

**An introduction to Fractional Calculus
Exercises**

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Ex.1 Prove that

$$n! = \int_0^\infty e^{-u} u^n du = \int_0^1 (-\ln t)^n dt.$$

Ex.2 Show that

$$\int_0^1 e^{-u} u^{z-1} du = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+z)n!}$$

for $z \neq -n$ with $n = 0, 1, 2, \dots$

Hint: Use the series representation of the exponential function. This result leads to the analytical continuation of the Gamma function to the left half plane $\operatorname{Re}(z) \leq 0$ excluding the points $z = 0, -1, -2, \dots$. These points turn out to be simple poles with residue $R_n = (-1)^n/n!$ with $n = 0, 1, 2, \dots$

Ex.3 Using integration by parts, show that

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re}(z) > 0.$$

where the Gamma function is defined as

$$\Gamma(z) := \int_0^\infty e^{-u} u^{z-1} du, \quad \operatorname{Re}(z) > 0.$$

This means that the Gamma function satisfies a *finite difference equation* as above. Furthermore, it is sufficient to know the Gamma function in the strip $0 < \operatorname{Re}(z) \leq 1$ in order to compute it in all the half-plane $\operatorname{Re}(z) > 0$ by iteration ahead:

$$\Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z), \quad n \in \mathbb{N}.$$

Iterating behind, $\Gamma(z) = \Gamma(z+1)/z$, we can get the analytical continuation of the Gamma function in the left half plane (except in the simple poles $z = 0, -1, -2, \dots$).

Ex.4 Introduce the substitution $st = u$ in the integral (Laplace transform) below to show the result

$$\mathfrak{L}[t^\alpha; s] := \int_0^\infty e^{-st} t^\alpha dt = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}$$

with $\operatorname{Re}(\alpha) > -1$ and $\operatorname{Re}(s) > 0$.

Ex.5 Considering $\operatorname{Re}(\nu) > 0$, $\operatorname{Re}(\mu) > 0$ and $\operatorname{Re}(z) > 0$ show the result

$$\int_0^\infty e^{-zt^\mu} t^{\nu-1} dt = \frac{1}{\mu} \frac{\Gamma(\nu/\mu)}{z^{\nu/\mu}} = \frac{1}{\nu} \frac{\Gamma(1 + \nu/\mu)}{z^{\nu/\mu}}.$$

Ex.6 Verify the following inequalities

$$x^\alpha < e^x < x^x,$$

for $x \gg 1$ and any $\alpha > 0$.

Furthermore, based on Stirling asymptotic formula

$$n! \simeq \sqrt{2\pi e}^{-n} n^{n+1/2}, n \rightarrow \infty,$$

estimate the order of magnitude of $10!$

Ex.7 The convolution between two (sufficiently well-behaved) functions $f(t)$ and $g(t)$ with $t \in \mathbb{R}$ is defined by the integral

$$f(t) \star g(t) := \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau.$$

a) Show the validity of the commutative property

$$f(t) \star g(t) = g(t) \star f(t),$$

so that we can write

$$f(t) \star g(t) \equiv \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau \equiv \int_{-\infty}^{+\infty} f(t - \tau) g(\tau) d\tau.$$

b) If both functions are *causal* (namely vanishing for $t < 0$) show

$$f(t) \star g(t) \equiv \int_0^t f(\tau) g(t - \tau) d\tau \equiv \int_0^t f(t - \tau) g(\tau) d\tau.$$

In view of their applications in the framework of Fourier transforms ($t \in \mathbb{R}$) and Laplace transforms ($t \in \mathbb{R}^+$) the two convolutions are usually referred to as *Fourier convolution* and *Laplace convolution*, respectively.

Ex.8 Prove the following result for the Laplace convolution between power-law functions

$$t^{p-1} \star t^{q-1} := \int_0^t \tau^{p-1} (t - \tau)^{q-1} d\tau = t^{p+q-1} B(p, q)$$

with $\operatorname{Re}(p) > 0$ and $\operatorname{Re}(q) > 0$. Here $B(p, q)$ denotes the Beta function

$$B(p, q) := \int_0^1 u^{p-1} (1 - u)^{q-1} du = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)}.$$

For which values of p and q such convolution integral is independent on t ? Verify the result

$$t^{-1/2} \star t^{-1/2} = \int_0^t \frac{d\tau}{\sqrt{\tau} \sqrt{t - \tau}} = \pi.$$

Ex.9 Using Ex.8 and defining the so-called *Gel'fand-Shilov function*

$$\phi_\lambda(t) := \begin{cases} \frac{t^{\lambda-1}}{\Gamma(\lambda)} & t > 0, \\ 0 & t < 0, \end{cases}$$

with $\operatorname{Re}(\lambda) > 0$, show the result for the Laplace convolution

$$\phi_\lambda(t) \star \phi_\mu(t) = \phi_{\lambda+\mu}(t).$$

Remark: This result will lead to the semi-group property of the Riemann-Liouville fractional integral.

Ex.10 Using the definition of the *ascending Pochhammer's symbols*

$$(z)_n := z(z+1)(z+2) \cdots (z+n-1), \quad n \in \mathbb{N}$$

and the *descending Pochhammer's symbols*

$$(z)_{-n} := z(z-1)(z-2) \cdots (z-n+1), \quad n \in \mathbb{N}$$

prove for $n \in \mathbb{N}$ that

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \quad (z)_{-n} = \frac{\Gamma(z+1)}{\Gamma(z-n+1)}.$$

Furthermore verify that

$$(-z)_{-n} = (-1)^n (z)_n, \quad n \in \mathbb{N},$$

both starting from the definition of Pochhammer's symbols and from their expression in terms of the Gamma function. For the last purpose, take into account the *Reflection formula*

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin \pi z}.$$

Remark 1: Pochhammer's symbols are commonly defined in the ascending way, so that when we speak about Pochhammer's symbols without attribute we mean in the ascending way. Note that descending Pochhammer's symbols are used in the definition of the binomial coefficients, see Ex.11.

Remark 2: Pochhammer's symbols are defined for all $z \in \mathbb{C}$ but care is required when the argument of the Gamma functions is a non-positive integer number for which the functions exhibit simple poles so that the corresponding reciprocal functions are vanishing.

Ex.11 Express the binomial coefficients in terms of Gamma functions

Solution: We have

$$\binom{\alpha}{n} := \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} = \frac{(\alpha)_{-n}}{n!} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1) \Gamma(n + 1)}.$$

For $\alpha = m \in \mathbb{N}$, we find the classical result

$$\binom{m}{n} = \frac{m!}{(m - n)! n!}, \quad m = 1, 2, \dots, n.$$

Ex.12 Recalling the power series representations of the Mittag-Leffler functions in one, two and three parameters with $z \in \mathbb{C}$, show that they are entire functions.

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \operatorname{Re}(\alpha) > 0; \quad (E1)$$

$$E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \beta \in \mathbb{C}; \quad (E2)$$

$$E_{\alpha, \beta}^{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\alpha n + \beta)} z^n, \quad \operatorname{Re}(\alpha) > 0, \beta \in \mathbb{C}, \gamma > 0; \quad (E3)$$

Hint: Prove that the radius of convergence of the corresponding power series is infinite by using the Stirling asymptotic formula for the (ratios between) Gamma functions.

Ex.13 Using the series representation of the Mittag-Leffler functions in Ex.12 prove

$$E_{1,2}(z) = \frac{e^z - 1}{z},$$

$$E_{2,1}(z^2) = \cosh(z), \quad E_{2,1}(-z^2) = \cos(z),$$

$$E_{2,2}(z^2) = \frac{\sinh(z)}{z}, \quad E_{2,2}(-z^2) = \frac{\sin(z)}{z}$$

Ex.14 Using the series representation of the Mittag-Leffler functions in Ex.12 prove

$$E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2 E_{2\alpha,\beta}(z^2),$$

$$E_{\alpha,\beta}(z) - E_{\alpha,\beta}(-z) = 2z E_{2\alpha,\alpha+\beta}(z^2).$$

Specific examples are

$$E_{1,1}(z) + E_{1,1}(-z) = 2 E_{2,1}(z^2) \iff e^z + e^{-z} = 2 \cosh(z),$$

$$E_{1,1}(z) - E_{1,1}(-z) = 2z E_{2,2}(z^2) \iff e^z - e^{-z} = 2 \sinh(z).$$

Ex.15 Prove the following Laplace transform pairs for the auxiliary functions of Mittag-Leffler type defined below

$$e_{\alpha}(t; \lambda) := E_{\alpha}(-\lambda t^{\alpha}) \div \frac{s^{\alpha-1}}{s^{\alpha} + \lambda} = \frac{s^{-1}}{1 + \lambda s^{-\alpha}}, \quad (LT - E1)$$

$$e_{\alpha,\beta}(t; \lambda) := t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha}) \div \frac{s^{\alpha-\beta}}{s^{\alpha} + \lambda} = \frac{s^{-\beta}}{1 + \lambda s^{-\alpha}}, \quad (LT - E2)$$

$$e_{\alpha,\beta}^{\gamma}(t; \lambda) := t^{\beta-1} E_{\alpha,\beta}^{\gamma}(-\lambda t^{\alpha}) \div \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} + \lambda)^{\gamma}} = \frac{s^{-\beta}}{(1 + \lambda s^{-\alpha})^{\gamma}}. \quad (LT - E3)$$

where we have used the sign \div for the juxtaposition of a function depending on t with its Laplace transform depending on s .

Hint: The above Laplace transform pairs can be formally obtained by transforming term by term the corresponding power series representations in t (use the result in Ex.4) and summing the resulting (geometric or binomial) series in s . This is legal because the original functions are entire of exponential type.

Solution: For convenience let us sketch the proof for (LT-E3) that contains (LT-E2) for $\delta = 1$ and (LT-E1) for $\beta = \gamma = 1$.

$$e_{\alpha,\beta}^{\gamma}(t; \lambda) \div \int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}^{\gamma}(-\lambda t^{\alpha}) dt = \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \left(\frac{\lambda}{s^{\alpha}} \right)^n.$$

On the other hand, using the binomial series and Ex.10, Ex.11, we have for $|z| < 1$:

$$(1+z)^{-\gamma} = \sum_{n=0}^{\infty} \binom{-\gamma}{n} z^n = \sum_{n=0}^{\infty} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma-n)n!} z^n = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)n!} z^n.$$

Comparison between the two above equations yields the required Laplace transform pair.

Remark: We outline that the above auxiliary functions (for restricted values of the parameters) turn out to be *completely monotonic* (CM) functions so that they enter in some types of relaxation phenomena of physical relevance. We recall that a function $f(t)$ is CM in \mathbb{R}^+ if $(-1)^n f^{(n)}(t) \geq 0$ similarly with e^{-t} . For a Bernstein theorem, more generally they are expressed in terms of a (generalized) real Laplace transform with positive measure

$$f(t) = \int_0^\infty e^{-rt} K(r) dr, \quad K(r) \geq 0.$$

Limiting to the auxiliary function in three parameters, we can prove for $\lambda > 0$

$$e_{\alpha,\beta}^\gamma(t; \lambda) := t^{\beta-1} E_{\alpha,\beta}^\gamma(-\lambda t^\alpha) \text{ CM iff } \begin{cases} 0 < \alpha, \beta, \gamma \leq 1, \\ 0 < \alpha\gamma \leq \beta. \end{cases}$$

Ex.16 With $\alpha > 0$ show that

$$t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) = -\frac{d}{dt} E_\alpha(-t^\alpha).$$

Solution: By using the series expansions of the Mittag-Leffler functions entering in both sides the proof is as follows:

$$t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + \alpha)}$$

and

$$\begin{aligned} -\frac{d}{dt} E_\alpha(-t^\alpha) &= -\frac{d}{dt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\alpha n}}{\Gamma(\alpha n + 1)} = -\sum_{n=1}^{\infty} (-1)^n \frac{\alpha n t^{\alpha n - 1}}{\Gamma(\alpha n + 1)} \\ &= -\sum_{n=1}^{\infty} (-1)^n \frac{t^{\alpha n - 1}}{\Gamma(\alpha n)}. \end{aligned}$$

Then, setting $n = m + 1$ so that $m = n - 1$, we get

$$\begin{aligned} -\frac{d}{dt} E_\alpha(-t^\alpha) &= \sum_{m=0}^{\infty} (-1)^m \frac{t^{\alpha(m+1)-1}}{\Gamma(\alpha(m+1))} = t^{\alpha-1} \sum_{m=0}^{\infty} (-1)^m \frac{t^{\alpha m}}{\Gamma(\alpha m + \alpha)} \\ &= t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha). \end{aligned}$$

The equality, however, is more easily proved by using the Laplace transform pairs of the left and right hand sides, as follows:

$$\text{L.H.S. } t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) \div \frac{1}{s^\alpha + 1},$$

and, since $E_\alpha(0) = 1$,

$$\text{R.H.S. } -\frac{d}{dt} E_\alpha(-t^\alpha) \div -s \frac{s^{\alpha-1}}{s^\alpha + 1} + 1 = \frac{1}{s^\alpha + 1}.$$

Ex.17 Using the Laplace transform (LT-E1) in Ex.12, prove for $0 < \alpha < 1$:

$$E_\alpha(-t^\alpha) = \int_0^\infty e^{-rt} K_\alpha(r) dr \simeq \begin{cases} 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cdots & t \rightarrow 0^+, \\ \frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)} \cdots & t \rightarrow +\infty, \end{cases}$$

with

$$K_\alpha(r) = \frac{1}{\pi} \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1} = \frac{1}{\pi r} \frac{\sin(\alpha\pi)}{r^\alpha + 2\cos(\alpha\pi) + r^{-\alpha}} > 0. \quad (SE.1)$$

Hint: Inverting the Laplace transform according to the Bromwich formula we need to compute only a proper integral along the branch cut on the real semi-axis because the main branch of the Laplace transform does not exhibit any other singularities in the half plane., see [Gorenflo and Mainardi (1997)]. We obtain

$$E_\alpha(-t^\alpha) = \frac{1}{2\pi i} \int_{Br} e^{st} \frac{s^{\alpha-1}}{s^\alpha + 1} ds = \int_0^\infty e^{-rt} K_\alpha(r) dr$$

where

$$K_\alpha(r) = -\frac{1}{\pi} \text{Im} \left\{ \frac{s^{\alpha-1}}{s^\alpha + 1} \Big|_{s=r e^{i\pi}} \right\}.$$

A trivial computation leads to the two equivalent expressions in (SE.1) for $K_\alpha(r)$. The algebraic function $K_\alpha(r)$ turns to be positive for $0 < r < \infty$: in fact the numerator is positive being $\sin(\alpha\pi) > 0$ for $0 < \alpha < 1$ and the denominator is, for any α not integer, always positive being $(r^\alpha - 1)^2 \geq 0$. The positivity of such function (referred to as *the spectral function*) proves that our Mittag-Leffler function is CM for $0 < \alpha < 1$. For $\alpha = 1$ we recover $E_1(-t) = \exp(-t)$ because $K_1(r) = \delta(r-1)$.

The asymptotic behaviour for $t \rightarrow 0$ is of course simply derived by the Taylor series, but it can be *formally* derived from the first few terms by inverting the power series representation of the Laplace transform around $s \rightarrow \infty$. In fact

$$\frac{s^{\alpha-1}}{s^\alpha + 1} = \frac{1}{s} \frac{1}{1 + s^{-\alpha}} \simeq \frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}} + \dots \quad s \rightarrow \infty.$$

Similarly the asymptotic behaviour for $t \rightarrow +\infty$ can be *formally* derived by inverting the power series representation of the Laplace transform as $s \rightarrow 0^+$:

$$\frac{s^{\alpha-1}}{s^\alpha + 1} \simeq \frac{1}{s^{1-\alpha}} (1 - s^\alpha + \dots) \simeq \frac{1}{s^{1-\alpha}} - \frac{1}{s^{1-2\alpha}} + \dots \quad s \rightarrow 0^+.$$

Ex.18 Using the integral representation in Ex.15 prove with $\lambda > 0$ that

$$e_\alpha(t; \lambda) := E_\alpha(-\lambda t^\alpha) = \int_0^\infty e^{-rt} K_\alpha(r; \lambda) dr,$$

where

$$K_\alpha(r; \lambda) = \lambda^{1/\alpha} K_\alpha(\lambda^{1/\alpha} r).$$

Ex.19 For sufficiently well-behaved causal functions $f(t)$ with $t \in \mathbb{R}^+$, we denote in $(0, t)$ the integral operator of order $n \in \mathbb{N}$ J^n as the n -fold repeated integral, namely

$$J^n f(t) := \int_0^t \int_0^{\tau_{n-1}} \dots \int_0^{\tau_1} f(\tau) d\tau d\tau_1 \dots d\tau_{n-1}.$$

The resulting function turns out to be the primitive of order n of $f(t)$, i.e. $f_n(t)$ that is vanishing in $t = 0^+$ along with its derivatives of order $1, \dots, n-1$. Check the Cauchy formula

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t f(\tau) (t-\tau)^{(n-1)} d\tau = \Phi_n(t) * f(t),$$

where $\Phi_n(t) := t^{n-1}/\Gamma(n)$ with $t > 0$ denotes the Gel'fand-Shilov function of order n . For complementation we define $J^0 := I$, the identity operator.

Hint: The formula can easily be checked by using known properties of the Laplace transforms because $\Phi_n(t) \div 1/s^n$

Ex.20 For sufficiently well-behaved causal functions $f(t)$ with $t \in \mathbb{R}^+$, we denote in $(0, t)$ by D^n the derivative operator of order $n \in \mathbb{N}$, namely

$$D^n f(t) := f^{(n)}(t) := \frac{d^n f(t)}{dt^n}.$$

For complementation we define $D^0 := I$, the identity operator.

Extending (*formally*) the definition of the Gel'fand-Shilov functions for negative value of the index via generalized functions of Dirac type (impulsive functions)

$$\Phi_n(t) := t^{-n-1}/\Gamma(-n) = \delta^{(n)}(t) \div s^n, \quad n = 0, 1, \dots,$$

prove that

$$D^n f(t) = \Phi_{-n}(t) * f(t); = \int_{0^-}^{t^+} f(\tau) \delta^{(n)}(t - \tau) d\tau,$$

where the Laplace convolution, intended for generalized functions of Dirac type, includes the extremes $0, t$ as *internal points*.

Hint: We recall for the Dirac type functions the known property

$$\int_a^b f(\tau) \delta^{(n)}(\tau - \tau_0) d\tau = (-1)^n f^{(n)}(\tau_0), \quad a < \tau_0 < b.$$

Since $\delta^{(n)}(\tau - \tau_0) = (1)^n \delta^{(n)}(\tau_0 - \tau)$ we have

$$\int_{0^-}^{t^+} f(\tau) \delta^{(n)}(t - \tau) d\tau = (-1)^n \int_{0^-}^{t^+} f(\tau) \delta^{(n)}(\tau - t) d\tau,$$

the required identity is (formally) proved.

Ex.21 For sufficiently well-behaved causal functions $f(t)$ with $t \in \mathbb{R}^+$, prove that

$$D^n J^n f(t) = f(t), \quad J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad n \in \mathbb{N}.$$

This proves that in general $D^n J^n = I$ and $J^n D^n \neq I$, so the derivative of order n is *the left inverse* (but not necessarily the right inverse) of the corresponding integral operator. Of course for both operators the semigroup property holds

$$D^m D^n = D^n D^m = D^{m+n}, \quad J^m J^n = J^n J^m = J^{m+n}, \quad m, n \in \mathbb{N}.$$

Hint: It is sufficient to consider the case $n = 1$ and then iterate the process by induction. In fact

$$\frac{d}{dt} \int_0^t f(\tau) d\tau = f(t), \quad \int_0^t f'(\tau) d\tau = f(t) - f(0^+).$$

Ex.22 Prove that for the power-law (integrable in \mathbb{R}^+) functions t^γ with $\gamma > -1$ we have

$$J^n t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 + n)} t^{\gamma+n}, \quad D^n t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - n)} t^{\gamma-n}, \quad \gamma > -1, t > 0.$$

Hint: To prove this is sufficient to recall from any course on calculus that

$$J^n t^\gamma = t^{\gamma+n}/(\gamma + 1)_n, \quad D^n t^\gamma = (\gamma)_{-n} t^{\gamma-n},$$

and then express Pchhammer's symbols in terms of Gamma functions according to Ex.10.

Ex.23 The Riemann-Liouville integral of order $\alpha > 0$ for a sufficiently well-behaved causal function $f(t)$ is defined as straightforward generalization of the n -fold repeated integral of Ex.19, namely

$$J^\alpha f(t) = \Phi_\alpha(t) * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha > 0.$$

In the above convolution integral, we refer to $\Phi_\alpha(t - \tau)$ as the Abelian kernel (so named after the Abel integral equations). We note that the Abelian kernel is weakly singular for $0 < \alpha < 1$.

For complementation we define $J^0 := I$ (Identity operator), *i.e.* we mean $J^0 f(t) = f(t)$.

Prove the following Laplace transform pair for the Riemann-Liouville integral

$$J^\alpha f(t) \div \frac{\tilde{f}(s)}{s^\alpha}, \quad \alpha \geq 0,$$

which is the straightforward generalization of the corresponding formula for the n -fold repeated integral.

Ex.24 Prove

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 + \alpha)} t^{\gamma+\alpha}, \quad \alpha \geq 0, \quad \gamma > -1, \quad t > 0,$$

which is the straightforward generalization of the corresponding formula for the n -fold repeated integral.

Ex.25 Prove the semi-group property for the the Riemann-Liouville integral

$$J^\alpha J^\beta = J^{\alpha+\beta}, \quad \alpha, \beta \geq 0,$$

which implies the commutative property $J^\beta J^\alpha = J^\alpha J^\beta$.

Ex.26 The Riemann-Liouville derivative of order $\alpha > 0$ for a sufficiently well-behaved causal function $f(t)$ is defined as the *the left inverse* of the corresponding fractional integral of order α , that is

$$D^\alpha J^\alpha f(t) = f(t), \quad \alpha > 0.$$

In order to have, for α not integer, representations with integrable kernels the only choice is to consider $m - 1 < \alpha \leq m$ with $m \in \mathbb{N}$ and define

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t), \quad \text{with } m - 1 < \alpha \leq m,$$

namely

$$D^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

For complementation we define $D^0 = I$.

We easily verify in view of the semigroup property of the fractional integral

$$D^\alpha J^\alpha = D^m J^{m-\alpha} J^\alpha = D^m J^m = I.$$

We note that for any non integer $\alpha > 0$ the kernel in the R-L fractional derivative is always weakly singular. In particular for $0 < \alpha < 1$ we have

$$D^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1,$$

Ex.27 Prove

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha \geq 0, \quad \gamma > -1, \quad t > 0,$$

which is the straightforward generalization of the corresponding formula for the derivative of order n .

This identity can be proved by using the definition of the fractional derivative in terms of a fractional integral, Ex.26, by recalling the results in Ex.22 and Ex.24.

Note the remarkable fact that when α is not integer ($\alpha \notin \mathbb{N}$) the fractional derivative $D^\alpha f(t)$ is not zero for the constant function $f(t) \equiv 1$. In fact we have

$$D^\alpha(1) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, \quad t > 0,$$

which identically vanishes for $\alpha \in \mathbb{N}$, due to the poles of the Gamma function in the points $0, -1, -2, \dots$.

Ex.28 Prove that the operator of the Riemann-Liouville derivative is continuous with respect to the integer order, namely, if $m - 1 < \alpha \leq m$ with $m \in \mathbb{N}$

$$\alpha \rightarrow (m - 1)^+ \quad D^\alpha f(t) \rightarrow D^{m-1} f(t) = f^{(m-1)}(t).$$

In fact, in view of the definition in Ex.26,

$$D^\alpha f(t) \rightarrow D^m J^1 f(t) = D^{m-1} f(t) = f^{(m-1)}(t).$$

Ex. 29 Prove the following statement about functions which for $t > 0$ admit the same Riemann-Liouville fractional derivative of order α

$$D^\alpha f(t) = D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^m c_j t^{\alpha-j},$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, and the coefficients c_j are arbitrary constants.

Ex.30 In addition to the Riemann-Liouville derivative we introduce the so-called *the Caputo derivative* of order $\alpha > 0$ with $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ for a sufficiently well-behaved causal function $f(t)$, denoted by ${}_*D^\alpha$, and defined by

$${}_*D^\alpha f(t) := J^{m-\alpha} D^m f(t), \quad \text{with } m - 1 < \alpha \leq m.$$

Thus it differs from the R-L derivative in that the processes of differentiation and fractional integration are interchanged, so that it explicitly reads

$${}_*D^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

In this way the Caputo derivative of a constant is zero. To distinguish the Caputo derivative from the Riemann-Liouville derivative we have decorated it with the additional apex $*$. For non-integer α the definition of the Caputo derivative requires the absolute integrability of the derivative of order m . Whenever we use the operator ${}_*D^\alpha$ we (tacitly) assume that this condition is met.

Now it is worthwhile to exhibit the relation between the two fractional derivative. It turns out

$${}_*D^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)},$$

and therefore, recalling the R-L fractional derivative of the power functions in Ex.27,

$${}_*D^\alpha f(t) = D^\alpha \left[f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!} \right].$$

We recognize that for non integer order the *Caputo fractional derivative* represents a sort of regularization in the time origin for the *Riemann-Liouville fractional derivative*. We also note that for its existence all the limiting values,

$$f^{(k)}(0^+) := \lim_{t \rightarrow 0^+} D^k f(t), \quad k = 0, 1, \dots, m-1,$$

are required to be finite. In the special case $f^{(k)}(0^+) \equiv 0$, we recover the identity between the two fractional derivatives.

Let us now restrict our attention for $0 < \alpha < 1$ (i.e. $m = 1$) where we have

$$\begin{aligned} {}_*D^\alpha f(t) &:= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau \\ &= D^\alpha f(t) - f(0^+) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = D^\alpha [f(t) - f(0^+)]. \end{aligned}$$

We limit to sketch the proof of the relationship between the two fractional derivatives for this case. It is clear that for this purpose we must starting from the R-L derivative and use the general formula

$$\frac{\partial}{\partial t} \int_{a(t)}^{b(t)} f(t, \tau) d\tau = f[t, b(t)] \frac{db}{dt} - f[t, a(t)] \frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(t, \tau)}{\partial t} d\tau.$$

However we offer an alternative (formal) proof playing with operator symbols and using the commutative property of the fractional integrals. Let us recall the following identities

$$D^\alpha := D^1 J^{1-\alpha}, \quad {}_*D^\alpha = J^{1-\alpha} D^1,$$

$$D^1 J^1 f(t) = f(t), \quad J^1 D^1 f(t) = f(t) - f(0^+).$$

Then

$$\begin{aligned} D^\alpha [f(t) - f(0^+)] &= D^\alpha J^1 D^1 f(t) = D^1 J^{1-\alpha} J^1 D^1 f(t) = D^1 J^1 J^{1-\alpha} D^1 f(t) \\ &= J^{1-\alpha} D^1 f(t) := {}_*D^\alpha f(t). \end{aligned}$$

Ex.31 Analyze the different behaviour of the R-L and Caputo fractional derivatives at the end points of the interval $(m-1, m)$ namely when the order is any positive integer.

With this exercise we complement the result in Ex.28 for the R-L derivative. We note that whereas for $\alpha \rightarrow m^-$ both derivatives reduce to D^m , due to the fact that the operator $J^0 = I$ commutes with D^m , for $\alpha \rightarrow (m-1)^+$ we have

$$\alpha \rightarrow (m-1)^+ : \begin{cases} D^\alpha f(t) \rightarrow D^m J^1 f(t) = D^{(m-1)} f(t) = f^{(m-1)}(t), \\ {}_*D^\alpha f(t) \rightarrow J^1 D^m f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+). \end{cases}$$

As a consequence, formally speaking, we can say that D^α is, with respect to its order α , an operator continuous at any positive integer, whereas ${}_*D^\alpha$ is an operator only left-continuous.

The above behaviours have induced us to keep for the Riemann-Liouville derivative the same symbolic notation as for the standard derivative of integer order, while for the Caputo derivative to decorate the corresponding symbol with subscript $*$.

Ex.32 Complement the Ex.29 analyzing the different behaviours of two functions that have the same fractional derivative in the R-L and Caputo sense. We can prove that, with $m-1 < \alpha \leq m$, and c_j arbitrary constants,

$$D^\alpha f(t) = D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^m c_j t^{\alpha-j},$$

$${}_*D^\alpha f(t) = {}_*D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^m c_j t^{m-j}.$$

On this respect, we observe that the Caputo derivative for non integer order $m-1 < \alpha < m$ satisfies the same rule as the ordinary derivative of order m .

Ex.33 Provide examples for which the fractional derivative of non integer order

$$D^\alpha D^\beta \neq D^\beta D^\alpha, \quad D^\alpha D^\beta \neq D^{\alpha+\beta}.$$

Some examples are provided in [Gorenflo and Mainardi (1997)], see below. However in the specialized literature there are conditions ensuring that

$$D^\alpha D^\beta = D^\beta D^\alpha = D^{\alpha+\beta},$$

namely when α, β and $\alpha + \beta$ are all non-integer numbers. In conclusion, in case of non-integer orders the semi-group property does not hold in general at variance with the cases for standard derivatives of integer order, as outlined in Ex.21.

To show how the *Law of Exponents* does not necessarily hold for the R-L fractional derivative, we provide two simple examples (with power functions) for which

$$\begin{cases} \text{(a)} & D^\alpha D^\beta f(t) = D^\beta D^\alpha f(t) \neq D^{\alpha+\beta} f(t), \\ \text{(b)} & D^\alpha D^\beta g(t) \neq D^\beta D^\alpha g(t) = D^{\alpha+\beta} g(t). \end{cases}$$

In the example (a) let us take $f(t) = t^{-1/2}$ and $\alpha = \beta = 1/2$. Then, using Ex.24, we get $D^{1/2} f(t) \equiv 0$, $D^{1/2} D^{1/2} f(t) \equiv 0$, but $D^{1/2+1/2} f(t) = D f(t) = -t^{-3/2}/2$.

In the example (b) let us take $g(t) = t^{1/2}$ and $\alpha = 1/2$, $\beta = 3/2$. Then, again using Ex.24, we get $D^{1/2} g(t) = \sqrt{\pi}/2$, $D^{3/2} g(t) \equiv 0$, but $D^{1/2} D^{3/2} g(t) \equiv 0$, $D^{3/2} D^{1/2} g(t) = -t^{3/2}/4$ and $D^{1/2+3/2} g(t) = D^2 g(t) = -t^{3/2}/4$.

Ex.34

We recall that under suitable conditions the Laplace transform of the m -derivative of $f(t)$ is given by

$$\begin{aligned} \mathfrak{L}\{D^m f(t); s\} &= s^m \tilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} f^{(k)}(0^+) \\ &= s^m \tilde{f}(s) - \sum_{k=0}^{m-1} s^k f^{(m-1-k)}(0^+), \end{aligned}$$

where

$$f^{(k)}(0^+) := \lim_{t \rightarrow 0^+} D^k f(t).$$

For the Caputo derivative of order α with $m - 1 < \alpha \leq m$ we have

$$\mathfrak{L}\{*_D^\alpha f(t); s\} = s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0^+),$$

$$f^{(k)}(0^+) := \lim_{t \rightarrow 0^+} D^k f(t).$$

The corresponding rule for the Riemann-Liouville derivative of order α is

$$\mathfrak{L}\{D^\alpha f(t); s\} = s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} g^{(k)}(0^+),$$

$$g^{(k)}(0^+) := \lim_{t \rightarrow 0^+} D^k g(t), \quad \text{where } g(t) := J^{(m-\alpha)} f(t).$$

Hint: Let us prove the above rules for $0 < \alpha < 1$.

For the Caputo derivative let us put $h(t) := D^1 f(t)$ so that

$$\mathfrak{L}[*_D^\alpha f(t); s] = \mathfrak{L}[J^{1-\alpha} D^1 f(t); s] = \frac{\tilde{h}(s)}{s^{1-\alpha}} = \frac{s\tilde{f}(s) - f(0^+)}{s^{1-\alpha}} = s^\alpha \tilde{f}(s) - f(0^+).$$

For the R-L derivative let us put $g(t) := J^{1-\alpha} f(t)$ so that

$$\begin{aligned} \mathfrak{L}[D^\alpha f(t); s] &= \mathfrak{L}[D^1 J^{1-\alpha} f(t); s] = s\tilde{g}(s) - g(0^+) \\ &= s \frac{\tilde{f}(s)}{s^{1-\alpha}} - g(0^+) = s^\alpha \tilde{f}(s) - J^{1-\alpha} f(0^+). \end{aligned}$$

Remark: We point out the major utility of the Caputo fractional derivative with respect to the R-L fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the corresponding rules for Laplace transformation. In fact the rule for R-L derivative requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral. However, we must note that, when all the limiting values $f^{(k)}(0^+)$ for $k = 0, 1, 2, \dots$ are finite and the order is not integer, we can prove that the corresponding $g^{(k)}(0^+)$ vanish so that the formula for R-L derivative simplifies into

$$\mathfrak{L}\{D^\alpha f(t); s\} = s^\alpha \tilde{f}(s), \quad m-1 < \alpha < m.$$

We suggest to verify this result for $0 < \alpha < 1$ by considering that $g(t) = J^{1-\alpha} f(t)$ as $t \rightarrow 0^+$ is vanishing necessarily when $f(t)$ is sufficiently well-behaved around $t = 0$ with $f(0^+)$ finite.

Ex.35 *On fractional differential equation for anomalous relaxation.* The different roles played by the R-L and Caputo derivatives and the Mittag-Leffler functions are clear when one wants to consider the corresponding fractional generalization of the first-order differential equation governing the phenomenon of (exponential) relaxation. Recalling (in non-dimensional units) the initial value problem

$$\frac{du}{dt} = -u(t), \quad t \geq 0, \quad \text{with } u(0^+) = 1,$$

whose solution is

$$u(t) = \exp(-t).$$

We consider the following generalizations involving the Caputo and R-L fractional derivatives with $\alpha \in (0, 1)$:

$$(a) \quad *_D^\alpha u(t) = -u(t), \quad t \geq 0, \quad \text{with } u(0^+) = 1,$$

and

$$(b) \quad D^\alpha u(t) = -u(t), \quad t \geq 0, \quad \text{with } \lim_{t \rightarrow 0^+} J^{1-\alpha} u(t) = 1.$$

In analogy with the standard problem we solve the problems (a) and (b) with the Laplace transform technique, using respectively the rules given for the Caputo and R-L derivative. In the case (a) we get

$$\tilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha + 1} \implies u(t) = E_\alpha(-t^\alpha),$$

whereas in the case (b) we get

$$\tilde{u}(s) = \frac{1}{s^\alpha + 1} = 1 - s \frac{s^{\alpha-1}}{s^\alpha + 1} \implies u(t) = t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) = -\frac{d}{dt} E_\alpha(-t^\alpha).$$

Ex.36 Taking $u(0^+)$ finite, prove the following chain of equivalent fractional equations for $t \geq 0$

$${}_*D^\alpha u(t) = -u(t) \implies u(t) - u(0^+) = -J^\alpha u(t) \implies \frac{du}{dt} = -D^{1-\alpha} u(t),$$

where ${}_*D^\alpha$, D^α , J^α denote the Caputo fractional derivative, the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral, respectively, with $0 < \alpha \leq 1$.

Solution: We denote

$$\begin{aligned} (1) \quad & {}_*D^\alpha u(t) = -u(t), \\ (2) \quad & u(t) - u(0^+) = -J^\alpha u(t), \\ (3) \quad & \frac{du}{dt} = -D^{1-\alpha} u(t). \end{aligned}$$

To prove (1) \implies (2) we recall ${}_*D^\alpha = J^{1-\alpha} D^1$ so that, applying to both sides of (1) the operator J^α , we get

$$J^\alpha J^{1-\alpha} D^1 u(t) = J^1 D^1 u(t) = u(t) - u(0^+).$$

To prove (2) \implies (3) we recall $D^{1-\alpha} = D^1 J^\alpha$ so that, applying to both sides of (2) the operator D^1 , we get

$$D^1 [u(t) - u(0^+)] = \frac{du}{dt} = -D^1 J^\alpha u(t) = -D^{1-\alpha} u(t).$$

The equivalence of the three fractional equations can also be proved in the Laplace domain

$$\begin{aligned} (1) \quad & s^\alpha \tilde{u}(s) - s^{\alpha-1} u(0^+) = -\tilde{u}(s) \implies \tilde{u}(s) = u(0^+) \frac{s^{\alpha-1}}{s^\alpha + 1}, \\ (2) \quad & \tilde{u}(s) - u(0^+)/s = -\tilde{u}(s)/s^\alpha \implies \tilde{u}(s) = u(0^+) \frac{s^{\alpha-1}}{s^\alpha + 1}, \\ (3) \quad & s \tilde{u}(s) - u(0^+) = -s^{1-\alpha} \tilde{u}(s) \implies \tilde{u}(s) = u(0^+) \frac{s^{\alpha-1}}{s^\alpha + 1}. \end{aligned}$$

As a matter of fact the use of the Laplace transform yields the same solution

$$u(t) = u(0^+) E_\alpha(-t^\alpha).$$

Main references

- Gorenflo, R. and Mainardi, F. (1997). Fractional calculus: integral and differential equations of fractional order, in: Carpinteri, A. and Mainardi, F. (Editors) *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien, pp. 223–276. [E-print: <http://arxiv.org/abs/0805.3823>]
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- Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London and World Scientific, Singapore.