Semi-Markov Models for High-Frequency Finance

Enrico Scalas

November 2011

Chapter 1

Continuous-time Random Walks and Fractional Diffusion Models

1.1 Introduction

This chapter contains a rather pedantic introduction to continuous-time random walks (CTRWs) and their link to space-time fractional diffusion. The material covered below is standard and rather elementary in the analysis of stochastic processes, but applied scientists may have not been previously exposed to it, at least in this form. The basic idea is to derive a theorem relating the continuous or hydrodynamic limit of CTRWs to the space-time fractional diffusion as quickly as possible. The cost to be paid is generality, even if we give pointers to the literature helping in filling gaps. We assume that our readers are more interested in basic ideas rather than in mathematical details. Incidentally, theorem 1.3.11 shows how the Caputo time derivative naturally emerges when some known results on normal diffusion are generalized to fractional diffusion. One can go on and use an equivalent formulation in terms of Riemann-Liouville fractional derivatives, but there is no point in doing that. A similar presentation of this material, but with more emphasis on compound Poisson process and a discussion of the so-called Montroll-Weiss equation [58] can be found in a chapter of a recent collective book [70].

CTRWs are used in physics to model single particle (tracer) diffusion when the tracer time of residence in a site is much larger than the time needed to jump to another site [58, 74, 75, 57, 78, 54, 55]. CTRWs are phenomenological models and do not include microscopic theories for tracer motion. However, the reader is warned that the processes called CTRWs in the literature on physics and chemical physics are known as generalized compound Poisson processes or compound renewal processes in the mathematical literature and they have a long history. Compound Poisson processes can be used to approximate Lévy processes. Indeed, as discussed by Feller in Chapter 17 of the second volume of his book [20] any infinitely divisible distribution can be approximated by a compound Poisson distribution. The importance of these processes prompted de Finetti to devote part of his second volume in probability theory to compound Poisson processes as well [16]. In theorem 1.2.51, we will derive a distribution given by equation 1.67. When $P(n, t)$ is the distribution of the Poisson process, equation 1.67 is also known as generalized Poisson distribution and it was discussed by Feller in his 1943 paper [19]. From the modelling viewpoint, this distribution is quite versatile. In the words of Feller [19]:

> Consider independent random events for which the simple Poisson distribution may be assumed, such as: telephone calls, the occurrence of claims in an insurance company, fire accidents, sickness and the like. With each event there may be associated a random variable X. Thus, in the above examples, X may represent the length of the ensuing conversation, the sum under risk, the damage, the cost (or length) of hospitalization, respectively.

The applications to insurance problems were indeed available at the beginning of the XXth Century [44, 15]. Already in 1943, Feller also wrote [19]:

> In view of the above examples, it is not surprising that the law, or special cases of it, have been discovered, by various means and sometimes under disguised forms, by many authors.

This process of rediscovery went on also after Feller's paper; as outlined above, it is the case of physics, where X is interpreted as tracer's position. More recently, for financial applications, X is seen as the log-return for a stock [71, 48, 69, 49]. More on that will be presented in the next chapter.

1.2 The Definition of Continuous-Time Random Walks

In this section, we shall formally define CTRWs as random walks subordinated to a counting renewal process. Essentially, we need two basic ingredients:

1. the random walk X_n;
2. the counting process $N(t)$.

Let us begin with the random walk. This is a stochastic process given by a sum of independent and identically distributed (i.i.d.) random variables.

Definition 1.2.1 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y) = \mathbb{P}(Y \leq y)$, then the process*

X_n defined as

$$X_0 = 0 \quad (1.1)$$

$$X_n = \sum_{i=1}^{n} Y_i, \; n \geq 1 \quad (1.2)$$

is a random walk.

In this book, we will not deal with issues of existence of stochastic processes, however it is useful to see that the definition is not void. Let us consider the following example.

Example 1.2.2 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of Bernoullian random variables with*

$$F_Y(y) = \frac{1}{2}\theta(y) + \frac{1}{2}\theta(y-1) \quad (1.3)$$

*where $\theta(x)$ is the càdlàg (*continue à droite, limite à gauche*) version of Heaviside function. This means that, for all $i \geq 1$, one has $Y_i = 0$ with probability $1/2$ or $Y_i = 1$ with probability $1/2$. In this case, the random walk X_n is just the number of* successes *up to time step n. Its one-point distribution is given by the binomial distribution of parameters $1/2$ and n. In other words, one can write $X_n \sim \mathrm{Bin}(1/2, n)$, or, more explicitly*

$$P(k,n) = \mathbb{P}(X_n = k) = \binom{n}{k}\frac{1}{2^n}. \quad (1.4)$$

In example 1.2.2, we are able to derive the one-point distribution function $P(k,n)$ of the random walk X_n using a well-known result of elementary probability theory. Elementary probability theory is helpful in deriving a general formula for the cumulative distribution function $F_{X_n}(x) = \mathbb{P}(X_n \leq x)$ and for a generic random walk, as well.

Theorem 1.2.3 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$. Then the cumulative distribution function of the corresponding random walk X_n is given by the n-fold convolution of $F_Y(y)$, in symbols one gets*

$$F_{X_n}(x) = F_Y^{*n}(x). \quad (1.5)$$

Proof Let us use induction on n for $n \geq 2$. To see that the formula is true for $n = 2$, let us consider the random variable $X_2 = Y_1 + Y_2$. One has the following chain of equalities

$$F_{X_2}(x) = \mathbb{P}(X_2 \leq x) = \mathbb{P}(Y_1 + Y_2 \leq x) = \mathbb{E}(I_{Y_1+Y_2 \leq x}) = \mathbb{E}\left(I_{\{Y_1 \in \mathbb{R}\} \cap \{Y_2 \leq x - Y_1\}}\right) = \mathbb{E}\left(I_{Y_1 \in \mathbb{R}} I_{Y_2 \leq x - Y_1}\right) = \int_{-\infty}^{+\infty} dF_Y(y_1) \int_{-\infty}^{x-y_1} dF_Y(y_2) = \int_{-\infty}^{+\infty} dF_Y(y_1) F_Y(x - y_1) = F_Y^{*2}(x). \quad (1.6)$$

Now suppose that the formula is true for $n-1$ and let us prove that it holds also for n. The inductive hypothesis is $F_{X_{n-1}}(x) = F_Y^{*(n-1)}(x)$. Taking into account definition 1.2.1, one has $X_n = X_{n-1} + Y_n$ and X_{n-1} is independent of Y_n, therefore, as before, we can write

$$
\begin{aligned}
F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}(X_{n-1} + Y_n \leq x) = \mathbb{E}(I_{X_{n-1}+Y_n \leq x}) = \\
\mathbb{E}\left(I_{\{X_{n-1}\in\mathbb{R}\}\cap\{Y_n \leq x - X_{n-1}\}}\right) = \mathbb{E}\left(I_{X_{n-1}\in\mathbb{R}} I_{Y_n \leq x - X_{n-1}}\right) = \\
\int_{-\infty}^{+\infty} dF_Y^{*(n-1)}(u) \int_{-\infty}^{x-u} dF_Y(w) = \int_{-\infty}^{+\infty} dF_Y^{*(n-1)}(u) F_Y(x-u) = F_Y^{*n}(x).
\end{aligned}
\tag{1.7}
$$

The latter chain of equalities completes the inductive proof. $\blacksquare$

Remark 1.2.4 *In the previous proof, I_A denotes the indicator function for set A. Moreover, the following facts are used: for any event A, one has that $\mathbb{P}(A) = \mathbb{E}(I_A)$ and, for two events A, B, $I_{A\cap B} = I_A I_B$.*

Remark 1.2.5 *Note that one has $F_Y^{*1}(x) = F_Y(x)$. The meaning of $F_Y^{*0}(x)$ is more interesting. Indeed, it is possible to show that $F_Y^{*0}(x) = \theta(x)$ where $\theta(x)$ denotes the càdlàg version of the Heaviside function.*

Remark 1.2.6 *The convolution used in equation* (1.5) *is called Lebesgue-Stieltjes convolution or convolution of measures. The following corollary connects this convolution to the usual convolution of functions for absolutely continuous measures (probability measures admitting a probability density function). Some authors use the symbol $\star$ to denote the convolution of measures [20].*

Corollary 1.2.7 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$. Assume that the probability density function $f_Y(y) = dF_Y(y)/dy$ exists. Then the probability density function of the corresponding random walk X_n is given by the n-fold convolution of $f_Y(y)$, in symbols one can write*

$$f_{X_n}(x) = f_Y^{*n}(x) \tag{1.8}$$

Proof It is sufficient to derive equation (1.5) in order to get this result. Recall that if $f_Y(x)$ exists, one can write $dF_Y(x) = f_Y(x)dx$. $\blacksquare$

Remark 1.2.8 *Note that the derivative of Heaviside function, $\theta(x)$, is a generalized function [77] known as Dirac delta, $\delta(x)$.*

Theorem 1.2.3 shows that the one-point measure of random walk is the n-fold convolution of the *jump* measure. For this reason, it is useful to use Fourier transforms when dealing with sums of independent (and identically distributed) random variables. Given a random variable X, one can define its characteristic function as follows.

Definition 1.2.9 *Let X be a random variable, its characteristic function $\widehat{f}(\kappa)$ is given by*

$$\widehat{f}_X(\kappa) = \mathbb{E}\left(\mathrm{e}^{i\kappa X}\right). \tag{1.9}$$

Theorem 1.2.10 *If the random variable X has a probability density function $f_X(x)$, then its characteristic function is just the Fourier transform of the probability density function. In symbols, one has*

$$\widehat{f}_X(\kappa) = \mathcal{F}(f_X(x); \kappa) = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{i\kappa x} f_X(x) \tag{1.10}$$

Proof The proof immediately follows from the definition

$$\widehat{f}_X(\kappa) = \mathbb{E}\left(\mathrm{e}^{i\kappa X}\right) = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{i\kappa x} f_X(x), \tag{1.11}$$

an elementary result which will be very useful. $\blacksquare$

Remark 1.2.11 *The conditions on $f_X(x)$ for the existence of the Fourier transform are not too demanding. For instance, the Fourier transform may also exist for generalized functions. In the case of Dirac delta, one has*

$$\mathcal{F}(\delta(x); \kappa) = \int_{-\infty}^{+\infty} dx\, \delta(x)\mathrm{e}^{i\kappa x} = 1. \tag{1.12}$$

If Lebesgue-Stieltjes integrals are used, a generic probability density function $f_X(x)$ will be a non-negative generalized function satisfying the constraint

$$\int_{-\infty}^{+\infty} dx f_X(x) = 1, \tag{1.13}$$

and its Fourier transform will exist, as well [12]. It is possible to prove that the characteristic function has the following properties:

1. *it is continuous for every $\kappa \in \mathbb{R}$;*
2. *$\widehat{f}_X(0) = 1$;*
3. *$\widehat{f}_X(\kappa)$ is a positive semi-definite function.*

Of these three properties, only the third one needs further illustration. This is a rather technical condition. Take an arbitrary integer n and a set of real numbers $\kappa_1, \ldots, \kappa_n$, then build the matrix $a_{i,j} = \widetilde{f}(\kappa_i - \kappa_j)$. Then this matrix is positive semi-definite. A theorem due to Bochner shows that the converse is true, that is any function with the three properties above is the characteristic function of a random variable [7].

Remark 1.2.12 *The derivatives of the characteristic function in $\kappa = 0$ are related to the moments of the corresponding random variable. The reader can check that*

$$\mathbb{E}[Y] = -i \left. \frac{df_Y(\kappa)}{d\kappa} \right|_{k=0}, \tag{1.14}$$

and that

$$\mathbb{E}[Y^2] = - \left. \frac{d^2 f_Y(\kappa)}{d\kappa^2} \right|_{k=0}. \tag{1.15}$$

The following property of the Fourier transform shows why it is so important when studying the sum of independent random variables.

Theorem 1.2.13 *Let $f(x)$ and $g(x)$ be two functions with respective Fourier transforms $\widehat{f}(\kappa)$ and $\widehat{g}(\kappa)$, then the Fourier transform of their convolution $(f * g)(x)$ is the product of their Fourier transforms. In symbols, one has*

$$\mathcal{F}((f * g)(x); \kappa) = \widehat{f}(\kappa)\widehat{g}(\kappa). \tag{1.16}$$

Proof The proof of this theorem, with different levels of detail, can be found in any book on Fourier methods. See reference [12] for example. ∎

The convolution theorem for Fourier transform has an immediate consequence on the characteristic function of the random walk X_n.

Corollary 1.2.14 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$ and generalized probability distribution function denoted by $f_Y(y)$. Let X_n be the corresponding random walk. Then the characteristic function of the random walk is given by*

$$\widehat{f}_{X_n}(\kappa) = \mathbb{E}\left(\mathrm{e}^{ikX_n}\right) = [\widehat{f}_Y(\kappa)]^n. \tag{1.17}$$

Proof This statement is a direct consequence of corollary 1.2.7 and of the convolution theorem 1.16. ∎

The following example is a simple way to introduce the concept of stable random variables.

Example 1.2.15 *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with probability density function given by*

$$f_Y(y) = \frac{1}{\sqrt{4\pi}} \mathrm{e}^{-y^2/4}. \tag{1.18}$$

This is the normal (Gaussian) probability density function with expectation $\mu = 0$ and variance $\sigma^2 = 2$; in other terms, we have $Y_i \sim N(0, 2)$. Which is the characteristic function of the corresponding random walk X_n? For the Y_is, the characteristic function is

$$\widehat{f}_Y(\kappa) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} dy\, \mathrm{e}^{i\kappa y - y^2/4} = \mathrm{e}^{-k^2}. \tag{1.19}$$

Therefore, from equation (1.17)*, one gets for* X_n

$$\widehat{f}_{X_n}(\kappa) = \left[\mathrm{e}^{-k^2}\right]^n = \mathrm{e}^{-nk^2}. \tag{1.20}$$

This Fourier transform can be easily inverted to get the probability density function $p(x,n) = f_{X_n}(x)$

$$p(x,n) = f_{X_n}(x) = \frac{1}{\sqrt{4n\pi}}\,\mathrm{e}^{-x^2/4n}. \tag{1.21}$$

The reader can prove that when $Y_i \sim N(\mu, \sigma^2)$*, that is when one has*

$$f_Y(x) = \frac{1}{\sqrt{2\pi n\sigma^2}}\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2}, \tag{1.22}$$

the characteristic function for jumps is

$$\widehat{f}_Y(\kappa) = \mathrm{e}^{i\mu\kappa-\sigma^2\kappa^2/2}, \tag{1.23}$$

the characteristic function of the random walk is

$$\widehat{f}_{X_n}(\kappa) = \mathrm{e}^{in\mu\kappa-n\sigma^2\kappa^2/2}, \tag{1.24}$$

and its probability density function is

$$p(x,n) = f_{X_n}(x) = \frac{1}{\sqrt{2\pi n\sigma^2}}\,\mathrm{e}^{-(x-n\mu)^2/2n\sigma^2}. \tag{1.25}$$

By comparing equations (1.22) *and* (1.25)*, one can see that the* n*-fold convolution of a normal distribution is still a normal distribution, but with parameters rescaled by* n*. Another way of expressing this result is through the concept of stable random variable.*

Definition 1.2.16 *Let* Y_1 *and* Y_2 *be two independent and identically distributed random variables that can be seen as copies of a random variable* Y*. Then,* Y *is said to be stable or stable in the broad sense if for any constants* a *and* b*, the sum* $aY_1 + bY_2$ *is distributed as* $cY + d$ *for some constants* c *and* d*. If* $d = 0$*, then the random variable* Y *is called strictly stable [60].*

The reader might wish to check directly that normal random variables are stable, behaving as a sort of "fixed point" for the convolution. Another important property of stable random variables is that they are attractors: under suitable hypothesis, when convolutions are repeated infinitely many times, the limiting distribution is given by a stable random variable. This is essentially the content of the famous central limit theorem. We will now discuss it in its Lindenberg-Lévy version. For that, we need to define the convergence in distribution for a sequence of random variables. Let us first define the so-called weak convergence for sequences of function

Definition 1.2.17 *Let $\{F_i(x)\}_{i=1}^{\infty}$ be a sequence of cumulative probability distribution functions defined on $\mathbb{R}$. The sequence converges weakly to the cumulative probability distribution $F(x)$ and we write*

$$\lim_{i\to\infty} F_i(x) \stackrel{w}{=} F(x) \tag{1.26}$$

if one has

$$\lim_{i\to\infty} F_i(x) = F(x) \tag{1.27}$$

on all the $x \in \mathcal{C}_F$, where $\mathcal{C}_F$ is the continuity set of $F(x)$.

Once weak convergence is defined, we can use it to define convergence in distribution for random variables.

Definition 1.2.18 *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables. We say that the sequence converges in distribution to a random variable X, and we write*

$$\lim_{i\to\infty} X_i \stackrel{d}{=} X \tag{1.28}$$

if one has

$$\lim_{i\to\infty} F_{X_i}(x) = \mathbb{P}(X_i \leq x) \stackrel{w}{=} F_X(x). \tag{1.29}$$

The Lévy continuity theorem provides a necessary condition for the convergence in distribution.

Theorem 1.2.19 *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables with cumulative distribution functions $F_{X_i}(x)$. Let $\widehat{f}_{X_i}(\kappa) = \mathbb{E}(\mathrm{e}^{i\kappa X_i})$ be the corresponding caracteristic functions. Assume that there is a function $\widehat{f}(\kappa)$ such that for any $\kappa \in \mathbb{R}$ (pointwise), one has*

$$\lim_{i\to\infty} \widehat{f}_{X_i}(\kappa) = \widehat{f}(\kappa), \tag{1.30}$$

with $\widehat{f}(\kappa)$ continuous for $\kappa = 0$, $\widehat{f}(0) = 1$ and positive semi-definite. Then, there is a random variable X and a corresponding cumulative distribution function $F_X(x)$ such that $\widehat{f}(\kappa)$ is the characteristic function of X and

$$\lim_{i\to\infty} F_{X_i}(x) \stackrel{w}{=} F_X(x). \tag{1.31}$$

Proof A nice proof of this theorem is contained in a book by David Williams [80], chapter 18. ∎

Theorem 1.2.20 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables, such that their common expected value is $\mu_Y = \mathbb{E}(Y) < \infty$ and their common variance is $\sigma_Y^2 = \mathbb{E}[(Y - \mu_Y)^2] < \infty$. Let X_n be the corresponding random walk. Define the random variable $Z_n = \sqrt{n}(X_n/n - \mu_Y)/\sigma_Y$, then for $n \to \infty$, Z_n converges in distribution to a normally distributed random variable $Z \sim N(0,1)$.*

Proof As a consequence of the convolution theorem 1.2.13 and of equation (1.15), one gets for a generic $\kappa \in \mathbb{R}$

$$\widehat{f}_{Z_n}(\kappa) = \left[\widehat{f}_Y\left(\frac{\kappa}{\sqrt{n}\sigma_Y}\right)\right]^n = \left[1 - \frac{1}{2}\frac{\kappa^2}{n} + \mathrm{o}\left(\frac{\kappa^2}{n\sigma_Y^2}\right)\right]^n, \tag{1.32}$$

where the first two terms of McLaurin expansion for the characteristic function are highlighted. Therefore, one has that

$$\lim_{n\to\infty} \widehat{f}_{Z_n}(\kappa) = \mathrm{e}^{-\kappa^2/2}. \tag{1.33}$$

Equation (1.23) and the Lévy continuity theorem immediately imply that $Z_n \stackrel{d}{\to} Z$ for $n \to \infty$ with $Z \sim N(0,1)$. ∎

The central limit theorem can be generalized in different directions. In our opinion, the fastest path to understand the connection between fractional diffusion and continuous-time random walks is to use symmetric α-stable random variables [82, 34, 67, 60].

Definition 1.2.21 *A symmetric α-stable random variable Y_α has the following characteristic function*

$$\widehat{f}_{Y_\alpha}(\kappa) = \mathrm{e}^{-|\kappa|^\alpha}, \tag{1.34}$$

for $\alpha \in (0,2]$.

Remark 1.2.22 *In general, neither the cumulative distribution function $F_{Y_\alpha}(y)$ nor the probability density $f_{Y_\alpha}(y)$ can be written in terms of elementary functions. There are exceptions. As a consequence of equation (1.23), one has that $Y_2 \sim N(0,2)$. Another remarkable case is $\alpha = 1$ which coincides with the Cauchy distribution. Note that these distributions have infinite second moment for $\alpha \in (0,2)$, whereas their first moment is finite for $\alpha \in (1,2]$. In the applied literature, a scale parameter $h > 0$ is often included in the definition, and one writes $\widehat{f}_{Y_\alpha}(\kappa|h) = \mathrm{e}^{-|h\kappa|^\alpha}$. Sometimes, the scale parameter has the form $c = h^\alpha$, so that one has $\widehat{f}_{Y_\alpha}(\kappa|c) = \mathrm{e}^{-c|\kappa|^\alpha}$. It is a useful exercise to check that Y_α defined in 1.2.21 is indeed a stable random variable according to definition 1.2.16.*

Similarly to the Normal distribution, symmetric α-stable distributions are fixed points as well as attractors for the convolution. Gnedenko and Kolmogorov and Lévy proved a generalization of the central limit theorem, involving sums of independent and identically distributed random variables with infinite second moment [28, 43].

Theorem 1.2.23 *Let $\{Y\}_{i=1}^\infty$ be a sequence of i.i.d. random variables such that their characteristic function has the following behaviour in the neighborhood of $\kappa = 0$*

$$\widehat{f}_Y(\kappa) = 1 - |\kappa|^\alpha + \mathrm{o}(|\kappa|^\alpha). \tag{1.35}$$

Let X_n be the corresponding random walk and define $Z_n = X_n/n^{1/\alpha}$, then one has

$$\lim_{n\to\infty} Z_n \stackrel{d}{=} Z, \tag{1.36}$$

where Z is a symmetric α-stable distribution.

Proof Given a $\kappa \in \mathbb{R}$, one has the following chain of equalities

$$\widehat{f}_{Z_n}(\kappa) = \left[\widehat{f}_Y\left(\frac{\kappa}{n^{1/\alpha}}\right)\right]^n = \left[1 - \frac{|\kappa|^\alpha}{n} + \mathrm{o}\left(\frac{|\kappa|^\alpha}{n}\right)\right]^n, \tag{1.37}$$

so that

$$\lim_{n\to\infty} \widehat{f}_{Z_n}(\kappa) = \mathrm{e}^{-|k|^\alpha}. \tag{1.38}$$

The Lévy continuity theorem yields the thesis. ∎

A natural question is whether condition (1.35) is satisfied by some random variable. Indeed, random variables whose cumulative distribution function has a power-law behavior for $|y| \to \infty$ do satisfy (1.35). This result can be presented with different levels of sofistication (see e.g. [67]). Here, we present a simplified version; for more details, the reader can consult a paper by R. Gorenflo and E.A.A. Abdel-Rehim [29].

Theorem 1.2.24 *Let Y be a symmetric random variable and assume that its cumulative distribution function $F_Y(y)$ has the following asymptotic behavior for large y and for $\alpha \in (0, 2)$*

$$\lim_{y\to\infty} \frac{b}{\alpha}\frac{F_Y(y)}{1/y^\alpha} = 1, \tag{1.39}$$

with

$$b = \frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}, \tag{1.40}$$

then the characteristic function of Y satisfies condition 1.35.

Proof This theorem is proved in Chapter 8 of [6]. ∎

It is now possible to define the counting process $N(t)$, the second ingredient needed for the continuous-time random walk. Here, we shall only consider counting processes of renewal type.

Definition 1.2.25 *Let $\{J\}_{i=1}^\infty$ be a sequence of i.i.d. positive random variables interpreted as sojourn times between subsequent events arriving at random times. They define a renewal process whose epochs of renewal (time instants at which the events occur) are the random times $\{T\}_{n=0}^\infty$ defined by*

$$\begin{aligned} T_0 &= 0, \\ T_n &= \sum_{i=1}^n J_i. \end{aligned} \tag{1.41}$$

The name renewal process is due to the fact that at any epoch of renewal, the process starts again from the beginning.

Definition 1.2.26 *Associated to any renewal process, there is the process $N(t)$ defined as*

$$N(t) = \max\{n : T_n \leq t\} \tag{1.42}$$

counting the number of events up to time t.

Remark 1.2.27 *The counting process $N(t)$ is the Poisson process if and only if $J \sim \exp(\lambda)$, that is if and only if sojourn times are i.i.d. exponentially distributed random variables with parameter λ. Incidentally, this is the only case of Lévy and Markov counting process related to a renewal process (see Çinlar's book [10] for a proof of this statement).*

Remark 1.2.28 *We shall always assume that the counting process has càdlàg (*continue à droite et limite à gauche *i.e. right continuous with left limits) sample paths. This means that the realizations are represented by step functions. If t_k is the epoch of the k-th jump, we have $N(t_k^-) = k - 1$ and $N(t_k^+) = k$.*

Remark 1.2.29 *In equation (1.42),* max *is used instead of the more general* sup *as only processes with finite (but arbitrary) number of jumps in $(0, t]$ are considered here.*

Given the cumulative probability distribution function $F_J(t)$ for the sojourn times, one immediately gets the distribution for the corresponding epochs.

Theorem 1.2.30 *Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. sojourn times with cumulative distribution function $F_J(t)$, then one gets for the generic epoch T_n*

$$F_{T_n}(t) = F_J^{*n}(t). \tag{1.43}$$

Proof This theorem is the same as theorem 1.2.3 with T_n playing the role of X_n. The only difference is that the cumulative distribution function is non-vanishing only for positive support. ∎

Example 1.2.31 *Assume that $J \sim \exp(\lambda)$, then it can be proved by direct calculation that the epochs T_n follow the so-called Erlang distribution*

$$F_{T_n}(t) = 1 - \mathrm{e}^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}. \tag{1.44}$$

The distribution of the random variable $N(t)$ can be derived from the knowledge of $F_J(t)$ as well.

Theorem 1.2.32 *Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. sojourn times with cumulative distribution function $F_J(t)$, then one has*

$$P(n, t) = \mathbb{P}(N(t) = n) = (f_J^{*n} * \bar{F}_J)(t) = \int_0^t du f_J^{*n}(u) \bar{F}_J(t - u), \tag{1.45}$$

where $f_J(t)$ is the probability density function of sojourn times J and $\bar{F}_J(t) = 1 - F_J(t)$ is the complementary cumulative distribution function.

Proof The event $\{N(t) = n\}$ is equivalent to the event $\{T_n \leq t\} \cap \{T_{n+1} > t\} = \{T_n \leq t, T_{n+1} > t\}$. Further observe that $T_{n+1} = T_n + J_{n+1}$ and T_n and J_{n+1} are independent random variables. Now, the following chain of equalities holds true

$$\begin{aligned}
&\mathbb{P}(N(t) = n) = \mathbb{P}(T_n \leq t, T_{n+1} > t) = \mathbb{P}(T_n \leq t, J_{n+1} > t - T_n) = \\
&\quad = \mathbb{E}\left(I_{\{T_n \leq t\}} I_{\{J_{n+1} > t - T_n\}}\right) = \int_{T_n \leq t} du\, f_J^{*n}(u) \int_{J_{n+1} > t - T_n} dw\, f_J(w) = \\
&\quad = \int_0^t du\, f_J^{*n}(u) \int_{t-u}^{\infty} dw\, f_j(w) = \int_0^t du\, f_J^{*n}(u)[1 - F_J(t-u)] = \\
&\quad = \int_0^t du f_J^{*n}(u) \bar{F}_J(t-u) = (f_J^{*n} * \bar{F}_J)(t). \quad (1.46)
\end{aligned}$$

In the above derivation, I_A denotes the indicator function of event A. Moreover, we have used the fact that $\mathbb{P}(A) = \mathbb{E}(I_A)$ and that $\mathbb{P}(A \cup B) = \mathbb{E}(I_A I_B)$. Finally, the independence of T_n and J_{n+1} implies that their joint probability density function is the product of the two marginals $f_{T_n, J_{n+1}}(u, w) = f_{T_n}(u) f_{J_{n+1}}(w) = f_J^{*n}(u) f_J(w)$. ∎

For positive random variables, the Laplace transform plays the same role as the Fourier transform.

Definition 1.2.33 *Let Y be a positive random variable, then its Laplace transform is*

$$\widetilde{f}_Y(s) = \mathbb{E}(\mathrm{e}^{-sY}), \quad (1.47)$$

with $s \in \mathbb{C}$.

Theorem 1.2.34 *Let $f_Y(y)$ (with $y > 0$) denote the (generalized) probability density function of a positive random variable Y, then, for $s \in \mathbb{C}$, the Laplace transform is given by*

$$\widetilde{f}_Y(s) = \mathbb{E}(\mathrm{e}^{-sY}) = \mathcal{L}(f_Y(y); s) = \int_0^{\infty} dy\, f_Y(y) e^{-st}. \quad (1.48)$$

Proof This is an immediate consequence of the definition (and indeed it could be incorporated in the definition itself). If $f_Y(y)$ is a generalized function, equation (1.48) is often called the Laplace-Stieltjes transform of the random variable or the Laplace-Stieltjes transform of the cumulative distribution function $F_Y(y)$. ∎

Remark 1.2.35 *Let $s = \mathrm{Re}(s) + i\mathrm{Im}(s)$, then one can write*

$$\mathcal{L}(f_Y(y); s) = \widetilde{f}_Y(s) = \int_0^{\infty} dy\, f_Y(y) \mathrm{e}^{-\mathrm{Re}(s) y} \mathrm{e}^{i(-\mathrm{Im}(s))y}. \quad (1.49)$$

in other words, the Laplace transform, can be seen as the Fourier transform calculated for $\kappa = -\mathrm{Im}(s)$ for the function $g_Y(y)$ which is 0 for $y < 0$ and *equals* $f_Y(y)\mathrm{e}^{-\mathrm{Re}(s)y}$ *for* $y > 0$*. For values of* s *in which* $g_Y(y) \in L^1(\mathbb{R})$*, the Laplace transform exists. A classical reference on Laplace transforms is the book by Widder [79].*

Not surprisingly, a convolution theorem holds true also for Laplace transforms.

Theorem 1.2.36 *Let* $f(t)$ *and* $g(t)$ *be two functions with positive support and with respective Laplace transforms* $\widetilde{f}(s)$ *and* $\widetilde{g}(s)$*, then the Laplace transform of their convolution* $(f * g)(t)$ *is the product of their Laplace transforms. In symbols, one has*

$$\mathcal{L}((f * g)(t); s) = \widetilde{f}(s)\widetilde{g}(s). \tag{1.50}$$

Proof This theorem is proved in any textbook on Laplace transforms, see e.g. the book by LePage [42]. ∎

It is now possible to define the fractional Poisson process [64, 41, 72, 46] as a counting renewal process.

Definition 1.2.37 *The Mittag-Leffler renewal process is the sequence* $\{J_{\beta,i}\}_{i=1}^{\infty}$ *of positive independent and identically distributed random variables with complementary cumulative distribution function* $\bar{F}_{J_\beta}(t)$ *given by*

$$\bar{F}_{J_\beta}(t) = E_\beta(-t^\beta), \tag{1.51}$$

where $E_\beta(z)$ *is the one-parameter Mittag-Leffler function.*

Remark 1.2.38 *The one-parameter Mittag-Leffler function in* (1.51) *is a generalization of the exponential function. It is defined by the following series*

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \tag{1.52}$$

The Mittag-Leffler function coincides with the exponential function for $\beta = 1$*. The function* $E_\beta(-t^\beta)$ *is completely monotonic and it is* 1 *for* $t = 0$*. This means that it is a legitimate survival function.*

Remark 1.2.39 *The function* $E_\beta(-t^\beta)$ *is approximated by a stretched exponential for* $t \to 0$*:*

$$E_\beta(-t^\beta) \simeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \simeq \mathrm{e}^{-t^\beta/\Gamma(\beta+1)}, \;\; \text{for } 0 < t \ll 1, \tag{1.53}$$

and by a power-law for $t \to \infty$*:*

$$E_\beta(-t^\beta) \simeq \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, \;\; \text{for } t \gg 1. \tag{1.54}$$

Remark 1.2.40 *For applications, it is often convenient to include a scale parameter in the definition* (1.51)*, and one can write*

$$\bar{F}_{J_\beta}(t) = E_\beta\left(-(t/\gamma_t)^\beta\right). \qquad (1.55)$$

The scale factor can be introduced in different ways, and the reader is warned to pay attention to its definition. The assumption $\gamma_t = 1$ *made in* (1.51) *is equivalent to a change of time unit.*

Theorem 1.2.41 *The counting process* $N_\beta(t)$ *associated to the renewal process defined by equation* (1.51) *has the following distribution*

$$P_\beta(n,t) = \mathbb{P}(N_\beta(t) = n) = \frac{t^{\beta n}}{n!} E_\beta^{(n)}(-t^\beta), \qquad (1.56)$$

where $E_\beta^{(n)}(-t^\beta)$ *denotes the* n*-th derivative of* $E_\beta(z)$ *evaluated at the point* $z = -t^\beta$.

Proof The Laplace transform of $P_\beta(0,t) = \bar{F}_{J_\beta}(t) = E_\beta(-t^\beta)$ is given by [62]

$$\widetilde{P}_\beta(0,s) = \frac{s^{\beta-1}}{1+s^\beta}. \qquad (1.57)$$

Therefore, the Laplace transform of the probability density function $f_{J_\beta}(t) = -dP_\beta(0,t)/dt$ is given by (see e.g. [42] for the Laplace transform of the derivative)

$$\widetilde{f}_{J_\beta}(s) = \frac{1}{1+s^\beta}; \qquad (1.58)$$

recalling equation (1.45) and the convolution theorem for Laplace transforms 1.2.36, one immediately has

$$\widetilde{P}_\beta(n,s) = \frac{1}{(1+s^\beta)^n} \frac{s^{\beta-1}}{1+s^\beta}. \qquad (1.59)$$

Using equation (1.80) in Podlubny's book [62] for the inversion of the Laplace transform in (1.59), one gets the thesis (1.56). ∎

Remark 1.2.42 *The previous theorem was proved by Scalas* et al. *[72, 46]. Notice that* $N_1(t)$ *is the Poisson process with parameter* $\lambda = 1$*. Recently, Meerschaert* et al. *[50] proved that the fractional Poisson process* $N_\beta(t)$ *coincides with the process defined by* $N_1(D_\beta(t))$ *where* $D_\beta(t)$ *is the functional inverse of the standard* β*-stable subordinator. The latter process was also known as fractal time Poisson process. This result unifies different approaches to fractional calculus [3, 50].*

Remark 1.2.43 *For* $0 < \beta < 1$*, the fractional Poisson process is semi-Markov, but not Markovian and is not Lévy. The process* $N_\beta(t)$ *is not Markovian as the only Markovian counting process is the Poisson process [10]. It is not Lévy as its distribution is not infinitely divisible. The reader not familiar with infinite divisibility can jump to definition 1.3.4.*

It is now possible to define continuous-time random walks as compound renewal processes.

Definition 1.2.44 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative probability distribution $F_Y(y)$ and let X_n be the corresponding random walk. We give to the Y_is the meaning of jump widths for a diffusing particle. Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative probability distribution $F_J(t)$ and with the meaning of sojourn times. Let $N(t)$ be the corresponding counting process. We then define the following stochastic process*

$$X(t) = X_{N(t)} = \sum_{i=1}^{N(t)} Y_i \tag{1.60}$$

and we call it compound renewal process or continuous-time random walk (CTRW).

Remark 1.2.45 *The CTRW is a random walk X_n subordinated to a counting process, that is a random sum of independent random variables. Note that we have not said if the couple (J_i, X_i) consists of independent random variables. If this is the case, we have an uncoupled (or decoupled) CTRW. This is the simplest case in which durations are independent of jumps. This remark leads us to consider a particular class of non-Markovian stochastic processes, the so-called semi-Markov processes [10, 35, 21, 22]. In the following, the reader will find a quick-and-dirty introduction to semi-Markov processes.*

Definition 1.2.46 *A Markov renewal process is a two-component Markov chain $\{X_n, T_n\}_{n=0}^{\infty}$, where X_n, $n \geq 0$ is a Markov chain and T_n, $n \geq 0$ is the n-th epoch of a renewal process, homogeneous with respect to the second component and with transition probability given by*

$$\mathbb{P}(X_{n+1} \in A, J_{n+1} \leq t | X_0, \ldots X_n, J_1, \ldots, J_n) = \mathbb{P}(X_{n+1} \in A, J_{n+1} \leq t | X_n), \tag{1.61}$$

where $A \subset \mathbb{R}$ is a Borel set and $J_{n+1} = T_{n+1} - T_n$.

Remark 1.2.47 *We will also assume homogeneity with respect to the first component. On other words, if $X_n = x$, the probability on the right-hand side of equation (1.61) does not explicilty depend on n.*

Remark 1.2.48 *The positive function*

$$Q(x, A, t) = \mathbb{P}(X_{n+1} = y \in A, J_{n+1} \leq t | X_n = x), \tag{1.62}$$

is called semi-Markov kernel with $x \in \mathbb{R}$, $A \subset \mathbb{R}$ a Borel set, and $t \geq 0$.

Definition 1.2.49 *Let $N(t)$ denote the counting process defined as in equation (1.42), the stochastic process $X(t)$ defined as*

$$X(t) = X_{N(t)} \tag{1.63}$$

is the semi-Markov process associated to the Markov renewal process X_n, T_n, $n \geq 0$.

Theorem 1.2.50 *Compound renewal processes are semi-Markov processes with semi-Markov kernel given by*

$$Q(x, A, t) = P(x, A)F_J(t), \tag{1.64}$$

where $P(x, A)$ *is the Markov kernel (a.k.a. Markov transition function or transition probability kernel) of the random walk*

$$P(x, A) \stackrel{\text{def}}{=} \mathbb{P}(X_{n+1} = y \in A | X_n = x), \tag{1.65}$$

and $F_J(t)$ *is the cumulative probability distribution function of sojourn times. Moreover, let* $f_Y(y)$ *denote the probability density function of jumps, one has*

$$P(x, A) = \int_{A-x} f_Y(u)\, du, \tag{1.66}$$

where $A - x$ *is the set of values in* A *translated of* x *towards left.*

Proof The compound renewal process is a semi-Markov process by construction, where the couple X_n, T_n, $n \geq 0$ defining the corresponding Markov renewal process is made up of a random walk X_n, $n \geq 0$ with $X_0 = 0$ and a renewal process with epochs given by T_n, $n \geq 0$ with $T_0 = 0$. Equation (1.64) is an immediate consequence of the independence between the random walk and the renewal process. Finally, equation (1.66) is the standard Markov kernel of a random walk whose jumps are i.i.d. random variables with probability density function $f_Y(y)$. $\blacksquare$

The cumulative distribution function of an uncoupled compound renewal process can be obtained by means of purely probabilistic considerations.

Theorem 1.2.51 *Let* $\{Y\}_{i=1}^{\infty}$ *be a sequence of i.i.d. real-valued random variables with cumulative distribution function* $F_Y(y)$ *and let* $N(t)$, $t \geq 0$ *denote a counting process independent of the previous sequence and such that the number of events in the interval* $[0, t]$ *is a finite but arbitrary integer* $n = 0, 1, \ldots$. *Let* $X(t)$ *denote the corresponding compound renewal process. Then if* $P(n, t) = \mathbb{P}(N(t) = n)$, *the cumulative distribution function of* $X(t)$ *is*

$$F_{X(t)}(x, t) = \sum_{n=0}^{\infty} P(n, t) F_Y^{*n}(x), \tag{1.67}$$

where $F_Y^{*n}(x)$ *is the* n*-fold convolution of* $F_Y(y)$.

Proof Assume that $X(0) = 0$ and that, at time t, there have been $N(t)$ jumps, with $N(t)$ assuming integer values starting from 0 ($N(t) = 0$ means no jumps up to time t). Consider a realization of $N(t)$, that is suppose one has $N(t) = n$. This means that

$$X(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{i=1}^{n} Y_i = X_n. \tag{1.68}$$

In this case, one finds

$$F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq x\right) = F_Y^{*n}(x). \tag{1.69}$$

Given the independence between $N(t)$ and the Y_is, one further has that

$$\mathbb{P}(X_n \leq x, N(t) = n) = \mathbb{P}(N(t) = n)\mathbb{P}(X_n \leq x) = P(n,t)F_Y^{*n}(x). \tag{1.70}$$

The events $\{X_n \leq x, N(t) = n\}$ for $n \geq 0$ are mutually exclusive and exhaustive, and this yields

$$\{X(t) \leq x\} = \cup_{n=0}^{\infty}\{X_n \leq x, N(t) = n\}, \tag{1.71}$$

and, for any $m \neq n$,

$$\{X_m \leq x, N(t) = m\} \cap \{X_n \leq x, N(t) = n\} = \varnothing. \tag{1.72}$$

Calculating the probability of the two sides in equation (1.71) and using equation (1.70) and the axiom of infinite additivity leads to

$$\begin{aligned} F_{X(t)}(x,t) &= \mathbb{P}(X(t) \leq x) = \mathbb{P}\left(\cup_{n=0}^{\infty}\{X_n \leq x, N(t) = n\}\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X_n \leq x, N(t) = n) = \sum_{n=0}^{\infty} P(n,t)F_Y^{*n}(x). \end{aligned} \tag{1.73}$$

which is our thesis. $\blacksquare$

Remark 1.2.52 *For $n = 0$, one assumes $F_{Y_0}^{*0}(y) = \theta(y)$ where $\theta(y)$ is the Heaviside function. Note that $P(0,t)$ is nothing else but the survival function at $y = 0$ of the counting process. Therefore, equation (1.67) can be equivalently written as*

$$F_{X(t)}(x,t) = P(n,0)\,\theta(x) + \sum_{n=1}^{\infty} P(n,t)F_Y^{*n}(x), \tag{1.74}$$

where $\theta(x)$ is the càdlàg version of Heaviside step function.

Remark 1.2.53 *The series (1.67) is uniformly convergent for $x \neq 0$ and for any value of $t \in (0,\infty)$. This statement can be proved using Weierstrass M test. For $x = 0$ there is a jump in the cumulative distribution function of amplitude $P(0,t)$.*

Example 1.2.54 *As an example of compound renewal process, consider the case in which $Y_i \sim \mathcal{N}(\mu,\sigma^2)$, so that their cumulative distribution function is*

$$F_Y(y) = \Phi(y|\mu,\sigma^2) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{y-\mu}{\sqrt{2\sigma^2}}\right)\right), \tag{1.75}$$

where

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \mathrm{e}^{-u^2}\,du \tag{1.76}$$

is the error function. In this case, the convolution $F_Y^{*n}(x)$ *is given by* $\Phi(x|n\mu, n\sigma^2)$*. The sojourn times are* $J_i \sim \exp(\lambda)$ *and one finds*

$$P(n,t) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}. \tag{1.77}$$

As a consequence of theorem 1.2.51 one gets

$$F_{X(t)}(x,t) = \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\Phi(x|n\mu, n\sigma^2). \tag{1.78}$$

Equation (1.78) *can be directly used for numerical estimates of* $F_{X(t)}(x,t)$*.*

Corollary 1.2.55 *In the same hypotheses as in Theorem 1.2.51, the probability density* $f_{X(t)}(y,t)$ *of the process* $X(t)$ *is given by*

$$f_{X(t)}(x,t) = P(0,t)\,\delta(x) + \sum_{n=1}^{\infty} P(n,t) f_Y^{*n}(x), \tag{1.79}$$

where $f_Y^{*n}(x)$ *is the n-fold convolution of the probability density function* $f_Y(y) = dF_Y(y)/dy$*.*

Proof The sought probability density function is $f_{X(t)}(x,t) = dF_{X(t)}(x,t)/dy$; equation (1.79) is the formal derivative of equation (1.67). If $x \neq 0$, there is no singular term and the series converges uniformly ($f_Y^{*n}(x)$ is bounded and Weierstrass M test applies), therefore, for any x the series converges to the derivative of $F_{X(t)}(x,t)$. This is so also in the case $x = 0$ for $n \geq 1$ and the jump in $x = 0$ gives the singular term of weight $P(0,t)$ (see equation (1.74)). ∎

Among all the compound renewal processes, we will need the compound fractional Poisson process [41].

Definition 1.2.56 *Let* $\{Y_i\}_{i=1}^{\infty}$ *be a sequence of i.i.d. random variables with cumulative distribution function given by* $F_Y(y)$ *and let* $N_\beta(t)$ *be the fractional Poisson process, then the process* $X_\beta(t)$ *defined as*

$$X_\beta(t) = X_{N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_i \tag{1.80}$$

is called compound fractional Poisson process.

Remark 1.2.57 *The process* $X_1(t)$ *coincides with the compound Poisson process of parameter* $\lambda = 1$*.*

Theorem 1.2.58 *Let* $X_\beta(t)$ *be a compound fractional Poisson process, then*

1. *its cumulative distribution function* $F_{X_\beta(t)}(x,t)$ *is given by*

$$F_{X_\beta(t)}(x,t) = E_\beta(-t^\beta)\theta(x) + \sum_{n=1}^{\infty} \frac{t^{\beta n}}{n!} E_\beta^{(n)}(-t^\beta) F_Y^{*n}(x); \tag{1.81}$$

2. *its probability density* $f_{X_\beta(t)}(x,t)$ *function is given by*

$$f_{X_\beta(t)}(x,t) = E_\beta(-t^\beta)\delta(x) + \sum_{n=1}^{\infty} \frac{t^{\beta n}}{n!} E_\beta^{(n)}(-t^\beta) f_Y^{*n}(x); \tag{1.82}$$

3. *its characteristic function* $\widehat{f}_{X_\beta(t)}(\kappa,t)$ *is given by*

$$\widehat{f}_{X_\beta(t)}(\kappa,t) = E_\beta\left[t^\beta(\widehat{f}_Y(\kappa) - 1)\right]. \tag{1.83}$$

Proof The first two equations (1.81) and (1.82) are a straightforward consequence of Theorem 1.2.51, Corollary 1.2.55 and Theorem 1.2.41. Equation (1.83) is the Fourier transform of (1.82). ∎

Remark 1.2.59 *For* $0 < \beta < 1$*, the compound fractional Poisson process is neither Markovian nor Lévy (see Remark 1.2.43). However, it is a semi-Markov process as a consequence of theorem 1.2.50.*

It is now possible to discuss the relationship between CTRWs and the space-time fractional diffusion equation. This will be the subject of the next section.

1.3 Fractional Diffusion and Limit Theorems

In order to link the processes introduced in the previous section to fractional diffusion, let us first consider the following Cauchy problem.

Theorem 1.3.1 *Consider the Cauchy problem for the space-time fractional diffusion equation with* $0 < \alpha \leq 2$ *and* $0 < \beta \leq 1$

$$\begin{aligned} {}^{\mathtt{R}}D_x^\alpha u_{\alpha,\beta}(x,t) &= {}^{\mathtt{C}}D_t^\beta u_{\alpha,\beta}(x,t) \\ u_{\alpha,\beta}(x,0^+) &= \delta(x), \end{aligned} \tag{1.84}$$

then the Green function

$$u_{\alpha,\beta}(x,t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \tag{1.85}$$

where

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa\, \mathrm{e}^{-i\kappa u} E_\beta(-|\kappa|^\alpha), \tag{1.86}$$

solves the Cauchy problem [47, 72].

Proof In equation (1.84), ${}^{\mathtt{R}}D_x^\alpha$ denotes the symmetric Riesz operator [66] whose Fourier symbol is $-|\kappa|^\alpha$; more precisely, for a suitable function $f(x)$ one can write

$$\mathcal{F}({}^{\mathtt{R}}D_x^\alpha f(x); \kappa) = -|\kappa|^\alpha \widehat{f}(\kappa). \tag{1.87}$$

Moreover, ${}^{\mathtt{C}}D_t^\beta$ is the Caputo derivative, whose Laplace symbol is given by

$$\mathcal{L}({}^{\mathtt{C}}D_t^\beta g(t); s) = s^\beta \widetilde{g}(s) - s^{\beta-1} g(0^+), \tag{1.88}$$

where g is a function whose Laplace trasform exists. The application of the Laplace-Fourier transform to equation (1.84) implies that

$$-|\kappa|^\alpha \widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) = s^\beta \widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) - s^{\beta-1}, \tag{1.89}$$

so that the Laplace-Fourier transform of the sought Green function is

$$\widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) = \frac{s^{\beta-1}}{|\kappa|^\alpha + s^\beta}. \tag{1.90}$$

A comparison between (1.57) and (1.90) immediately shows that the inversion of the Laplace transform gives the Fourier transform of $u_{\alpha,\beta}(x, t)$

$$\widehat{u}_{\alpha,\beta}(\kappa, t) = E_\beta(-|\kappa|^\alpha t^\beta). \tag{1.91}$$

A further inversion of the Fourier transform leads to the thesis. ∎

Remark 1.3.2 *In the above derivation, the role of Fourier and Laplace transforms is interchangeable. One can first invert the Fourier transform and then the Laplace transform and get the same result.*

Remark 1.3.3 *The Green function $u_{\alpha,\beta}(x, t)$ is a probability density function for any $t > 0$, that is*

$$\int_{-\infty}^{+\infty} dx\, u_{\alpha,\beta}(x, t) = 1. \tag{1.92}$$

When $\alpha = 2$ and $\beta = 1$, the Riesz symmetric derivative coincides with the second derivative with respect to x: ${}^{\mathtt{R}}D_x^2 = \partial^2/\partial x^2$ and the Caputo derivative becomes the first derivative with respect to time: ${}^{\mathtt{C}}D_t^1 = \partial/\partial t$. Then, equation (1.84) defines the Cauchy problem for ordinary diffusion, with diffusion coefficient equal to 1; the corresponding Green function is

$$u_{2,1}(x, t) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-x^2/4t}. \tag{1.93}$$

Definition 1.3.4 *A random variable X is called infinitely divisible if, for any n, it can be written as the sum of n independent and identically distributed random variables. Equivalently, the corresponding cumulative distribution function is called infinitely divisible if, for any n, it can be written as $F_X(x) = F_{Y_n}^{*n}(x)$, that is there exists a random variable Y_n whose n-fold convolution gives $F_X(x)$.*

Remark 1.3.5 *Stable distributions are infinitely divisible, but the converse is not true, not every infinitely divisible distribution is stable.*

Remark 1.3.6 *The Poisson distribution and the compound Poisson distribution are infinitely divisible and it can be shown that any infinitely divisible distribution is the limit of compound Poisson distributions.*

Theorem 1.3.7 *For $0 < \alpha \le 2$, and $\beta = 1$, the characteristic function of the Green function is*

$$\widehat{u}_{\alpha,1}(\kappa, t) = \mathrm{e}^{-|\kappa|^{\alpha} t} \tag{1.94}$$

and it is infinitely divisible.

Proof For $\beta = 1$, the Mittag-Leffler function coincides with the exponential function and equation (1.91) yields equation (1.94). In order to show that the random variable $U_{\alpha,1}(t)$ is infinitely divisible for every t, one has to show that its cumulative distribution function is the n-fold convolution of n identical distribution functions. But then, it is enough to choose a random variable whose characteristic function is given by

$$[\mathrm{e}^{-|\kappa|^{\alpha} t}]^{1/n} = \mathrm{e}^{-|\kappa|^{\alpha} t/n}, \tag{1.95}$$

to prove infinite divisibility of $U_{\alpha,1}(t)$. ∎

Remark 1.3.8 *It can be proved that to every infinitely divisible distribution there corresponds a unique càdlàg extension of a Lévy process. In the case under scrutiny, $u_{2,1}(x, t)$ corresponds to the Wiener process whose increments follow the $N(0, 2t)$ distribution. The density $u_{\alpha,1}(x, t)$ for $0 < \alpha < 2$ corresponds to processes (called Lévy flights in the applied literature) that generalize the Wiener process and whose increments follow the symmetric α-stable distribution. For more information on infinitely divisible distribution, Lévy processes and related pseudo-differential operators the reader can consult the following references [20, 4, 68, 76, 32, 2].*

A simple way to understand the connection between CTRWs and space-time fractional diffusion is to consider the following stochastic process.

Definition 1.3.9 *Let $\{Y_{\alpha,i}\}_{i=1}^{\infty}$ be a sequence of i.i.d. symmetric α-stable distributions whose characteristic function is given by*

$$\widehat{f}_{Y_\alpha}(\kappa) = \mathrm{e}^{-|k|^{\alpha}} \tag{1.96}$$

and let $X_{\alpha,n}$ be the corresponding random walk. The following compound fractional Poisson process

$$X_{\alpha,\beta}(t) = X_{\alpha,N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_{\alpha,i} \tag{1.97}$$

is the fractional compound Poisson process with symmetric α-stable jumps.

Corollary 1.3.10 *The characteristic function of the fractional compound Poisson process with symmetric α-stable jumps is given by*

$$\widehat{f}_{X_{\alpha,\beta}(t)}(\kappa) = E_\beta \left[t^\beta \left(\mathrm{e}^{-|\kappa|^\alpha} - 1 \right) \right]. \quad (1.98)$$

Proof This result is an immediate consequence of equations (1.83) and (1.96). ∎

If properly rescaled, the random variable $X_{\alpha,\beta}(t)$ can be made to converge weakly to $U_{\alpha,\beta}(t)$, the random variable whose distribution is characterized by the probability density function (1.85) that solves the Cauchy problem for the space-time fractional diffusion equation (1.84). The trick is to build a sequence of random variables whose characteristic function converges to (1.91). Indeed, we can prove the following theorem.

Theorem 1.3.11 *Let $X_{\alpha,\beta}(t)$ be a compound fractional Poisson process with symmetric α-stable jumps and let h and r be two scaling factors such that*

$$X_{\alpha,n}(h) = hY_{\alpha,1} + \ldots + hY_{\alpha,n} \quad (1.99)$$

$$T_{\beta,n}(r) = rJ_{\beta,1} + \ldots + rJ_{\beta,n}, \quad (1.100)$$

and

$$\lim_{h,r\to 0} \frac{h^\alpha}{r^\beta} = 1, \quad (1.101)$$

with $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. Given the assumption on the jumps $Y_{\alpha,i}$, for $h \to 0$, one has

$$\widehat{f}_{Y_\alpha}(h\kappa) = 1 - h^\alpha |\kappa|^\alpha + \mathrm{o}(h^\alpha |\kappa|^\alpha), \quad (1.102)$$

then, for $h, r \to 0$ with $h^\alpha / r^\beta \to 1$, $f_{hX_{\alpha,\beta}(rt)}(x,t)$ weakly converges to $u_{\alpha,\beta}(x,t)$, the Green function of the fractional diffusion equation.

Proof In order to prove weak convergence, it suffices to show the convergence of the characteristic function (1.83) as a consequence of the Lévy continuity theorem 1.2.19. Indeed, one can write

$$\widehat{f}_{hX_{\alpha,\beta}(rt)}(\kappa, t) = E_\beta \left[-\frac{t^\beta}{r^\beta} \left(\mathrm{e}^{-h^\alpha|\kappa|^\alpha} - 1 \right) \right] \overset{h,r\to 0}{\to} E_\beta(-t^\beta |\kappa|^\alpha), \quad (1.103)$$

which completes the proof and establishes the connection between CTRWs and the space-time fractional diffusion equation. ∎

Remark 1.3.12 *This result can be generalized to compound fractional Poisson processes with heavy tails both in the jump and in the sojourn time distributions. A more general proof can be found in [72]. The relationship between fractional diffusion and CTRWs is discussed in several physics papers with different levels of detail [23, 65, 14, 66]. Hilfer and Anton realized the important role played by the Mittag-Leffler function in this derivation and rigorously discussed the link between fractional diffusion and CTRWs [30].*

Remark 1.3.13 *Up to now, we have focused on the (weak) convergence of random variables and not of stochastic processes. The convergence of processes is delicate as one must use techniques related to functional central limit theorems in appropriate functions spaces [33]. Let* $L_\alpha(t)$ *denote the symmetric* α*-stable Lévy process. Equation* (1.91) *is the characteristic function of the process* $U_{\alpha,\beta}(t) = L_\alpha(D_\beta(t))$*, that is of the symmetric* α*-stable Lévy process subordinated to the inverse* β*-stable subordinator,* $D_\beta(t)$*, the functional inverse of the* β*-stable subordinator. This remark leads to conjecture that* $U_{\alpha,\beta}(t)$ *is the functional limit of* $X_{\alpha,\beta}(t)$*, the compound fractional Poisson process with* α*-stable jumps. This conjecture can be found in a paper by Magdziarz and Weron [45] and is proved in Meerschaert* et al. *[50] using the methods discussed by Meerschaert and Scheffler [52].*

Chapter 2

Applications of Continuous-Time Random Walks to Finance and Economics

2.1 Introduction

In financial markets, when one considers tick-by-tick trades, not only price fluctuations, but also waiting times between consecutive trades vary at random. This fact is pictorially represented in figure 2.1. In this figure, the value of the FTSE MIB Index is plotted for trades occurring on February 3rd, 2011. The FTSE MIB Index is a weighted average of the prices of the thirty more liquid stocks in the Italian Stock Exchange and it is updated every time a trade occurs. This is a consequence of trading rules. In many regulated financial markets trading is performed by means of the so-called continuous-double auction. Here, we just present the basic idea of this microstructural market mechanism for an order driven market; details may vary from stock exchange to stock exchange. For every stock traded in the exchange there is a *book* where orders are registered. Traders can either place buy orders (*bids*) or sell orders (*asks*) and this explains why the auction is called double: There are two sides. Morover, orders can be placed at any time, and, for this reason, the auction is called continuous. There are many different kinds of orders, but the typical order is the *limit order*. A bid limit order is an order to buy $q_b^{(T)}$ units of the share at a price not larger than a limit price selected by the trader $p_b^{(T)}$, where T is a label identifying the trader. An ask limit order is an order to sell $q_a^{(T)}$ units of the share at a price not smaller than a limit price selected by the trader $p_a^{(T)}$. The couples $(p_b^{(T)}, q_b^{(T)})$ are stored in the book and ordered from the best bid to the

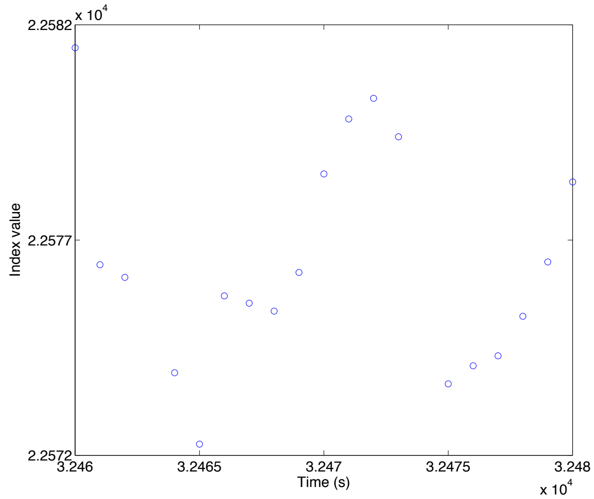

Figure 2.1: Tick-by-tick price fluctuations. As eplained in the text, this is the FTSE MIB Index recorded on February 3rd, 2011. Time is given in seconds since midnight.

worst bid, the best bid being $\widehat{p}_b = \max_{T \in I_b}(p_b^{(T)})$, where I_b is the set of traders placing bids. The couples $(p_a^{(T)}, q_a^{(T)})$ are also stored in the book and ordered from the best ask to the worst ask, the best ask being $\widehat{p}_a = \min_{T \in I_a}(p_a^{(T)})$, where I_a is the set of traders placing asks. At a generic time t, one has that $\widehat{p}_a(t) > \widehat{p}_b(t)$. The difference

$$s(t) = \widehat{p}_a(t) - \widehat{p}_b(t) \tag{2.1}$$

is called *bid-ask spread*. Occasionally, a trader may accept an existing best bid or best ask, and the i-th trade takes place at the epoch t_i. This is called a *market order*. Market rules specify which are the priorities for limit orders placed at the same price and what to do when the quantity requested in a market order is not fully available at the best price. Several authors use the mid-price defined as

$$p_m(t) = \frac{\widehat{p}_b(t) + \widehat{p}_a(t)}{2} \tag{2.2}$$

in order to summarize and study the above process. Both the bid-ask spread and the mid-price can be represented as step functions varying at random times. Jumps in spread and midprice may occur when a better limit order enters the book or when a trade takes place. Another important process is the one of realized trades represented in figures 2.1 and 2.2. In figure 2.2, a càdlàg representation of the process is given, the so-called *previous tick interpolation*, where it is assumed that the price remains constant between two consecutive trades. With these definitions in mind, we can now show how CTRWs can be used in

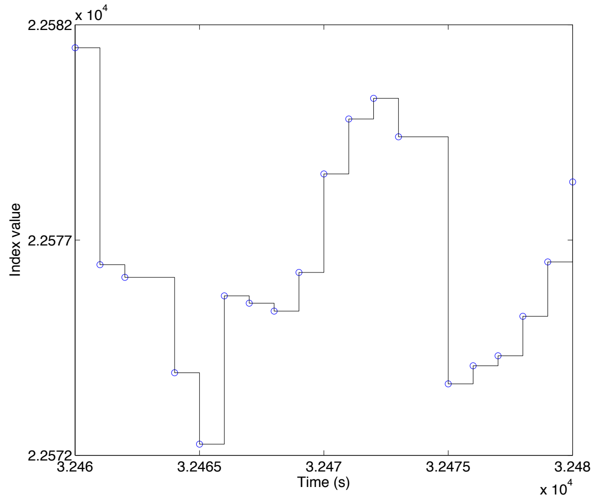

Figure 2.2: Tick-by-tick price fluctuations represented as a càdlàg step function. FTSE MIB Index, February 3rd, 2011. These are the same data as in figure 2.1.

financial modelling [69, 49]. Figures 2.3 to 2.5 represent consecutive magnifications of the same price process and they show how scaling breaks down. In particular, visual inspection shows that scaling is no longer valid already at the time scale of figure 2.5. In other words, the behaviour of high-frequency price fluctuations cannot be described by self-similar, or self-affine, or even multifractal processes.

2.2 Models of price fluctuations in financial markets

Let $p(t)$ represent the price of an asset at time t. Define $p_0 = p(0)$ the initial price. The variable

$$x(t) = \log\left[\frac{p(t)}{p_0}\right] \tag{2.3}$$

is called *logarithmic price* or *log-price* or even *logarithmic return* or *log-return*. With this choice, one has $x(0) = 0$. Now, let t_i be the epoch of the i-th trade and $p(t_i)$ the corresponding price, then the variable

$$\xi_i = \log\left[\frac{p(t_i)}{p(t_{i-1})}\right] \tag{2.4}$$

is called *tick-by-tick log-return*. Let $n(t)$ represent the number of trades from the market opening, up to time t, then the relationship between the log-price

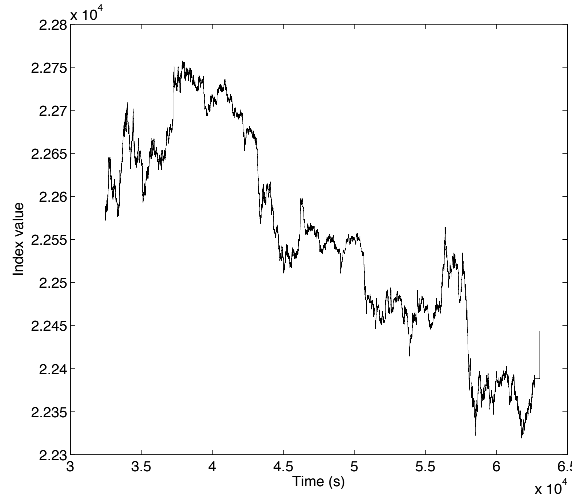

Figure 2.3: Tick-by-tick price fluctuations represented as a càdlàg step function. FTSE MIB Index: A whole trading day, February 3rd 2011.

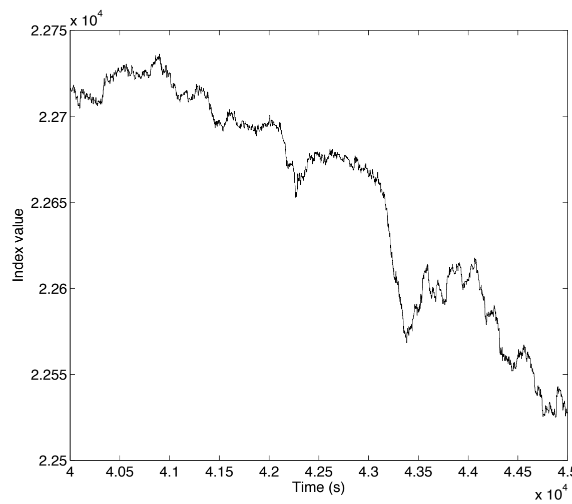

Figure 2.4: Tick-by-tick price fluctuations represented as a càdlàg step function. A zoom of the data of figure 2.3.

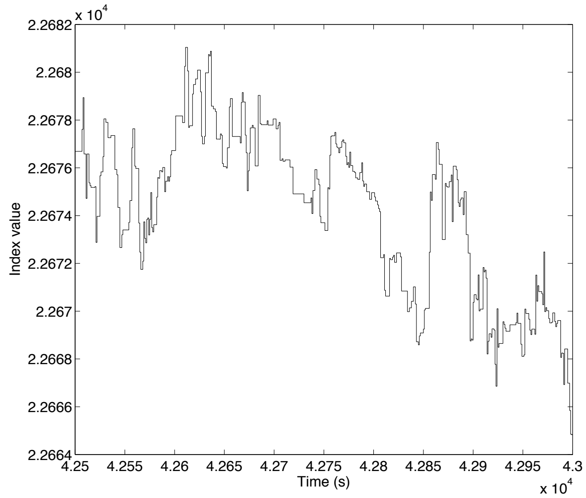

Figure 2.5: Tick-by-tick price fluctuations represented as a càdlàg step function. A zoom of the data of figure 2.4.

and the tick-by-tick log-returns is

$$x(t) = \sum_{i=1}^{n(t)} \xi_i. \tag{2.5}$$

The reason for using these variables in finance is as follows. If price fluctuations are small compared to the price, one can see that the tick-by-tick log-return is very close to the tick-by-tick return $r_i = [p(t_i) - p(t_{i-1})/p(t_{i-1})]$. In symbols, one can write

$$\xi_i = \log\left[\frac{p(t_i)}{p(t_{i-1})}\right] \approx \frac{p(t_i) - p(t_{i-1})}{p(t_{i-1})} = r_i. \tag{2.6}$$

However, returns are not additive, one cannot write the return from a time t to a time $t + \Delta t$ as the sum of tick-by-tick returns, whereas this is possible for log-returns. Equation (2.5) can be compared with equation (1.60) in Chapter 1. It becomes natural to interpret $x(t)$ as a realization of a CTRW or of a compound (renewal) process $X(t)$, $n(t)$ as a realization of a counting (renewal) process $N(t)$ and ξ_i as a value of a random variable Y_i. The simplest possible choice for $x(t)$ is the normal compound Poisson process (NCPP) discussed in Chapter 1, with normally distributed tick-by-tick log-returns (jumps) $\xi \sim N(\mu_\xi, \sigma_\xi^2)$ and exponentially distributed durations (sojourn times) $\tau_i = t_i - t_{i-1} \sim \exp(\lambda)$. However, the normal compound Poisson process is falsified by the following empirical findings on high frequency data:

1. The empirical distribution of tick-by-tick log-returns is leptokurtic, whereas the NCPP assumes a normal distribution which is mesokurtic.

2. The empirical distribution of durations is non-exponential with excess standard deviation [17, 18, 48, 63, 72], whereas the NCPP assumes an exponential distribution.

3. The autocorrelation of absolute log-returns decays slowly [63], whereas the NCPP assumes i.i.d. log-returns.

4. Log-returns and waiting times are not independent [63, 51], whereas the NCPP assumes their independence.

5. Volatility and activity vary during the trading day [5], whereas the NCPP assumes they are constant.

Compound renewal processes take into account facts (1) and (2) as well as fact (4) [51], but they are falsified by fact (3), as they assume independent returns and durations, and by point (5), as they assume identically distributed returns and durations.

2.3 Simulation

Simulation of CTRWs is not difficult [24, 26]. A typical algorithm for uncoupled CTRWs uses the following steps:

1. generate n values for durations from your favorite distribution and store them in a vector;

2. generate n values for tick-by-tick log-returns from your favorite distribution and store them in a second vector;

3. generate the epochs by means of a cumulative sum of the duration vector;

4. generate the log-prices by means of a cumulative sum of the tick-by-tick log-return vector.

If one wishes to simulate $x(t)$, the value of the log-price at time t, it is enough to include a control statement in the above algorithm to ensure that it runs until the sum of durations is less or equal than t. These algorithms generate single realizations of the process either for n jumps or until time t respectively. If many independent runs are performed, one can approximate the distribution of $x(t)$ or any other finite dimensional distribution of the process. An algorithm for the fractional compound process with jumps distributed according to the symmetric α-stable distribution is given in Appendix 3. In that example, for the variables ξ_i, we have used the standard transformation method by Chambers, Mallows and Stuck [11] for $\alpha \in (0, 2]$

$$\xi_i = \gamma_x \left(\frac{-\log(u)\cos(\phi)}{\cos[(1-\alpha)\phi]} \right)^{1-1/\alpha} \frac{\sin(\alpha\phi)}{\cos\phi}, \tag{2.7}$$

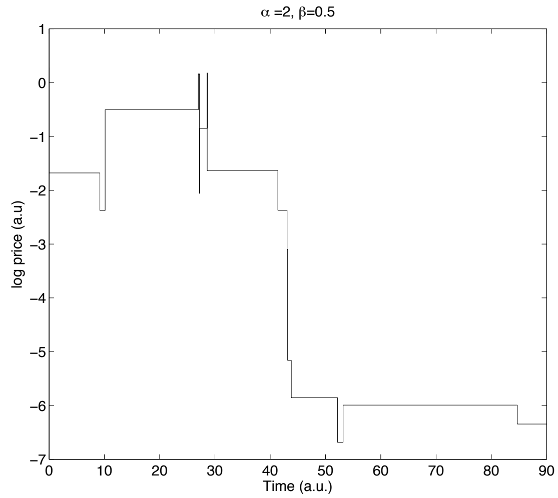

Figure 2.6: Compound fractional Poisson process simulation for $\alpha = 2$ and $\beta = 0.5$.

where γ_x is a scale factor, u is a uniformly distributed random variable between 0 and 1 and $\phi = \pi(v - 1/2)$, with v uniformly distributed between 0 and 1 and not depending on u. For $\alpha = 2$, equation (2.7) reduces to $\xi_i = 2\gamma\sqrt{-\log(u)}\sin(\phi)$, that is to the Box-Muller algorithm for normally distributed random numbers. The algorithm for the generation of Mittag-Leffler distributed τ_is with $\beta \in (0, 1]$ is (see [61, 37, 40, 39, 38, 36, 24])

$$\tau_i = -\gamma_t \log(u) \left(\frac{\sin(\beta\pi)}{\tan(\beta\pi v)} - \cos(\beta\pi) \right)^{1/\beta}, \tag{2.8}$$

where u and v are independent uniformly distributed random variables with values between 0 and 1. For $\beta = 1$ equation (2.8) reduces to $\tau_i = -\gamma_t \log(u)$, that is to the standard transformation formula for the exponential distribution. The resuls of simulations based on this algorithm are represented in figures from 2.6 to 2.11 for the following couples of parameters $(\alpha = 2, \beta = 0.5)$, $(\alpha = 2, \beta = 0.99)$, $(\alpha = 1.95, \beta = 0.5)$, $(\alpha = 1.95, \beta = 0.5)$, $(\alpha = 1, \beta = 0.5)$, and $(\alpha = 1, \beta = 0.5)$. Visual inspection shows that larger jumps in time are more likely smaller values of β and larger jumps in log-price are expected for smaller values of α.

2.4 Option pricing

In sections 2.1 and 2.2, we gave arguments in favor of using CTRWs as models of tick-by-tick price fluctuations in financial markets. We have also seen the

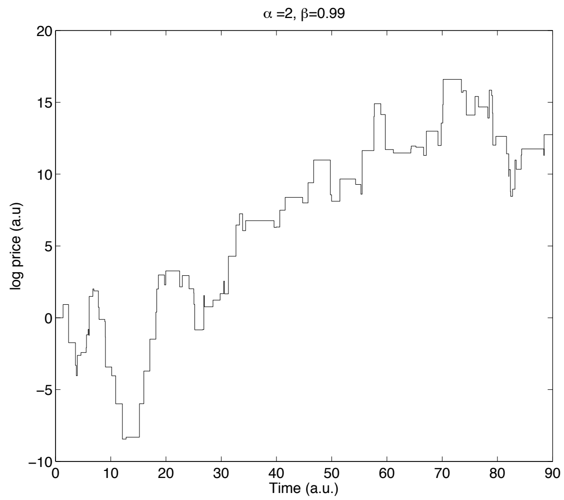

Figure 2.7: Compound fractional Poisson process simulation for $\alpha = 2$ and $\beta = 0.99$.

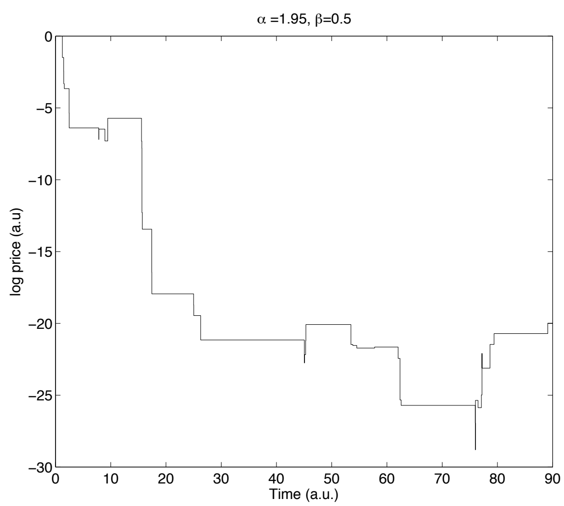

Figure 2.8: Compound fractional Poisson process simulation for $\alpha = 1.95$ and $\beta = 0.5$.

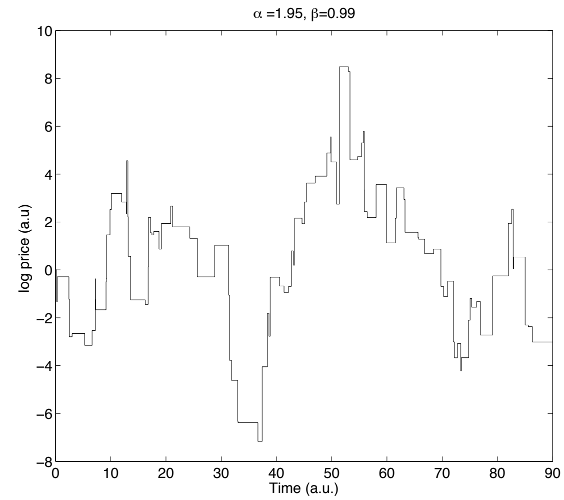

Figure 2.9: Compound fractional Poisson process simulation for $\alpha = 1.95$ and $\beta = 0.99$.

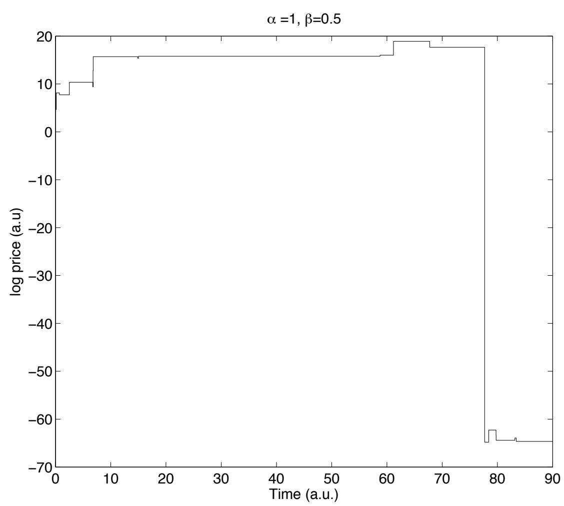

Figure 2.10: Compound fractional Poisson process simulation for $\alpha = 1$ and $\beta = 0.5$.

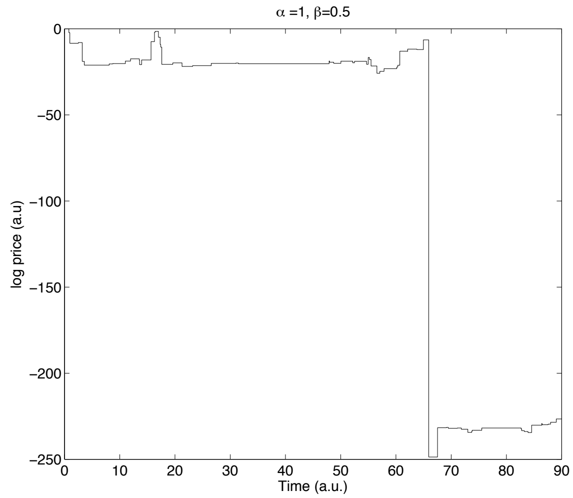

Figure 2.11: Compound fractional Poisson process simulation for $\alpha = 1$ and $\beta = 0.99$.

limits of uncoupled CTRWs as market models. Now, let us suppose that we have an underlying asset whose log-price fluctuations are described by equation (2.5). In other words, we assume that log-price fluctuations follow a compound renewal process. Furthermore, we assume that these fluctuations represent the intra-day behavior of an asset, such as a share traded in a stock exchange. For an intra-day time horizon, we can safely assume that the *risk-free* interest rate is $r_F = 0$. This would be the return rate of a zero-coupon bond. Even if such a return rate were $r_Y = 10\%$ on a yearly time horizon, meaning that the State issuing this financial instrument is close to default (so that, it would not be so riskless, after all) or that the inflation rate is quite high, then the interest rate for one day would be $r_d \approx 1/(10 \cdot 200) = 5 \cdot 10^{-4}$ (200 is the typical number of working days in a year) and this number has still to be divided by 8 (number of trading hours) and by 3600 (number of seconds in one hour) in order to get an approximate interest rate for a time horizon of 1 second. This gives $r_s \approx 1.7 \cdot 10^{-8}$. On the other hand, typical tick-by-tick returns in a stock exchange are larger than the tick divided by the price of the share. Even if we assume that the share is worth 100 monetary units, with a 1/100 tick size (the minimum price difference allowed), we will have a return r larger than $1 \cdot 10^{-4}$ and much larger than r_s. Therefore, it is safe to assume a risk free interest rate $r = 0$ for intra-day hedging.

Hedging is performed through special contracts called options whose price is assumed to depend on the price of the underlying contract. A detailed discussion of these contracts is outside the scope of the present book. However, it is possible to present the basic ideas on option pricing, before turning to our high-frequency problem. The interest reader can consult the introductory books by Hull [31] and by Willmott [81]. One of the simplest option contract is the so-called *plain-vanilla European call*. This is the right (not the obligation) to buy an asset at a future time at a given price K called the *strike price* at a future date T called the *maturity*. If, at maturity, the asset price $p(T)$ is larger than K then, in principle, the option holder can exercise the option, pay K to the option writer to get one unit of asset and resell the assets on the market thus realizing a profit of $p(T)-K$ for each asset unit. On the other side, if $p(T) < K$, it does not make sense to exercise the option. So, one has that the option payoff at maturity is given by

$$C(T) = \max(p(T) - K, 0). \tag{2.9}$$

The problem to be addressed is the following. Suppose you are at time $t < T$ and you want to get a plain vanilla option contract. Which is its fair price? In order to give a feeling on how to solve this problem, we shall consider a simplified version: the so-called one-period binomial option pricing. The price of an asset is $p_0 = p(0)$ at time $t = 0$ and it can either go up or down at the next time step $t = 1$. Assume that $p_1^+ = p(1) = p_0 u$ with probability q and $p_1^- = p(1) = p_0 d$ with probability $1 - q$, where u is the up factor and d is the down factor. For the sake of simplicity, we shall further assume that the risk free interest rate is $r_F = 0$ during this period. The two factors, u and d cannot assume arbitrary values. Since we want that $0 < p_1^- < p_1^+$, this means that $0 < d < u$. Moreover, we want to avoid arbitrage, a trading strategy according to which one can get money out of nothing. Suppose indeed, that $d \geq 1$ and that we take from a bank p_0 monetary units to buy one share at time $t = 0$, then at time $t = 1$, the value of our share will be $p_1 \geq dp_0$, then by selling it and giving back p_0 monetary units to the bank, we will surely get a net profit of $p_1 - p_0 \geq dp_0 - p_0 = (d-1)p_0 \geq 0$ as $d \geq 1$. Therefore, to avoid arbitrage, we must take $d < 1$. Similarly, assume that $u \leq 1$, then one could borrow an asset share at time $t = 0$, then sell it for p_0 units of money and put the money in a bank. Now, at time $t = 1$, we could use this money to pay the share we borrowed at $t = 0$, since $p_1 \leq up_0$, in the end we would realize a certain profit $p_0 - p_1 \geq p_0 - up_0 = (1 - u)p_0 \geq 0$ since $u \leq 1$. In this case, to avoid arbitrage, we must have $u > 1$. In the end, we must require that $0 < d < 1 < u$. Now, if our strike price is $p_1^- < K < p_1^+$, our payoff at time $t = 1$ will be $C_1^+ = C(1) = p_1^+ - K$ with probability q and $C_1^- = C(1) = 0$ with probability $1 - q$. It is possible to prove that the option price $C(0)$ at $t = 0$ is given by the following conditional expectation

$$C(0) = \mathbb{E}_{\widetilde{\mathbb{Q}}}[C(1)|I(0)], \tag{2.10}$$

where $I(0)$ represents the information available at time $t = 0$ and $\widetilde{\mathbb{Q}}$ represents the equivalent martingale measure. Two probability measures are *equivalent* if each one is absolutely continuous with respect to the other. A probability measure $\mathbb{P}$ is absolutely continuous to respect to measure $\mathbb{Q}$ if its null set is contained in the null set of $\mathbb{Q}$. The null sets of two equivalent measures do coincide. An elementary introduction to these concepts can be found in a book by T. Mikosch [56]. Among all the equivalent measures, the equivalent martingale measure is the one for which the price process is a martingale, meaning that the price process is integrable and

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[p(1)|I(0)] = p(0). \tag{2.11}$$

In our case, the martingale measure is given by $\tilde{q} = (1-d)/(u-d)$ and $1-\tilde{q} = (u-1)/(u-d)$ and the option price is given by

$$C(0) = \frac{1-d}{u-d} C_1^+. \tag{2.12}$$

The martingale measure can be found by simple algebraic manipulations imposing equation (2.11). Indeed, one has that $\mathbb{E}_{\tilde{\mathbb{Q}}}[p(1)|I(0)] = \tilde{q}up_0 + (1-\tilde{q})dp_0$, and imposing (2.11) immediately leads to $\tilde{q} = (1-d)/(u-d)$. Note that equation (2.10) means that also C is a martingale under the measure $\tilde{\mathbb{Q}}$. Technically speaking $I(0)$ in equation (2.10) is the *filtration* at time $t=0$. A filtration is a non decreasing family of σ-algebras which represents the information available at a certain time, see [56] for a rigorous definition of this concept. Equation (2.12) can be derived from the fact that it is possible to replicate the option in terms of a suitable non-financing portfolio coupled with a no-arbitrage argument. This derivation shows that equations (2.10) and (2.12) give the *optimal* option price in term of fairness. However, it is not always possible to extend the arguments leading to the martingale option price when more general assumptions on the process followed by the price of the underlying asset are made. However, it is often possible to compute the martingale price in many cases of practical interest and this is done by *quants* in everyday financial practice [59]. In 1976, R. Merton solved the problem of finding the option martingale price for an underlying whose log-price follows the NCPP [53]. The idea behind Merton's derivation is as follows. Assume that $t=0$ is a renewal point, that the risk free interest rate is $r_F = 0$ and denote the price at $t=0$ by $S_0 = S(0)$, the strike price by K and the maturity by T. The NCPP assumption means that the underlying log-price follows the process

$$X(t) = \log(S_0) + \sum_{i=1}^{N(t)} Y_i, \tag{2.13}$$

where $Y_i \sim N(\mu, \sigma^2)$ and $N(t)$ is the Poisson process. The price should follow the process $S(t) = S_0 \mathrm{e}^{X(t)}$. This is not a martingale, however. In order to find the option martingale price, let us consider the situation in which there are exactly n jumps from 0 and T. In this case, one has to study the processes

$$X_n = \log(S_0) + \sum_{i=1}^{n} Y_i \tag{2.14}$$

and

$$S_n = S_0 \mathrm{e}^{X_n} = S_0 \prod_{i=1}^{n} \mathrm{e}^{Y_i}. \tag{2.15}$$

Notice that the random variables e^{Y_i} follow the log-normal distribution. The process defined by X_n is not a martingale, but the equivalent martingale measure can be found by imposing that the process

$$S'_n = S_0 \mathrm{e}^{X_n + na} \tag{2.16}$$

is a martingale and this leads to

$$a = -\log[\mathbb{E}(\mathrm{e}^Y)], \tag{2.17}$$

where

$$\mathbb{E}(\mathrm{e}^Y) = \mathrm{e}^{\mu+\sigma^2/2} \tag{2.18}$$

so that

$$a = -(\mu + \sigma^2/2). \tag{2.19}$$

The option price at $n = 0$ is thus given by

$$C_n(S_0, K, \mu, \sigma^2) = \mathbb{E}_{\mathbb{S}'}[C(T)|I(0)], \tag{2.20}$$

where the expected value is computed according to the measure defined by the process (2.16). For the plain vanilla European call option with $C(T) = \max(S(T) - K, 0)$, a straightforward calculation leads to

$$C_n(S_0, K, \mu, \sigma^2) = N(d_{1,n})S_0 - N(d_{2,n})K, \tag{2.21}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy\, \mathrm{e}^{-y^2/2} \tag{2.22}$$

is the standard normal cumulative distribution function and

$$d_{1,n} = \frac{\log(S_0/K) + n(\mu + \sigma^2/2)}{\sqrt{n}\sigma}, \tag{2.23}$$

$$d_{2,n} = d_{1,n} - \sigma\sqrt{n}. \tag{2.24}$$

Given the independence between jumps and durations, one can now write the option price at $t = 0$ as

$$C(0) = \mathrm{e}^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} C_n(S_0, K, \mu, \sigma^2), \tag{2.25}$$

where λ is the intensity of the Poisson process. To obtain equation (2.25), it is enough to notice that the probability of having n jumps in the time interval $[0, T]$ is given by the Poisson distribution of parameter λT and that one can go from S_0 to $S(T)$ in any number of steps n. Equation (2.25) can be generalized to renewal processes simply by replacing the Poisson distribution with the counting distribution $P(n, t)$. In the Mittag-Leffler case with $\gamma_t = 1$, one can write (see equation (1.56))

$$C(0) = \sum_{n=0}^{\infty} \frac{t^{\beta n}}{n!} E_{\beta}^{(n)}(-t^{\beta})\, C_n(S_0, K, \mu, \sigma^2). \tag{2.26}$$

Merton's result has been recently revisited and it is still the object of active research [13]. Note that this method works when the random variable e^Y has finite first moment. This is the case when all the moments of the tick-by-tick log-returns Y are finite as in the normal case discussed above.

2.5 Other applications

This chapter focuses on the application of uncoupled continuous-time random walks in high-frequency financial data modelling. For the sake of simplicity, we have not discussed the coupled case, but this is covered in reference [51]. Morevoer, it is possible to use the program described in section 2.3 for scenario simulation and speculative option pricing [73].

As discussed in section 1.1, it is not surprising that CTRWs can also be applied elsewhere. A standard application is to insurance [44, 15], where the capital $R(t)$ of an insurance company can be written as

$$R(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \tag{2.27}$$

and where u is the initial capital of the company, c is the rate of capital increase, $N(t)$ is the random number of claims Y_i that the company has paid since inception. In this case, ruin is the interesting phenomenon. It takes place the first time that $R(t) = 0$, that is when the capital of the insurance company vanishes. In this framework, one can define the time to ruin as the following hitting time

$$\tau(u) = \inf \left\{ t : \; u + ct - \sum_{i=1}^{N(t)} Y_i < 0 \right\}. \tag{2.28}$$

Two interesting quantities are the probability $q(u)$ of ruin in infinite time and the probability of ruin in a finite time T. These two quantities are defined as follows, respectively:

$$q(u) = \mathbb{P}(\tau(u) < \infty), \tag{2.29}$$

and

$$q(u, T) = \mathbb{P}(\tau(u) < T). \tag{2.30}$$

It is always possible to study these quantities by means of Monte Carlo simulations, using the algorithm of Appendix 3 or a suitable modification.

Another interpretation of the random variables is in terms of economic growth. Let us be as general as possible and denote by S a suitable "size". This size has the meaning of wealth, firm size, city size, etc., depending on the scientific context. Then, according to Gibrat's approach [27], one can define the log-size $X = \log(S)$ and write it as a sum of exogenous shocks Y_i

$$X_n = X_0 + \sum_{i=1}^{n} Y_i. \tag{2.31}$$

For large n, if the shocks have finite first and second moments, X_n approximately follows the normal distribution as a consequence of central limit theorems (see theorem 1.2.20 for a simple version). This means that the size S_n approximately follows the log-normal distribution [1]. If the growth shocks arrive at random times, equation 2.31 can be replace by the familiar equations for CTRWs with non-homogeneous initial position

$$X(t) = X_0 + \sum_{i=1}^{N(t)} Y_i. \tag{2.32}$$

This method was used by Italian economists to study firm growth and size distribution [8, 25, 9].

Chapter 3

Monte Carlo simulation of CTRW

Two implementations are presented (in R and In Matlab) of a Monte Carlo program to simulate CTRWs according to the algorithm described in section 2.3. The program below, generates and plots a single realization of a CTRW with a given number of jumps and durations. Even if this is a very simple algorithm, it consists of three parts. The first part is the generator of independent and indentically distributed Mittag-Leffler deviates according to equation (2.8). Then, Lévy α-stable deviates are generated following equation (2.7). Finally, cumulative sums give the position coordinates and the epochs and positions are plotted as a function of the epochs. This routine can be easily modified with suitable external cycles to generate many realizations up to a given time t and estimate the probability density $f_{X(t)}(x,t)$ from the histogram of realized positions $X(t)$. This was explicitly done in reference [24].
Matlab implementation for the Monte Carlo simulation of CTRW.

```
%Plot of a single CTRW realization

%Generation of Mittag-Leffler deviates
%See Fulger, Scalas, Germano 2008 and references therein

n=100; %number of points
gammat=1; %scale parameter
beta=0.99; %ML parameter

u1=rand(n,1); %uniform deviates
v1=rand(n,1); %uniform deviates

%Generation of symmetric alpha stable deviates

tau=-gammat*log(u1).*(sin(beta*pi).
/tan(beta*pi*v1)-cos(beta*pi)).^(1/beta);
```

```
gammax=1; %scale parameter
alpha=1.95; %Levy parameter

u2=rand(n,1); %uniform deviates
v2=rand(n,1); %uniform deviates
phi=pi*(v2-0.5);

xi=gammax*(sin(alpha*phi)./cos(phi)).
*(-log(u2).*cos(phi)./cos((1-alpha)*phi)).^(1-1/alpha);

%Random walk

x=cumsum(xi');
x=[0 x];

%Epochs

t=cumsum(tau');
t=[0 t];

stairs(t,x) %plots ctrw
```

R implementation for the Monte Carlo simulation of CTRW.

```
# Plot of a single CTRW realization

# Generation of Mittag-Leffler deviates
# See Fulger, Scalas, Germano 2008 and references therein

n <- 10000 #number of points
gammat <- 1 #scale parameter
beta <- 0.95 #ML parameter

u1 <- runif(n) #uniform deviates
v1 <- runif(n) #uniform deviates

tau <- -gammat * log(u1) * (sin(beta * pi)/tan(beta * pi * v1) -
cos(beta * pi))^(1/beta)

# Generation of symmetric alpha-stable deviates
# See Fulger, Scalas, Germano 2008 and references therein

gammax <- 1 #scale parameter
alpha <- 1.95 #Levy parameter

u2 <- runif(n)
v2 <- runif(n)
phi <- pi*(v2 - 0.5)

xi <- gammax *(sin(alpha*phi)/cos(phi))*
(-log(u2)*cos(phi)/cos((1-alpha)*phi))^(1-1/alpha)
```

```
# histogram of xi
# hist(xi)

# Random walk

x <- cumsum(xi)

# Epochs

t <- cumsum(tau)

# Stairplot

plot(t,x,type="s")
```

Bibliography

[1] J. Aitchison and J. A. Brown, *The Lognormal Distribution*, Cambridge Univ. Press, Cambridge, 1957.

[2] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge Univ. Press, Cambridge, 2009.

[3] L. Beghin and E. Orsingher, *Fractional Poisson process and related random motions*, Electron. J. Probab. **14** (2009), 1790–1826.

[4] J. Bertoin, *Lévy Processes*, Cambridge Univ. Press, Cambridge, 1998.

[5] W. Bertram, *An empirical investigation of Australian Stock Exchange data*, Phys. A **341** (2004), 533–546.

[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge Univ. Press, Cambridge, 1987.

[7] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Akademische Verlagsgesellschaft, Leipzig, 1932.

[8] G. Bottazzi and A. Secchi, *Why are distributions of firm growth rates tent shaped?*, Economics Letters **80** (2003), 415–420.

[9] ______, *Explaining the distribution of firm growth rates*, The RAND Journal of Economics **37** (2006), 235–256.

[10] E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, Englewood Cliffs, 1975.

[11] J. M. Chambers, C. L. Mallows, and B. W. Stuck, *A method for simulating stable random variables*, J. Amer. Statist. Ass. **71** (1976), 340–344.

[12] D. C. Champeney, *A Handbook of Fourier Theorems*, Cambridge Univ. Press, Cambridge, 1987.

[13] G. H. Cheang and C. Chiarella, *A modern view on Merton's jump-diffusion model*, 2011, University of Technology, Sydney, Quantitative Finance Research Centre, Research paper 287.

[14] A. Compte, *Stochastic foundations of fractional dynamics*, Phys. Rev. E **53** (1996), 4191–4193.

[15] H. Cramér, *On the Mathematical Theory of Risk*, Skandia Jubilee Volume, Stockholm, 1930.

[16] B. de Finetti, *Theory of Probability*, vol. II, Wiley, New York, 1975.

[17] R. Engle and J. Russel, *Forecasting the frequency of changes in quoted foreign exchange prices with the autoregressive conditional duration model*, Journal of Empirical Finance **4** (1997), 187–212.

[18] ———, *Autoregressive conditional duration: A new model for irregularly spaced transaction data*, Econometrica **66** (1998), 1127–1162.

[19] W. Feller, *On a general class of "contagious" distributions*, Ann. Math. Stat. **14** (1943), 389–400.

[20] ———, *Introduction to Probability Theory and its Applications*, vol. II, Wiley, New York, 1971.

[21] O. Flomenbom and J. Klafter, *Closed-form solutions for continuous time random walks on finite chains*, Phys. Rev. Lett. **95** (2005), 098105.

[22] O. Flomenbom and R. J. Silbey, *Path PDFs for semi-Markovian random walks*, Phys. Rev. E **76** (2007), 041101.

[23] H. C. Fogedby, *Langevin equations for continuous time Lévy flights*, Phys. Rev. E **50** (1994), 1657–1660.

[24] D. Fulger, E. Scalas, and G. Germano, *Monte Carlo simulation of uncoupled continuous-time random walks yielding a stochastic solution of the space-time fractional diffusion equation*, Phys. Rev. E **77** (2008), 021122.1–021122.7.

[25] E. Gaffeo, D. Delli Gatti, C. Di Guilmi, G. Giulioni, M. Gallegati, and A. Palestrini, *Business cycle fluctuations and firms' size distribution dynamics*, Adv. Complex Syst. **7** (2004), 223–240.

[26] G. Germano, M. Politi, E. Scalas, and R. L. Schilling, *Stochastic calculus for uncoupled continuous-time random walks*, Phys. Rev. E **79** (2009), 066102–066113.

[27] R. Gibrat, *Les inégalités economiques, application: Aux inégalités des richesses, à la concentrations des entreprises, aux populations des villes, aux statistiques des familles, etc., d'une loi nouvelle, al loi de l'effet proportionnel*, Librairie du Recueil Sirey, Paris, 1931.

[28] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, MA, 1954.

[29] R. Gorenflo and E. A. A. Abdel-Rehim, *From power laws to fractional diffusion: The direct way*, Vietnam Journal of Mathematics **32 (SI)** (2004), 65–75.

[30] R. Hilfer and L. Anton, *Fractional master equations and fractal time random walks*, Phys. Rev. E **51** (1995), R848–R851.

[31] J. C. Hull, *Options, Futures and Other Derivatives*, Prentice Hall, Upper Saddle River, 2005.

[32] N. Jacob, *Pseudo-differential Operators and Markov Processes*, vol. III, Imperial College Press, London, 2005.

[33] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, New York, 2002.

[34] A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α-Stable Stochastic Processes*, Marcel Dekker, New York, 1994.

[35] J. Janssen and R. Manca, *Semi-Markov Risk Models for Finance, Insurance and Reliability*, Springer, New York, 2007.

[36] K. Jayakumar, *Mittag-Leffler process*, Math. Comput. Modelling **37** (2003), 1427–1434.

[37] T. J. Kozubowski, *Mixture representation of Linnik distribution revisited*, Stat. Probab. Lett. **38** (1998), 157–160.

[38] ______, *Computer simulation of geometric stable distributions*, J. Comput. Appl. Math. **116** (2000), 221–229.

[39] ______, *Exponential mixture representation of geometric stable distributions*, Annals of the Institute of Statistical Mathematics **52** (2000), 231–238.

[40] T. J. Kozubowski and S. T. Rachev, *Univariate geometric stable distributions*, J. Comput. Anal. Appl. **1** (1999), 177–217.

[41] N. Laskin, *Fractional Poisson process*, Commun. Nonlinear Sci. Numer. Simul. **8** (2003), 201–213.

[42] W. R. LePage, *Complex Variables and the Laplace Transform for Engineers*, Dover Publ., New York, 1980.

[43] P. P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris, 1954.

[44] P. Lundberg, *Approximerad Framställning av Sannolikehetsfunktionen. å terförsäkering av Kollektivrisker*, Almqvist & Wicksell, Uppsala, 1903.

[45] M. Magdziarz and K. Weron, *Anomalous diffusion schemes underlying the Cole-Cole relaxation: The role of the inverse α-stable subordinator*, Phys. A **367** (2006), 1–6.

[46] F. Mainardi, R. Gorenflo, and E. Scalas, *A fractional generalization of the Poisson process*, Vietnam Journal of Mathematics **32 (SI)** (2004), 53–64.

[47] F. Mainardi, Yu. Luchko, and G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. **4** (2001), 153–192.

[48] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, *Fractional calculus and continuous-time finance: The waiting-time distribution*, Phys. A **287** (2000), 468–481.

[49] J. Masoliver, M. Montero, J. Perelló, and G. H. Weiss, *The continuos time random walk formalism in financial markets*, Journal of Economic Behavior and Organization **61** (2006), 577–591.

[50] M. M. Meerschaert, E. Nane, and P. Vellaisamy, *The fractional Poisson process and the inverse stable subordinator*, Electron. J. Probab. **16** (2011), 1600–1620.

[51] M. M. Meerschaert and E. Scalas, *Coupled continuous time random walks in finance*, Phys. A **370** (2006), 114–118.

[52] M. M. Meerschaert and H. P. Scheffler, *Limit distributions for sums of independent random vectors: Heavy tails in theory and practice*, Wiley, New York, 2001.

[53] R. C. Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics **3** (1976), 125–144.

[54] R. Metzler and J. Klafter, *The random walk guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), 1–77.

[55] ______, *The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics*, J. of Phys. A: Math. General **37** (2004), R161–R208.

[56] T. Mikosch, *Elementary Stochastic Calculus with Finance in View*, World Sci., Singapore, 1999.

[57] E. W. Montroll and H. Scher, *Random walks on lattices IV: Continuous-time random walks and influence of absorbing boundary conditions*, J. Stat. Phys. **9** (1973), 101–135.

[58] E. W. Montroll and G. H. Weiss, *Random walks on lattices II*, J. Math. Phys. **6** (1965), 167–181.

[59] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer, New York, 2002.

[60] J. P. Nolan, *Stable Distributions: Models for Heavy Tailed Data*, Birkhäuser, Boston, 2012.

[61] A. G. Pakes, *Mixture representations for symmetric generalized Linnik laws*, Stat. Probab. Lett. **37** (1998), 213–221.

[62] I. Podlubny, *Fractional differential equations*, Acad. Press, San Diego, 1999.

[63] M. Raberto, E. Scalas, and F. Mainardi, *Waiting-times and returns in high-frequency financial data: An empirical study*, Phys. A **314** (2002), 749–755.

[64] O. N. Repin and A. I. Saichev, *Fractional Poisson law*, Radiophys. Quantum Electron. **43** (2000), 738–741.

[65] H. E. Roman and P. A. Alemany, *Continuous-time random walks and the fractional diffusion equation*, J. Phys. A **27** (1994), 3407–3410.

[66] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations: solutions and applications*, Chaos **7** (1997), 753–764.

[67] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman & Hall, London, 1994.

[68] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, Cambridge, 1999.

[69] E. Scalas, *The application of continuous-time random walks in finance and economics*, Phys. A **362** (2006), 225–239.

[70] ———, *A class of CTRWs: Compound fractional Poisson processes*, Fractional Dynamics (R. Metzler, ed.), World Sci., Singapore, 2011, pp. 351–372.

[71] E. Scalas, R. Gorenflo, and F. Mainardi, *Fractional calculus and continuous-time finance*, Phys. A **284** (2000), 376–384.

[72] ———, *Uncoupled continuous-time random walks: Solutions and limiting behavior of the master equation*, Phys. Rev. E **69** (2004), 011107.

[73] E. Scalas, R. Gorenflo, F. Mainardi, and M. M. Meerschaert, *Speculative option valuation and the fractional diffusion equation*, Proceedings of the 1st IFAC Workshop on Fractional Differentiation and its Applications (A. Le Méhauté, J. A. Tenreiro Machado, J. C. Trigeassou, and J. Sabatier, eds.), ENSEIRB, Bordeaux, 2004.

[74] H. Scher and M. Lax, *Stochastic transport in a disordered solid I. Theory*, Phys. Rev. B **7** (1973), 4491–4502.

[75] ———, *Stochastic trasport in a disordered solid II. Impurity conduction*, Phys. Rev. B **7** (1973), 4506–4519.

[76] W. Schoutens, *Lévy Processes in Finance*, Wiley, New York, 2003.

[77] L. M. Schwartz, *Théorie des distributions*, Hermann, Paris, 1950.

[78] M. F. Shlesinger, *Random processes*, Encyclopedia of Applied Physics (G. L. Trigg, ed.), vol. 16, VCH, New York, 1996, pp. 45–70.

[79] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, 1941.

[80] D. Williams, *Probability with Martingales*, Cambridge Univ. Press, Cambridge, 1991.

[81] P. Wilmott, *Derivatives: The Theory and Practice of Financial Engineering*, Wiley, New York, 1998.

[82] V. M. Zolotarev, *One-dimensional Stable Distributions*, Am. Math. Soc., Providence, 1986.