx_1 &\le \Big(\int_{\mathbb{R}^2} |\varphi(x_1,y)|^2dx_1dy\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2} |\frac{\partial \varphi}{\partial x_2}(x_1,y)|^2dx_1dy\Big)^{\frac{1}{2}}\\ &\le |\varphi|_{L^2(\mathbb{R}^2)}|\frac{\partial \varphi}{\partial x_2}|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

In the same way we have

$$\int_{-\infty}^{+\infty} \mu(x_2)dx_2 \le |\varphi|_{L^2(\mathbb{R}^2)}|\frac{\partial \varphi}{\partial x_1}|_{L^2(\mathbb{R}^2)}.$$

Therefore

$$|\varphi|^4_{L^4(\mathbb{R}^2)} \le 4|\varphi|_{L^2(\mathbb{R}^2)}|\frac{\partial \varphi}{\partial x_2}|_{L^2(\mathbb{R}^2)}|\varphi|_{L^2(\mathbb{R}^2)}|\frac{\partial \varphi}{\partial x_1}|_{L^2(\mathbb{R}^2)} \le 2|\varphi|^2_{L^2(\mathbb{R}^2)}||\varphi||^2_{H^1(\mathbb{R}^2)}.$$

This finishes the proof of Lemma 3.1.2.
Notations.
We will denote by $< .,. >$ the duality between V' and V. Notice that when $f \in H^{-1}(\Omega)^N$, then the mapping $w \to < f, w >_{H^{-1}(\Omega)^N, H_0^1(\Omega)^N}$ is linear continuous on V and therefore defines (in a non unique way) an element of V' that we will still denote by f. We will also denote the duality between $H^{-1}(\Omega)^N$ and $H_0^1(\Omega)^N$ by $< \cdot, \cdot >$.
We will write A the operator defined by

$$\forall u, v \in V, \quad a(u, v) =< Au, v > .$$

Then it is immediate to see that

$$A \in \mathcal{L}(V, V').$$

Now for $N \leq 4$ we can write

$$\forall u, v, w \in V, \quad b(u, v, w) =< B(u, v), w >$$

where

$$(u, v) \to B(u, v)$$

is bilinear continuous from $V \times V$ in V' and we have

$$||B(u, v)||_{V'} \leq C||u||_V ||v||_V.$$

When $u \in L^2(0, T; V)$ and $v \in L^2(0, T; V)$ it is clear that $B(u, v) \in L^1(0, T; V')$.

Lemma 3.1.3 *In dimension $N = 2$, when $u, v \in L^2(0, T; V) \cap L^\infty(0, T; H)$, then*

$$B(u, v) \in L^2(0, T; H^{-1}(\Omega)^N).$$

Proof.
We have $< B(u, v), w >= - < B(u, w), v >$ so that

$$| < B(u, v), w > | \leq C|u|_{L^4(\Omega)} |v|_{L^4(\Omega)} ||w||_{H_0^1(\Omega)^N}.$$

Then

$$||B(u, v)||_{H^{-1}(\Omega)^N} \leq C|u|_{L^4(\Omega)} |v|_{L^4(\Omega)} \leq C|u|^{\frac{1}{2}}_{L^2(\Omega)} ||u||^{\frac{1}{2}}_V |v|^{\frac{1}{2}}_{L^2(\Omega)} ||v||^{\frac{1}{2}}_V.$$

This implies

$$\int_0^T ||B(u, v)||^2_{H^{-1}(\Omega)^N} dt \leq C||u||_{L^\infty(0,T;H)} ||v||_{L^\infty(0,T;H)} ||u||_{L^2(0,T;V)} ||v||_{L^2(0,T;V)},$$

which proves the Lemma 3.1.3.

Lemma 3.1.4 *In dimension $N = 3$, when $u, v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ we have*

$$B(u,v) \in L^{\frac{4}{3}}(0,T;H^{-1}(\Omega)^N).$$

Proof.
We still have

$$||B(u,v)||_{H^{-1}(\Omega)^N} \leq C|u|_{L^4(\Omega)}|v|_{L^4(\Omega)}.$$

Then

$$\int_0^T ||B(u,v)||^{\frac{4}{3}}_{H^{-1}(\Omega)^N} dt \leq C \int_0^T |u|^{\frac{4}{3}}_{L^4(\Omega)} |v|^{\frac{4}{3}}_{L^4(\Omega)} dt.$$

But

$$\begin{aligned} |u|^4_{L^4(\Omega)} = \int_\Omega |u|^4 dx = \int_\Omega |u||u|^3 dx &\leq (\int_\Omega |u|^2 dx)^{\frac{1}{2}} (\int_\Omega |u|^6 dx)^{\frac{1}{2}} \\ &\leq |u|_{L^2(\Omega)} |u|^3_{L^6(\Omega)} \leq |u|_{L^2(\Omega)} ||u||^3_V. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^T ||B(u,v)||^{\frac{4}{3}}_{H^{-1}(\Omega)^N} dt &\leq \int_0^T |u|^{\frac{1}{3}}_{L^2(\Omega)} |v|^{\frac{1}{3}}_{L^2(\Omega)} ||u||_V ||v||_V dt \\ &\leq ||u||^{\frac{1}{3}}_{L^\infty(0,T;H)} ||v||^{\frac{1}{3}}_{L^\infty(0,T;H)} ||u||_{L^2(0,T;V)} ||v||_{L^2(0,T;V)} \end{aligned}$$

and

$$||B(u,v)||_{L^{\frac{4}{3}}(0,T;H^{-1}(\Omega)^N)} \leq C||u||^{\frac{1}{4}}_{L^\infty(0,T;H)} ||v||^{\frac{1}{4}}_{L^\infty(0,T;H)} ||u||^{\frac{3}{4}}_{L^2(0,T;V)} ||v||^{\frac{3}{4}}_{L^2(0,T;V)}.$$

It is now natural to formulate Navier-Stokes equations in the following way. We look for $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that (with $f \in L^2(0,T;V')$)

$$\frac{d}{dt}(u(t),w) + a(u(t),w) + b(u(t),u(t),w) = < f(t), w >, \quad \forall w \in V. \tag{3.1.11}$$

In order to give a sense to the initial condition we will need u to be continuous in time with values in some suitable space in order to impose

$$u(0) = u_0 \in H. \tag{3.1.12}$$

We have for the two dimensional case

Proposition 3.1.5 *In dimension $N = 2$, if $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and if u satisfies (3.1.11) with $f \in L^2(0,T;V')$, then*

$$u \in C([0,T],H), \quad \frac{du}{dt} \in L^2(0,T;V')$$

and we can impose initial condition (3.1.12) with $u_0 \in H$. These solutions will be called weak solutions of Navier-Stokes equations.
We then have

$$\frac{du}{dt} + Au + B(u,u) = f \quad in \quad L^2(0,T;V'), \tag{3.1.13}$$

$$u(0) = u_0, \tag{3.1.14}$$

and, using the results for Stokes equation, when $f \in L^2(0,T;H^{-1}(\Omega)^N)$, there exists a pressure $p \in H^{-1}(0,T;L^2(\Omega))$ such that

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u = f - \nabla p, \quad in \quad \Omega \times (0,T), \tag{3.1.15}$$

$$\operatorname{div} u = 0, \quad in \quad \Omega \times (0,T), \tag{3.1.16}$$

$$u = 0, \quad on \quad \Gamma \times (0,T), \tag{3.1.17}$$

$$u(0) = u_0. \tag{3.1.18}$$

Proof.
If $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ we have, from Lemma 3.1.3, $B(u,u) \in L^2(0,T;H^{-1}(\Omega)^N)$. We are then lead to the interpretation of Stokes problem with right hand side $f - B(u,u) \in L^2(0,T;H^{-1}(\Omega)^N)$ and the result follows immediately.

3.2 Existence and uniqueness in dimension $N = 2$

Uniqueness of weak solutions for Navier-Stokes equations in dimension $N = 3$ is one of the major open problems nowadays in mathematics. Of course we will not discuss this question here but we will give the uniqueness result in dimension $N = 2$ below.

Theorem 3.2.1 *If $f \in L^2(0,T;V')$ and $u_0 \in H$, there exists at most one solution u to the Navier-Stokes equations satisfying $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$.*

Proof.
Let us suppose that we have two solutions u^1 and u^2 in the admissible class. We then have

$$\frac{d}{dt}(u^1 - u^2) + A(u^1 - u^2) + B(u^1,u^1) - B(u^2,u^2) = 0,$$
$$(u^1 - u^2)(0) = 0.$$

Taking the scalar product with $(u^1 - u^2)$ we obtain

$$\frac{1}{2}\frac{d}{dt}|u^1 - u^2|_H^2 + \nu||u^1 - u^2||_V^2 + (B(u^1,u^1) - B(u^2,u^2), u^1 - u^2) = 0 \quad \text{a.e. in} \quad t.$$

But we have

$$(B(u^1,u^1)-B(u^2,u^2),u^1-u^2) = (B(u^1-u^2,u^1),u^1-u^2)+(B(u^2,u^1-u^2),u^1-u^2)$$

and

$$(B(u^2,u^1-u^2),u^1-u^2) = b(u^2,u^1-u^2,u^1-u^2) = 0.$$

Then

$$\begin{aligned}|(B(u^1,u^1)-B(u^2,u^2),u^1-u^2)| &\leq C|u^1-u^2|_{L^4(\Omega)}||u^1||_V|u^1-u^2|_{L^4(\Omega)}\\ &\leq C||u^1||_V|u^1-u^2|_H||u^1-u^2||_V.\end{aligned}$$

We then obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u^1-u^2|_H^2+\nu||u^1-u^2||_V^2 \leq C||u^1||_V|u^1-u^2|_H||u^1-u^2||_V\\ &\qquad\qquad \leq \nu||u^1-u^2||_V^2+\frac{C^2}{4\nu}||u^1||_V^2|u^1-u^2|_H,\\ &|(u^1-u^2)(0)|_H = 0.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u^1-u^2|_H^2 \leq \frac{C^2}{4\nu}||u^1||_V^2|u^1-u^2|_H,\\ &|(u^1-u^2)(0)|_H = 0,\end{aligned}$$

and as $u^1 \in L^2(0,T;V)$, $||u^1||_V^2 \in L^1(0,T)$.

Lemma 3.2.2 *(Gronwall inequality.) If $\varphi \geq 0$ satisfies*

$$\frac{d\varphi}{dt}(t) \leq \theta(t)\varphi(t) \quad on \quad (0,T)$$

with $\theta \in L^1(0,T)$, then we have

$$\forall t \in [0,T], \quad \varphi(t) \leq \varphi(0)e^{(\int_0^t \theta(s)ds)}.$$

Proof.
we have

$$\frac{d}{dt}\Big(\varphi(t)e^{-(\int_0^t \theta(s)ds)}\Big) \leq 0.$$

We can now use Gronwall inequality in our context and as $|(u^1-u^2)(0)|_H = 0$ we obtain

$$\forall t \in (0,T), \quad |(u^1-u^2)(t)|_H = 0.$$

This finishes the proof of the uniqueness result in dimension $N = 2$. We can now give the existence result.

Theorem 3.2.3 *If $f \in L^2(0,T,V')$ and $u_0 \in H$, there exists a (weak) solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ of the Navier-Stokes equations.*

Proof.
We will give the proof in the case of dimension $N = 2$ but the existence theorem for weak solutions is still true in dimension $N = 3$ and the proof requires the use of some interpolation spaces.
First step. The proof uses the Galerkin method with a special basis. Let us consider the special basis (w_n) of eigenfunctions of operator A as described in the previous chapter. Then (w_n) is orthonormal in H and orthogonal in V. Let us first consider the following "approximate problem" . First of all we define the finite dimensional space

$$V^m = \text{span}\{w_1, \cdots, w_m\}.$$

Then we look for a function u^m defined on $(0,T)$ with values in V^m such that almost everywhere in t and for every $j = 1, \cdots, m$

$$\frac{d}{dt}(u^m(t), w_j) + a(u^m(t), w_j) + b(u^m(t), u^m(t), w_j) =< f(t), w_j >, \tag{3.2.19}$$

$$u^m(0) = u_0^m, \tag{3.2.20}$$

where

$$u_0^m = \sum_{j=1}^m (u_0, w_j) w_j \in V^m$$

and

$$u_0^m \to u_0 \quad \text{in} \quad H \quad \text{when} \quad m \to +\infty.$$

Then

$$u^m(t) = \sum_{i=1}^m g_{im}(t) w_i,$$

and, as the (w_i) are orthonormal in H and orhtogonal in V, the system can be written as

$$\frac{dg_{jm}}{dt} + \lambda_j g_{jm} + G_j(t, g_{1m}, \cdots, g_{mm}) = 0, \ \forall j = 1, \cdots, m, \tag{3.2.21}$$

$$g_{jm}(0) = (u_0, w_j), \tag{3.2.22}$$

where

$$G_j(t, g_{1m}, \cdots, g_{mm}) = b(\sum_{i=1}^m g_{im}(t) w_i, \sum_{k=1}^m g_{km}(t) w_k, w_j) - < f(t), w_j > .$$

Using Cauchy-Lipschitz theorem, it is easy to show that this **nonlinear** system of ordinary differential equations has a unique solution locally in time and therefore on an interval of time $[0, T_m] \subset [0, T]$.

Second step. A priori estimates (I).

We will argue indifferently on the (g_{jm}) or on u^m as it gives the same thing. Multiplying equation for u^m by g_{jm} and summing up in j we get

$$\frac{1}{2}\frac{d}{dt}|u^m|_H^2 + a(u^m(t), u^m(t)) + b(u^m(t), u^m(t), u^m(t)) =< f(t), u^m(t) > .$$

As $b(u^m(t), u^m(t), u^m(t)) = 0$, this gives

$$\frac{1}{2}\frac{d}{dt}|u^m|_H^2 + \nu||u^m(t)||_V^2 \leq ||f(t)||_{V'}||u^m(t)||_V \leq \frac{\nu}{2}||u^m(t)||_V^2 + \frac{1}{2\nu}||f(t)||_{V'}^2,$$

and then

$$\frac{d}{dt}|u^m|_H^2 + \nu||u^m(t)||_V^2 \leq \frac{1}{\nu}||f(t)||_{V'}^2.$$

This implies that for every $t \in (0, T_m)$

$$|u^m(t)|_H^2 + \nu\int_0^t ||u^m(s)||_V^2 ds \leq |u_0^m|_H^2 + \frac{1}{\nu}\int_0^t ||f(s)||_{V'}^2 ds \leq |u_0^m|_H^2 + \frac{1}{\nu}\int_0^T ||f(s)||_{V'}^2 ds.$$

This gives an a priori bound on the norm $|u^m(t)|_H^2$ and therefore we can take $T_m = T$ for every m and we obtain

$$\forall t \in (0, T), \quad |u^m(t)|_H^2 \leq |u_0^m|_H^2 + \frac{1}{\nu}\int_0^T ||f(s)||_{V'}^2 ds \leq M$$

and

$$\nu\int_0^T ||u^m(s)||_V^2 ds \leq |u_0^m|_H^2 + \frac{1}{\nu}\int_0^T ||f(s)||_{V'}^2 ds \leq M.$$

This shows that (u^m) stays bounded in $L^\infty(0, T; H)$ and in $L^2(0, T, V)$. After extraction of a subsequence still denoted by (u^m), there exists $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ such that when $m \to +\infty$

$$u^m \rightharpoonup u \quad \text{in} \quad L^2(0, T; V) \text{ weakly},$$

$$u^m \rightharpoonup u \quad \text{in} \quad L^\infty(0, T, H) \text{ weak} * .$$

As we work in dimension $N = 2$, from Lemma 3.1.3 we see that $B(u^m, u^m)$ stays bounded in $L^2(0, T, V')$ and we can suppose that

$$B(u^m, u^m) \rightharpoonup g \quad \text{in} \quad L^2(0, T, V') \text{ weakly}.$$

The question is now : how to prove that $g = B(u,u)$?

Third step. A priori estimates (II).

Let us show that $\frac{du^m}{dt}$ stays bounded in $L^2(0,T,V')$. We have

$$< \frac{du^m}{dt}(t), w_j > + < Au^m(t), w_j > + < B(u^m(t), u^m(t)), w_j > = < f(t), w_j > .$$

Let us write $h_m = f - B(u^m, u^m) - Au^m$. We know from the previous estimates that h_m stays bounded in $L^2(0,T,V')$ and we have

$$< \frac{du^m}{dt}(t), w_j > = < h_m(t), w_j > .$$

Therefore

$$||\frac{du^m}{dt}||^2_{L^2(0,T,V')} = \int_0^T (\sum_{j=1}^m \frac{| < \frac{du^m}{dt}(t), w_j > |^2}{\lambda_j})dt \leq ||h_m||^2_{L^2(0,T;V')},$$

and $\frac{du^m}{dt}$ stays bounded in $L^2(0,T,V')$.

We now use the following compactness lemma.

Lemma 3.2.4 *Let V be* **compactly** *embedded in H and let (u^m) be a sequence of $L^2(0,T;V)$ such that*

- *(u^m) is bounded in $L^2(0,T;V)$.*
- *$(\frac{du^m}{dt})$ is bounded in $L^2(0,T;V')$.*

Then the sequence (u^m) is relatively compact in $L^2(0,T;H)$ and therefore, there exists a subsequence which converges (strongly) in $L^2(0,T;H)$.

Applying this lemma in our context, we see that (denoting again the subsequence by (u^m))

$$u^m \to u \quad \text{in} \quad L^2(0,T;H).$$

Let us then show that $B(u^m, u^m)$ converges to $B(u,u)$ in $L^2(0,T;V')$ weakly. It is sufficient to show that $g = B(u,u)$. This amounts to show that

$$\forall w \in L^2(0,T;V), \quad \int_0^T < B(u^m, u^m), w > dt \to \int_0^T < B(u,u), w > dt.$$

We have

$$\int_0^T < B(u^m, u^m), w > dt = \int_0^T b(u^m, u^m, w)dt = -\int_0^T b(u^m, w, u^m)dt$$
$$= -\sum_{i,j=1}^N \int_0^T \int_\Omega u_j^m \frac{\partial w_i}{\partial x_j} u_i^m dxdt.$$

As $\frac{\partial w_i}{\partial x_j}$ is a fixed function in $L^2(0,T;L^2(\Omega))$ we only have to show that for every $i,j = 1,\cdots,N$,

$$u_j^m u_i^m \to u_j u_i \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \text{ weakly.}$$

From Lemma 3.1.2, we know that (because we work in dimension $N = 2$) $L^2(0,T;V)\cap L^\infty(0,T;H) \subset L^4(0,T;L^4(\Omega))$ with continuous injection. As (u^m) is bounded in $L^2(0,T;V)\cap L^\infty(0,T;H)$ we see that (u^m) is bounded in $L^4(0,T;L^4(\Omega))$ and therefore

$$u^m \to u \quad \text{in} \quad L^4(0,T;L^4(\Omega)) \text{ weakly.}$$

Then $u_j^m u_i^m$ is bounded in $L^2(0,T;L^2(\Omega))$ and this implies that it converges weakly in this space. But we know that

$$u^m \to u \quad \text{in} \quad L^2(0,T;H) \text{ strongly}$$

so that

$$u_j^m u_i^m \to u_j u_i \quad \text{in} \quad L^1(0,T;L^1(\Omega)) \quad \text{strongly.}$$

Therefore

$$u_j^m u_i^m \rightharpoonup u_j u_i \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{weakly,}$$

and

$$B(u^m,u^m) \rightharpoonup B(u,u) \quad \text{in} \quad L^2(0,T;V') \text{ weakly.}$$

Fourth step. Passage to the limit.
Let $\varphi \in C^\infty([0,T])$ such that $\varphi(T) = 0$. After integration by parts we have

$$-\int_0^T (u^m(t),w_j)\varphi'(t)dt - (u_0^m,w_j)\varphi(0) + \int_0^T a(u^m(t),w_j)\varphi(t)dt$$
$$+\int_0^T < B(u^m(t),u^m(t)),w_j > \varphi(t)dt = \int_0^T < f(t),w_j > \varphi(t)dt.$$

For fixed j let m tend to $+\infty$. All terms have a limit and we obtain for every j

$$-\int_0^T (u(t),w_j)\varphi'(t)dt - (u_0,w_j)\varphi(0) + \int_0^T a(u(t),w_j)\varphi(t)dt$$
$$+\int_0^T < B(u(t),u(t)),w_j > \varphi(t)dt = \int_0^T < f(t),w_j > \varphi(t)dt.$$

As the (w_j) form a basis in V we then have for every $w \in V$ and every $\varphi \in C^\infty([0,T])$ with $\varphi(T) = 0$

$$-\int_0^T (u(t),w)\varphi'(t)dt - (u_0,w)\varphi(0) + \int_0^T a(u(t),w)\varphi(t)dt \tag{3.2.23}$$
$$+\int_0^T < B(u(t),u(t)),w > \varphi(t)dt = \int_0^T < f(t),w > \varphi(t)dt.$$

Now if we choose first $\varphi \in C_0^\infty(]0,T[)$ we obtain

$$\frac{d}{dt}(u(t),w) + a(u(t),w) + < B(u(t),u(t),w > = < f(t),w > \quad \text{in} \quad \mathcal{D}'(]0,T[),$$

with $f \in L^2(0,T;V')$ and $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Therefore $\frac{du}{dt} \in L^2(0,T;V')$ and $u \in C([0,T];H)$ and the above equation holds in $L^2(0,T)$.
Now we take a general $\varphi \in C^\infty([0,T])$ with $\varphi(T) = 0$ and $\varphi(0) \neq 0$. We multiply the above equation by φ, integrate on $(0,T)$ and integrate by parts and we obtain by comparison with the relation (3.2.23) that

$$\forall w \in V, \quad (u(0),w)\varphi(0) = (u_0,w)\varphi(0)$$

so that

$$u(0) = u_0.$$

Therefore u is a weak solution to the Navier-Stokes equations and Theorem 3.2.3 is proved.

3.3 Complements in dimension $N = 2$

First of all we will give a result of dependence with respect to the datas.

Theorem 3.3.1 *In dimension $N = 2$, let u^1 be the weak solution to Navier-Stokes equations corresponding to the datas u_0^1 and f^1 and u^2 be the weak solution corresponding to the datas u_0^2 and f^2. Then there exists a constant $C = C(u^2) > 0$ such that*

$$||u^1 - u^2||_{L^2(0,T;V)} + |u^1 - u^2|_{L^\infty(0,T;H)} \leq C(u^2)(|u_0^1 - u_0^2|_H + ||f^1 - f^2||_{L^2(0,T;V')}).$$

Proof.
We have by difference

$$\frac{d}{dt}(u^1 - u^2) + A(u^1 - u^2) + B(u^1,u^1) - B(u^2,u^2) = f^1 - f^2,$$
$$(u^1 - u^2)(0) = u_0^1 - u_0^2.$$

Multiplying by $u^1 - u^2$ we get

$$\frac{1}{2}\frac{d}{dt}|u^1(t) - u^2(t)|_H^2 + \nu||u^1(t) - u^2(t)||_V^2 \leq ||f^1(t) - f^2(t)||_{V'}||u^1(t) - u^2(t)||_V$$
$$+| < B(u^1,u^1)(t) - B(u^2,u^2)(t), u^1(t) - u^2(t) > |$$

As we have already seen,

$$| < B(u^1,u^1)(t) - B(u^2,u^2)(t), u^1(t) - u^2(t) > | \leq C|u^1(t) - u^2(t)|_H ||u^1(t) - u^2(t)||_V ||u^2(t)||_V.$$

From this we obtain

$$\frac{d}{dt}|u^1(t) - u^2(t)|_H^2 + \nu||u^1(t) - u^2(t)||_V^2 \leq \frac{2}{\nu}||f^1(t) - f^2(t)||_{V'}^2 + \frac{2C}{\nu}||u^2(t)||_V^2 |u^1(t) - u^2(t)|_H^2.$$

Now, as $||u^2||_V^2 \in L^1(0,T)$, using Gronwall inequality we get

$$\max_{t\in[0,T]} |u^1(t) - u^2(t)|_H^2 \leq C(u^2)(|u_0^1 - u_0^2|_H^2 + ||f^1 - f^2||_{L^2(0,T;V')}^2).$$

Then we also have

$$\nu \int_0^T ||u^1(t) - u^2(t)||_V^2 dt \leq C(u^2)(|u_0^1 - u_0^2|_H^2 + ||f^1 - f^2||_{L^2(0,T;V')}^2).$$

This finishes the proof of Theorem 3.3.1.
Next we give a regularity result again in dimension $N = 2$.

Theorem 3.3.2 *In dimension $N = 2$, if $f \in L^2(0,T;H)$ and $u_0 \in V$, then the weak solution (u,p) of Navier-Stokes equations satisfies*

$$u \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V), \quad \frac{\partial u}{\partial t} \in L^2(0,T;H), \quad p \in L^2(0,T,H^1(\Omega)_{/\mathbb{R}}).$$

Proof.
We use again the special basis (w_n) and we go back to the approximate problem set in the finite dimensional space V^m where we look for u^m such that

$$(\frac{du^m}{dt} + Au^m + B(u^m,u^m) - f, w_i) = 0, \quad i = 1,\cdots,m.$$

As $Aw_i = \lambda_i w_i$, we see that $Au^m(t) \in V^m$. So we have

$$(\frac{du^m}{dt} + Au^m + B(u^m,u^m) - f, Au^m) = 0.$$

This gives

$$\frac{\nu}{2}\frac{d}{dt}||u^m||_V^2 + |Au^m|_{L^2(\Omega)^N}^2 \leq |B(u^m,u^m) - f|_{L^2(\Omega)^N} |Au^m|_{L^2(\Omega)^N}.$$

But if $v \in H^2(\Omega)^N \cap H_0^1(\Omega)^N$ we have

$$|v_j \frac{\partial v_i}{\partial x_j}|_{L^2(\Omega)} \leq |v_j|_{L^4(\Omega)} |\frac{\partial v_i}{\partial x_j}|_{L^4(\Omega)}.$$

As $v_j \in H_0^1(\Omega)$ we have seen that

$$|v_j|_{L^4(\Omega)} \leq C|v_j|_{L^2(\Omega)}^{\frac{1}{2}} ||v_j||_{H^1(\Omega)}^{\frac{1}{2}}.$$

It can also be shown (again in dimension $N = 2$) that when $z \in H^1(\Omega)$

$$|z|_{L^4(\Omega)} \leq C(\Omega)|z|_{L^2(\Omega)}^{\frac{1}{2}} ||z||_{H^1(\Omega)}^{\frac{1}{2}}.$$

In order to see that we have to use an extension of z to $H^1(\mathbb{R}^N)$ which is continuous and also continuous from $L^2(\Omega)$ to $L^2(\mathbb{R}^N)$. We can then use this inequality for $z = \frac{\partial v_i}{\partial x_j}$. We then have

$$\begin{aligned}
&|v_j \frac{\partial v_i}{\partial x_j}|_{L^2(\Omega)} \leq |v_j|_{L^4(\Omega)} |\frac{\partial v_i}{\partial x_j}|_{L^4(\Omega)} \\
&\leq C|v_j|_{L^2(\Omega)}^{\frac{1}{2}} ||v_j||_{H^1(\Omega)}^{\frac{1}{2}} |\frac{\partial v_i}{\partial x_j}|_{L^2(\Omega)}^{\frac{1}{2}} ||\frac{\partial v_i}{\partial x_j}||_{H^1(\Omega)}^{\frac{1}{2}} \\
&\leq |v_j|_{L^2(\Omega)}^{\frac{1}{2}} ||v_j||_{H^1(\Omega)}^{\frac{1}{2}} ||v_i||_{H^1(\Omega)}^{\frac{1}{2}} ||v_i||_{H^2(\Omega)}^{\frac{1}{2}}.
\end{aligned}$$

Then

$$|B(v, v)|_{L^2(\Omega)^N} \leq C|w|_H^{\frac{1}{2}} ||w||_V |Aw|_{L^2(\Omega)^N}^{\frac{1}{2}}.$$

Using this inequality we have

$$\begin{aligned}
&\frac{\nu}{2} \frac{d}{dt} ||u^m||_V^2 + |Au^m|_{L^2(\Omega)^N}^2 \leq C|Au^m|_{L^2(\Omega)^N}^{\frac{3}{2}} |u^m|_H^{\frac{1}{2}} ||u^m||_V + |f|_{L^2(\Omega)^N} |Au^m|_{L^2(\Omega)^N} \\
&\leq \frac{1}{2}|Au^m|_{L^2(\Omega)^N}^2 + C|f|_{L^2(\Omega)^N}^2 + C|u^m|_H^2 ||u^m||_V^4.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{d}{dt} ||u^m||_V^2 + |Au^m|_{L^2(\Omega)^N}^2 \leq C|f|_{L^2(\Omega)^N}^2 + C|u^m|_H^2 ||u^m||_V^2 ||u^m||_V^2 \\
&||u^m(0)||_V \leq C.
\end{aligned}$$

We already know that $|u^m|_H^2 ||u^m||_V^2$ is bounded in $L^1(0, T)$. Then, from Gronwall inequality, we can say that u^m stays bounded in $L^\infty(0, T; V)$ and Au^m stays bounded in $L^2(0, T, L^2(\Omega)^N)$. As we already know that u^m converges to u, this implies

$$u \in L^\infty(0, T; V) \quad \text{and} \quad Au \in L^2(0, T; L^2(\Omega)^N).$$

From this we have $B(u,u) \in L^4(0,T;L^2(\Omega)^N)$. Using now the regularity result for Stokes equation with right hand side $f - B(u,u)$ we obtain

$$\frac{\partial u}{\partial t} \in L^2(0,T;H) \quad \text{and} \quad p \in L^2(0,T;H^1(\Omega)_{/\mathbb{R}}).$$

This gives Theorem 3.3.2.

Chapter 4

Stationnary Navier-Stokes equations

4.1 Existence results and some cases of uniqueness

Stationnary Navier-Stokes equations can be written as follows.

$$(4.1.1) \qquad -\nu \Delta u_i + \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} \quad \text{in} \quad \Omega,\ i = 1, \cdots, N,$$

$$(4.1.2) \qquad \operatorname{div} u = 0 \quad \text{in} \quad \Omega,$$

$$(4.1.3) \qquad u = 0 \quad \text{on} \quad \Gamma.$$

It is natural to set the problem in the following variational form. Let $f \in H^{-1}(\Omega)^N$. We look for $u \in V$ such that

$$(4.1.4) \qquad a(u, w) + b(u, u, w) = < f, w > \quad \forall w \in V,$$

$$(4.1.5) \qquad u \in V.$$

We will always here suppose that $N \leq 4$.

Remark 4.1.1 *As $N \leq 4$, for every $v \in V$ we have $B(v, v) \in V'$ (and $B(v, v) \in H^{-1}(\Omega)^N$) and*

$$||B(v, v)||_{V'} \leq C_0 ||v||_V^2$$

or

$$||B(v, v)||_{H^{-1}(\Omega)^N} \leq C_0 ||v||_V^2.$$

Let us start with the case of small datas f (or of large viscosity ν).

Theorem 4.1.2 *If $f \in H^{-1}(\Omega)^N$ and if $||f||_{H^{-1}(\Omega)^N} < \frac{\nu^2}{C_0}$, then there exists a unique solution u of Navier-Stokes equations with*

$$||u||_V \leq \frac{1}{\nu}||f||_{H^{-1}(\Omega)^N}.$$

Proof.
Let $v \in V$. Let us set

$$\tilde{a}(u,w) = a(u,w) + b(v,u,w).$$

Then $\tilde{a}(.,.)$ is a bilionear continuous form on $V \times V$ and

$$\tilde{a}(w,w) = a(w,w) + b(v,w,w) \geq \nu||w||_V^2.$$

From Lax-Milgram theorem, for any $F \in V'$ (we take $< F, w >=< f, w > \ \forall w \in V$) there exists a unique u such that

$$\begin{aligned} &\tilde{a}(u,w) =< F, w > \quad \forall w \in V, \\ &u \in V. \end{aligned}$$

Moreover we have

$$||u||_V \leq \frac{1}{\nu}||F||_{V'} \leq \frac{1}{\nu}||f||_{H^{-1}(\Omega)^N}.$$

Let us set

$$u = T(v,f).$$

The problem is then to show that $T(.,f)$ has a fixed point. We notice that for fixed f, $v \to T(v,f)$ maps the ball of V or radius $\frac{1}{\nu}||f||_{H^{-1}(\Omega)^N}$ into itself. Let us assume that $||f||_{H^{-1}(\Omega)^N} < \frac{\nu^2}{C_0}$ and let us show then that $T(.,f)$ is a strict contraction. Let us set

$$u = T(v,f) \quad \hat{u} = T(\hat{v},f).$$

We then have

$$A(u-\hat{u}) + B(v,u) - B(\hat{v},\hat{u}) = 0$$

so that

$$\begin{aligned} &\nu||u-\hat{u}||_V^2 =< B(\hat{v},\hat{u}) - B(v,u), u-\hat{u} > \\ &=< B(\hat{v}-v,u), u-\hat{u} > \leq C_0||\hat{v}-v||_V||u||_V||u-\hat{u}||_V. \end{aligned}$$

Therefore

$$||u-\hat{u}||_V \leq \frac{C_0}{\nu}||u||_V||v-\hat{v}||_V.$$

So $T(.,f)$ will be a strict contraction if $\frac{C_0||u||_V}{\nu} < 1$ but we have

$$\frac{C_0||u||_V}{\nu} \leq \frac{C_0||f||_{H^{-1}(\Omega)^N}}{\nu^2} < 1$$

and therefore $T(.,f)$ has a unique fixed point in the ball of radius $\frac{1}{\nu}||f||_{H^{-1}(\Omega)^N}$ which is a solution to the Navier-Stokes equations.
Let us now consider the general case $N \leq 4$.

Theorem 4.1.3 *For every $f \in H^{-1}(\Omega)^N$, there exists $u \in V$ solution of the Navier-Stokes equations.*
In the general case there is no uniqueness result.

Let us consider the special basis (w_n) of H and the finite dimensional space V^m already defined. We first look for $u^m \in V_m$ such that

$$a(u^m, w_j) + b(u^m, u^m, w_j) = < f, w_j >, \quad j = 1, \cdots, m.$$

In order to solve this problem we define $T_m(z^m) = v^m$ where

$$\begin{aligned} &a(v^m, w_j) + b(z^m, v^m, w_j) = < f, w_j >, \quad j = 1, \cdots, m, \\ &v^m \in V_m. \end{aligned}$$

This last problem has a unique solution and we have (as above)

$$||v^m||_V \leq \frac{||f||_{H^{-1}(\Omega)^N}}{\nu}.$$

So T_m maps this ball of V of radius $\frac{||f||_{H^{-1}(\Omega)^N}}{\nu}$ into itself and T_m is Lipschitz (see above) so it is continuous. From Brouwer fixed point theorem (we are here working in finite dimension), T_m has (at least) a fixed point which we call u^m which satisfies

$$a(u^m, w_j) + b(u^m, u^m, w_j) = < f, w_j >, \quad j = 1, \cdots, m.$$

As $b(u^m, u^m, u^m) = 0$ we see that

$$||u^m||_V \leq \frac{||f||_{H^{-1}(\Omega)^N}}{\nu}.$$

Therefore, we can extract a subsequence, still denoted (u^m) such that

$$u^m \rightharpoonup u \quad \text{in} \quad V \text{ weakly}.$$

From the compactness of the embedding $V \subset H$ we have

$$u^m \to u \quad \text{in} \quad H \text{ strongly.}$$

Now for every $v \in V$ we have

$$b(u^m, u^m, v) = -b(u^m, v, u^m) = -\sum_{i,j=1}^{N} \int_\Omega u_j^m \frac{\partial v_i}{\partial x_j} u_i^m dx.$$

We know that $u_j^m u_i^m \to u_j u_i$ in $L^1(\Omega)$ strongly and that $u_j^m u_i^m$ is bounded in $L^2(\Omega)$. Therefore we have

$$u_j^m u_i^m \rightharpoonup u_j u_i \quad \text{in} \quad L^2(\Omega) \text{ weakly}$$

and therefore, for every $v \in V$ we have

$$b(u^m, u^m, v) = -b(u^m, v, u^m) \to -b(u, v, u) = b(u, u, v).$$

Now, for fixed j we can pass to the limit in $m \to +nfty$ and obtain

$$\begin{aligned} & a(u, w_j) + b(u, u, w_j) =< f, w_j > \quad \forall j, \\ & u \in V, \end{aligned}$$

and u is a solution to Navier-Stokes equations.

4.2 Some particular flows

4.2.1 Couette flows

We consider a flow between two parallel planes $\{z = 0\}$ and $\{z = a\}$ with velocity 0 on the lower plane $\{z = 0\}$ and velocity $v_z = 0$ and $v_x = v_0$ on the plane $\{z = a\}$ and under the action of gravity.
We have the solution given by

$$v_z = 0, \quad v_x = v_x(z)$$

with

$$\nu \frac{\partial^2 v_x}{\partial z^2} = 0, \quad -\rho g = \frac{\partial p}{\partial z}$$

which gives

$$v_x = v_0 \frac{z}{a}, \quad v_z = 0, \quad \nabla_x p = 0.$$

4.2.2 Poiseuille flows

In dimension $N = 2$ we have a flow between two parallel planes $\{y = -a\}$ and $\{y = a\}$ with no-slip boundary condition under the effect of a constant pressure gradient in the variable x, $\frac{\partial p}{\partial x} = -K$.
We have the solution

$$v_x = v_x(y), \quad v_y = 0,$$

and

$$\nu \frac{\partial^2 v_x}{\partial y^2} + K = 0, \quad v_x(\pm a) = 0$$

which gives

$$v_x = \frac{K}{\nu}(-\frac{y^2}{2} + \frac{a^2}{2}).$$

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