Basque Center for Applied Mathematics
NUMERICAL APPROXIMATION OF FLUID PROBLEMS
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Hoja 1

Finite volumes: conservation laws

In this practice we briefly describe some basic concepts on conservatives schemes to solve the scalar one-dimensional conservation law

$$\partial_t u + \partial_x f(u) = 0,\ x \in \mathbb{R},\ t > 0, \tag{1}$$
$$u(x,0) = u^0(x). \tag{2}$$

We assume that f is a C^2 function, $u^0 \in L^\infty(\mathbb{R})$ and we set

$$a(u) = f'(u).$$

We consider a suitable discretization of the domain by considering a uniform spatial grid Δ with increment Δx and a time-step Δt. We set

$$\lambda = \frac{\Delta t}{\Delta x}.$$

We introduce a general 3-point explicit difference scheme of the form

$$v_j^{n+1} = H(v_{j-1}^n, v_j^n, v_{j+1}^n),\ \forall n \geq 0, j \in \mathbb{Z}, \tag{3}$$

where $H : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and v_j^n denotes an approximation of the exact solution u at the grid point $(x_j = j\Delta x, t_n = n\Delta t)$.

Now we define a particular class of difference schemes, known as conservative schemes, for which the approximations obtained converge to weak solutions of the continuous equation.

Definition 1 *The difference scheme (3) can be put in conservation form if there exists a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$H(v_{-1}, v_0, v_1) = v^0 - \lambda[g(v_{-1}, v_0) - g(v_0, v_1)]. \tag{4}$$

The function g is called the numerical flux.

If we define

$$g_{j+1/2}^n = g(v_j^n, v_{j+1}^n)$$

then, the numerical scheme (3) reads

$$v_j^{n+1} = v_j^n - \lambda(g_{j+1/2}^n - g_{j-1/2}^n). \tag{5}$$

Definition 2 *The difference scheme (5) is consistent with equation (1) if*

$$g(v, v) = f(v),\ \forall v \in \mathbb{R}. \tag{6}$$

Concerning the initial datum (2) we will consider any *suitable* discretization. A common choice is to take

$$v_{j,0} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u^0(x)\, dx, \tag{7}$$

where $x_{j+1/2} = (x_j + x_{j+1})/2$.

Finally the approximation by a conservative scheme of (1)-(2) is

$$v_j^{n+1} = v_j^n - \lambda(g_{j+1/2}^n - g_{j-1/2}^n),\ j \in \mathbb{Z},\ n \geq 0 \tag{8}$$

$$v_j^0 = v_{j,0}. \tag{9}$$

In order to study the convergence of the numerical approximations obtained with this discrete system we introduce a suitable representation of the discrete data as functions. Thus, for a given sequence (v_j^n) we introduce the piecewise constant function v_Δ defined in $(0,\infty) \times \mathbb{R}$ by

$$v_\Delta(t,x) = v_j^n,\ \ t \in [t_n, t_{n+1}),\ x \in (x_{j-1/2}, x_{j+1/2}). \tag{10}$$

Theorem 1 *(Lax-Wendroff) Assume that the difference scheme (5) is consistent with (1) and let* $v^0 = (v_{j,0})$ *be given by (7). Assume that there exists a sequence* $\Delta x \to 0$ *such that if* $\Delta t = \lambda \Delta x$ *(with* λ *constant)*

$$\|v_\Delta\|_{L^\infty((0,\infty)\times \mathbb{R})} \leq C,$$

v_Δ *converges in* $L^1_{loc}((0,\infty) \times \mathbb{R})$ *and a.e. to a function* u

Then u *is a weak solution of (1)-(2).*

The above theorem tell us that a difference scheme in conservation form which converges always converges to a weak solution.

The main questions now are:

- Find sufficient conditions to convergence.
- Find criteria which ensure that the limit is the unique entropy solution.
- Determine the order of accuracy of the difference scheme.

Some examples

The Lax-Friedrichs scheme is given by

$$\begin{cases} \frac{v_j^{n+1} - \frac{v_{j-1}^n + v_{j+1}^n}{2}}{\Delta t} + \frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{2\Delta x} = 0, & n = 0, ..., N \\ v_j^0 = v_{0,j}, \end{cases} \tag{11}$$

which can be put in conservation form with the numerical flux

$$g(u,v) = \frac{f(u) + f(v)}{2} - \frac{v-u}{2\lambda}.$$

In the linear case, $q = 1$ and this scheme is L^2-stable under the CFL condition.

The upwind scheme is given by

$$v_j^{n+1} = \begin{cases} v_j^n - \lambda(f(v_j^n) - f(v_{j-1}^n)), & \text{if } f' > 0 \\ v_j^n - \lambda(f(v_{j+1}^n) - f(v_j^n)), & \text{if } f' < 0 \end{cases}$$

In the linear case, $q = |a\lambda| = |\nu|$ and the scheme is L^2-stable under the CFL condition.

The Godunov scheme is based on the exact solution of local Riemann problems. The numerical flux is given by

$$g(u,v) = \begin{cases} \min_{w\in[u,v]} f(w), & \text{if } u \leq v \\ \max_{w\in[u,v]} f(w), & \text{if } v \leq u \end{cases}$$

In the linear case, it coincides with the upwind difference scheme.

The Murman-Roe is given by

$$g(u,v) = \frac{1}{2}(f(u) + f(v) - |a(u,v)|(v-u)),$$

The Lax-Wendroff scheme is given by

$$g(u,v) = \frac{1}{2}(f(u) + f(v) - \lambda|a(u,v)|(f(v) - f(u))),$$

Exercise 1 We consider the advection equation in a segment

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, & -1 < x < 1, \quad t > 0, \\ u(x,0) = u_0(x), & -5 < x < 5, \\ u(-5,t) = 0, & t > 0, \\ u(5,t) = 0, & t > 0, \end{cases}$$

Use different finite volume methods to approximate the solution. Consider the following particular cases for which the exact solution can be computed and compare with the numerical solution.

1. $$u_0(x) = \begin{cases} \frac{1}{2}cos(\pi x) \text{ if } x \in (-.5,.5) \\ 0 \text{ otherwise} \end{cases}$$

2. $$u_0(x) = \begin{cases} 1 \text{ if } x \in (-.5,.5) \\ 0 \text{ otherwise} \end{cases}$$

Exercise 2 *Solve the Burgers equation* $f(u) = u^2/2$ *with initial datum the step function and the previous ones.*