An Introduction to Viscosity Solutions: theory, numerics and applications

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OUTLINE OF THE COURSE:

- Lecture 1: Introduction to viscosity solutions
- Lecture 2: Approximation schemes for viscosity solutions
- Lecture 3: Approximation of optimal control problems via DP
- Lecture 4: Efficient methods and perspectives

OUTLINE OF THIS LECTURE:
Introduction to viscosity solutions

- Motivations (related to control problems)
- Viscosity solutions for 1st order PDEs
- Uniqueness for viscosity solutions
- Some properties of viscosity solutions
- Hopf–Lax representation formula
- Some hints on viscosity solutions for 2nd order PDEs
- Existence and uniqueness

Our approach to the solution of optimal control problems and games is based on Dynamic Programming, another approach is based on Pontryagin Maximum Principle.

By the Dynamic Programming Principle, we will derive the characterization of the value function in terms of a first order partial differential equation (PDE), the Bellman equation (or the Isaacs equation for games).

This approach is interesting for the continuous control problems as well for numerical purposes and can be applied to all classical control problems.

In one of the following lectures we will deal with the construction of approximation schemes via Discrete Dynamic Programming. We will also deal with the algorithms which allow to compute the value function, optimal controls in feedback form and the corresponding optimal trajectories.

Dynamics

$$\begin{cases} \dot{y}(t) = b(y(t), \alpha(t)) \\ y(t_0) = x_0 \end{cases}$$

where $\alpha(\cdot) \in \mathcal{A} \equiv \{\alpha : [0, +\infty[\to A, \text{ measurable}\}$ and $A \subset \mathbb{R}^M$ is compact

Cost

$$J_{(t_0,x_0)}(\alpha) \equiv \int_{t_0}^{t_f} f(y(s), \alpha(s)) \mathrm{e}^{-\lambda s} ds \ + \psi(y(t_f)), \quad \lambda > 0$$

Value function $v(t_0, x_0) \equiv \inf_{\alpha \in \mathcal{A}} J_{(t_0,x_0)}(\alpha)$.

Dynamics

$$\begin{cases} \dot{y}(t) = b(y(t), \alpha(t)) \\ y(0) = x \end{cases}$$

where $\alpha(\cdot) \in \mathcal{A} \equiv \{\alpha : [0, +\infty[\to A, \text{ measurable}\}$ and $A \subset \mathbb{R}^M$ is compact

Cost

$$J_x(\alpha) \equiv \int_0^\infty f(y(s), \alpha(s)) e^{-\lambda s} ds \ , \qquad \lambda > 0$$

Value function $v(x) \equiv \inf_{\alpha \in \mathcal{A}} J_x(\alpha)$.

The infinite horizon problem with state constraints

Dynamics

$$\begin{cases} \dot{y}(t) = b(y(t), \alpha(t)) \\ y(0) = x \end{cases}$$

where $y \in \mathbb{R}^N$. Now we require that $y_x(t) \in \Omega \subset \mathbb{R}^N$ for any t.

Admissible controls

$\alpha(\cdot) \in \mathcal{A}_x \equiv \{\alpha : [0, +\infty[\to A, \;$ measurable such that $y_x(t) \in \Omega\}$

where $A \subset \mathbb{R}^M$ is compact.

Cost

$$J_x(\alpha) \equiv \int_0^\infty f(y(s), \alpha(s)) \mathrm{e}^{-\lambda s} ds \ , \qquad \lambda > 0$$

Value function $v(x) \equiv \inf_{\alpha \in \mathcal{A}_x} J_x(\alpha)$.

The nonlinear minimum time problem

Dynamics

$$\begin{cases} \dot{y}(t) = b(y(t), \alpha(t)) \\ y(0) = x \end{cases}$$

$$y \in \mathbb{R}^N \quad \text{and } \alpha(\cdot) \in \mathcal{A}$$

where

$$\mathcal{A} \equiv \{\alpha(\cdot) : [0, +\infty[\to A \quad \text{measurable}\} \quad \text{and } A \subset \mathbb{R}^M \text{ is compact}$$

Target $\mathcal{T}$ is a compact set with nonempty interior

The minimum time function is:

$$T(x) \equiv \inf_{\alpha(\cdot)\in\mathcal{A}} \{t : \; y_x(t, \alpha(t)) \in \mathcal{T}\}$$

A priori T is not defined everywhere, its domain of definition is the reachable set ,

$$\mathcal{R} \equiv \left\{ x \in \mathbb{R}^N : \; T(x) < +\infty \right\}$$

Note that $\mathcal{R}$ is NOT given and can have a rather complicated shape even for simple dynamics. This is a free-boundary problem where we have to detect the couple $(T, \mathcal{R})$.

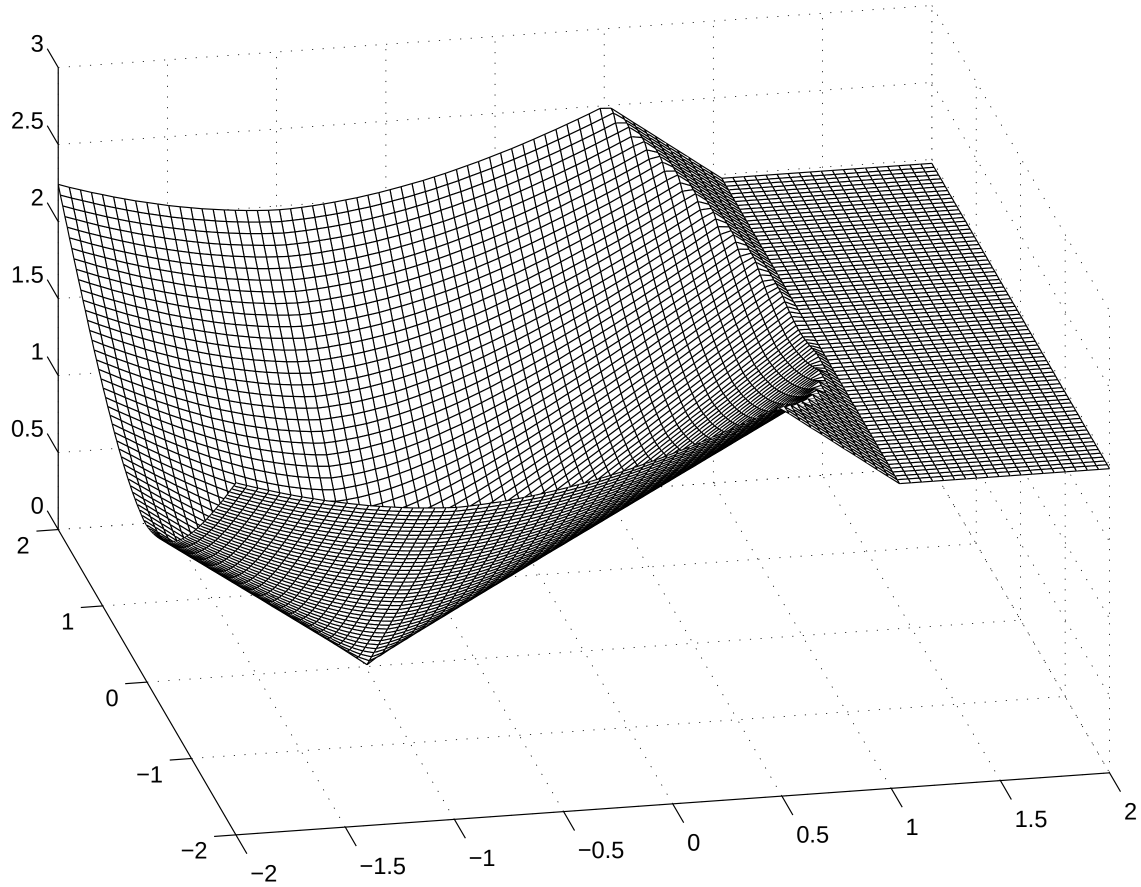
The link is the following

The value function is the unique viscosity solution of the Bellman equation associated to the problem via Dynamic Programming.

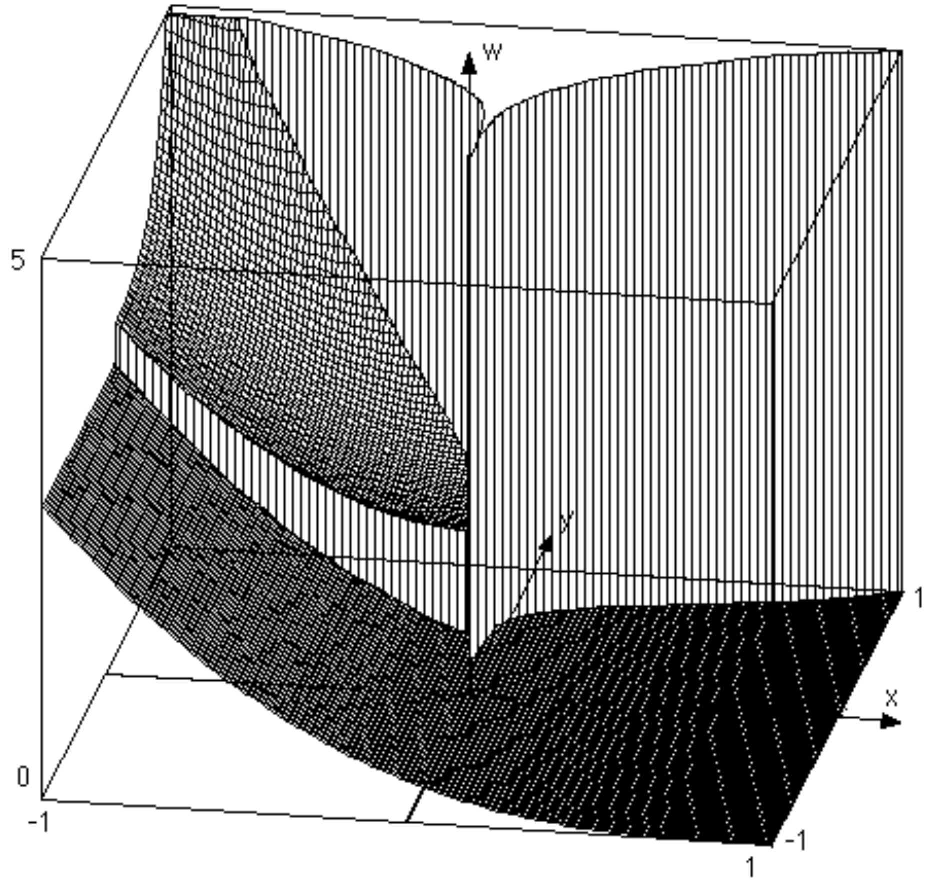
For the infinite horizon problem this is

$$\lambda v(x) + \max_{a \in A}\{-b(x,a) \cdot \nabla u(x) - f(x,a)\} = 0, \qquad x \in \mathbb{R}^N$$

Value function for a minimum time problem



Value function for a state constraint problem



Let us start with the stationary model problem

$$\begin{cases} H(x,u,Du) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{cases} \tag{HJ}$$

where $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian and g is a given boundary condition.
We want to define a good notion of weak solution since the problem is nonlinear and we cannot expect a classical solution to (HJ).

The previous examples have shown that we can have:

- jumps in the gradient of the solution
- jumps in the solution for the constrained problem and for differential games

The typical assumptions on the Hamiltonian $H(x, u, p)$ to set up the theory are

A1: $H(\cdot, \cdot, \cdot)$ is uniformly continuous
A2: $H(x, u, \cdot)$ is convex
A3: $H(x, \cdot, p)$ is monotone

Example: $|u_x| = 1$

Consider the following Dirichlet problem for the eikonal equation in 1 dimension

$$\begin{cases} |u_x| = 1 & \text{in } \Omega \equiv (-1,1) \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{E}$$

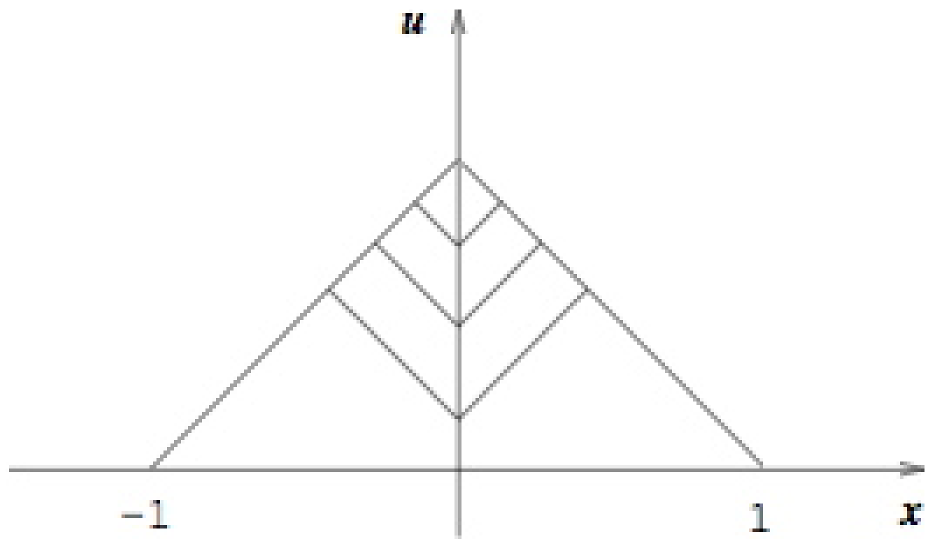
Obviously, $u_1(x) = x$ and $u_2(x) = -x$ satisfy the equation point-wise in (-1,1) but they do not satisfy the boundary conditions. Moreover, a classical solution can not exist due to Rolle Theorem.

Both functions $u_3(x) = |x| - 1$ and $u_4(x) = 1 - |x|$ satisfy the equation a.e. and satisfy the boundary conditions.

Example: $|u_x| = 1$

One can easily seen that there are infinitely many a.e. solutions of the equation.

In fact, collecting piecewise affine functions parallel to x or $-x$ one can built a new a.e. solution.



Example: $|u_x| = 1$

It is clear that the notion of "a.e. solution" gives too many solutions and it is unsuitable for a uniqueness result.

One possibility to select a solution is to regularize the problem first adding a second order term $-\varepsilon u_{xx}$ and then pass to the limit for $\varepsilon \to 0$.

This is called elliptic regularization.

Vanishing viscosity limit

Consider the second order problem

$$-\varepsilon u_{xx} + |u_x| = 1, \text{ in } (-1,1) \tag{1}$$

with the homogeneous boundary condition $u(-1) = u(1)$. This has a regular solution $u^\varepsilon \in C^2(-1,1)$ for every positive ε

We can pass to the limit at every $x \in \Omega$ getting

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = \overline{u}(x) \tag{2}$$

and define the limiting function $\overline{u}$ to be the weak solution of our problem (Kruzkov, 60s).

Note that this is the reason for the name.

Let us consider the "classical definition" for $u \in BUC(\Omega)$ (the space of Bounded Uniformly Continuous function over the open set Ω).

Note that the definition is direct (without any reference to a regularization and/or a limit) and LOCAL .

DEFINITION

$u \in BUC(\Omega)$ is a viscosity solution of

$$H(x, u, Du) = 0 \text{ in } \Omega$$

if and only if, for any $\varphi \in C^1(\Omega)$ the following conditions hold:

i) at every local maximum point $x_o \in \Omega$ for $u - \varphi$

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

i.e. u is a viscosity sub-solution.

ii) at every local minimum point $x_o \in \Omega$ for $u - \varphi$

$$H(x_0, u(x_0), D\varphi(x_0)) \geq 0$$

i.e. u is a viscosity super-solution.

Let us go back to the example.

Take any a.e solution u which has a local minimum at x_0 and choose (for example) $\varphi = costant$. Clearly, x_0 is a local minimum point for $u - \varphi$ so that we should have

$$|\varphi_x(x_0) \geq 1$$

which is FALSE since $\varphi_x \equiv 0$.

Conclusion: every a.e. solution having a local minimum point cannot be a viscosity super-solution.

The same argument will not work for sub-solutions.

Take any a.e solution u which has a local maximum at x_0 and choose (for example) $\varphi = costant$. Clearly, x_0 is a local maximum point for $u - \varphi$ so that we should have

$$|\varphi_x(x_0)| \leq 1$$

which is TRUE since $\varphi_x \equiv 0$.
The only viscosity solution of our problem is $v(x) = 1 - |x|$.

1. If u is a classical $C^1(\Omega)$ solution then it is also a viscosity solution.

2. If u is a regular viscosity solution then it is also a classical solution (i.e. satisfies the equation pointwise).

3. the viscosity solution u is the maximal sub-solution , i.e.

$$w \leq u, \quad \text{for any } w \in S \equiv \{\text{space of sub-solutions}\}$$

4. (stability) the viscosity solution u is the uniform limit of u^ε where

$$-\varepsilon u^\varepsilon_{xx} + H(x, u^\varepsilon, Du^\varepsilon) = 0$$

which means

$$u(x) = \lim_{\varepsilon \to 0^+} u^\varepsilon(x)$$

WARNING: as we have seen in the example, if u is a viscosity solution this DOES NOT IMPLY that $-u$ is a viscosity solution (unless u is regular).

The crucial point for viscosity solution is to prove uniqueness. This is done via a comparison principle (also called maximum principle).

THEOREM

Let $u, v \in BUC(\Omega)$ be respectively a sub and a super-solution for

$$H(x, u, Du) = 0 \qquad \text{in } \Omega$$

and let

$$u(x) \leq v(x) \qquad \text{for any } x \in \partial\Omega.$$

Then,

$$u(x) \leq v(x) \qquad \text{for any } x \in \Omega.$$

Uniquess via the comparison principle

The comparison principle is enough to get uniqueness.
In fact, let u and v be two viscosity solutions of the equation satisfying the same boundary condition, clearly they are (both) sub and super-solutions. Then, we have

$$u(x) \leq v(x) \text{ for any } x \in \Omega$$

AND, reverting the role of u and v, also

$$u(x) \geq v(x) \text{ for any } x \in \Omega$$

which implies $u(x) = v(x)$ in Ω.

Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be continuous and $\omega(\cdot)$ be a modulus of continuity.

Let us assume that

$$A4: \quad |H(x,u,p) - H(y,u,p)| \leq \omega(|x-y|(1+|p|))Q_R(x,y,u,p)$$

for any $x, y \in \Omega$, $-R \leq u \leq R$ and $p \in \mathbb{R}^n$, where

$$Q_R(x,y,u,p) \equiv \max\left(\varphi(H(x,u,p)), \varphi\left(H(y,u,p)\right)\right)$$

THEOREM

Let the assumptions (A1–A4) be satisfied, then the comparison principle holds for (HJ), i.e. the viscosity solution is unique.

The goal is to prove that $M = \max_{x \in \overline{\Omega}} (u - v)$ is negative.

We introduce a test function depending on two variables

$$\psi_\varepsilon(x, y) \equiv u(x) - v(y) - \frac{|x - y|^2}{\varepsilon^2}.$$

Due to the penalization term, we can expect that the maximum points $(x_\varepsilon, y_\varepsilon)$ for ψ_ε should have x_ε and y_ε close enough for ε small. Moreover, for $\varepsilon \to 0^+$ we have:

$$M_\varepsilon \to M, \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \text{ and } u(x_\varepsilon) - v(y_\varepsilon) \to M$$

Those properties allow to pass to the limit and get the comparison result.

Another peculiar point is the way boundary conditions are satisfied.

One can consider Dirichlet boundary conditions, Neumann boundary conditions and "state constraints" boundary conditions.
The typical compact form for boundary conditions is

$$\min(H(x,u(x),Du(x)),B(x,u(x),Du(x))) \leq 0 \qquad \text{on } \partial\Omega$$

$$\max(H(x,u(x),Du(x)),B(x,u(x),Du(x))) \geq 0 \qquad \text{on } \partial\Omega$$

where B represents the "boundary" operator.
The technical point is that the equation plays a role up to the boundary.

Examples of Boundary conditions

For example, the two classical Dirichlet and Neumann boundary conditions correspond to:

The typical compact form for boundary conditions is

$$B(x, u(x), Du(x))) \equiv u - g \qquad \text{on } \partial\Omega$$

$$B(x, u(x), Du(x))) \equiv \frac{\partial u}{\partial \eta} - g \qquad \text{on } \partial\Omega$$

WARNING: not all the boundary conditions are compatible with the equations.

Viscosity solutions for evolutive 1st order PDEs

A typical example is the evolutive equation related to the level-set formulation of a front propagating in the normal direction with a (known) velocity:

$$\begin{cases} u_t + c(x)|Du| = 0 & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \tag{3}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ must be a proper representation of the front Γ_0.

This means that u_0 has to change sign on $\Gamma \equiv \partial\Omega_0$

$$\begin{cases} u_0(x) < 0 & \text{in } \Omega_0^- \\ u_0(x) = 0 & \text{in } \Gamma \\ u_0(x) > 0 & \text{in } \Omega_0^+ \end{cases}$$

so that the front at time 0 is identified as the 0-level set of u_0.

Definition of viscosity solutions (evolutive case)

DEFINITION

$u \in BUC(\Omega \times (0,T))$ is a viscosity solution of

$$u_t + H(x,u,Du) = 0 \text{in } \Omega$$

if and only if, for any $\varphi \in C^1(\Omega \times (0,T))$ the following conditions hold:

i) at every local maximum point $(x_0,t_0) \in \Omega \times (0,T)$ for $u - \varphi$

$$\varphi_t(x_0,t_0) + (H(x_0,u(x_0,t_0),D\varphi(x_0,t_0)) \leq 0$$

i.e. u is a viscosity sub-solution.

ii) at every local minimum point $(x_0,t_0) \in \Omega \times (0,T)$ for $u - \varphi$

$$\varphi_t(x_0,t_0) + (H(x_0,u(x_0,t_0),D\varphi(x_0,t_0)) \geq 0$$

i.e. u is a viscosity super-solution.

It is interesting to note that in some cases we have a representation formula for the viscosity solution.
For example, let us consider the problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{EHJ}$$

where $H(\cdot)$ is continuous and convex and

$$\lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty \qquad \text{coercivity}$$

DEFINITION (Legendre-Fenchel conjugate)

$$H^*(p) = \sup_{q\in\mathbb{R}^n} \{p \cdot q - H(q)\} \text{ for } p \in \mathbb{R}^n$$

PROPERTIES

- $H^* : \mathbb{R}^n \to \mathbb{R}$
- $H = H^{**}$

A typical example is the following

$$H_2(p) = \frac{|p|^2}{2} \text{ for which } H_2^*(p) = \frac{|p|^2}{2}.$$

A similar construction works also for the case

$$H_1(p) = |p| \quad \text{(WARNING: } H_1 \text{ is not coercive!)}$$

In this case the Legendre-Fenchel conjugate is not defined everywhere, in fact

$$H_1^*(p) = \begin{cases} 0 & \text{for } |p| \leq 1 \\ +\infty & \text{elsewhere} \end{cases}$$

The unique viscosity solution of (EHJ) is given by the Hopf-Lax representation formula

$$v(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ v_0(y) + tH^* \left(\frac{x-y}{t} \right) \right\}$$

which after the change of variable $a = \frac{x-y}{t}$ can also be written as

$$v(x,t) = \inf_{a \in \mathbb{R}^n} \{ v_0(x - ta) + tH^*(a) \}$$

Another interesting remark is the link between the entropy solutions and viscosity solutions.

This link is only valid in $\mathbb{R}$. Consider the two problems, the evolutive Hamilton-Jacobi equation

$$\begin{cases} v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \mathbb{R} \end{cases} \tag{HJ}$$

and the associated conservation law

$$\begin{cases} u_t + H(u)_x = 0 & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \tag{CL}$$

Assume that

$$v_0(x) \equiv \int_{-\infty}^{x} u_0(\xi) d\xi$$

If u is the entropy solution of (CL), then

$$v(x,t) = \int_{-\infty}^{x} u(\xi, t) d\xi$$

is the unique viscosity solution of (HJ).
Viceversa, let v be the viscosity solution of (HJ), then $u = v_x$ is the unique entropy solution for (CL) (note that v is a.e. differentiable).
This link will be also useful for numerical purposes.

In several applications, e.g. to image processing and to games, it is natural to look for discontinous solutions. This can be done in the framework of viscosity solutions.

ENVELOPES

Let $z : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a bounded function.

Lower semi-continuous envelope

$$z_*(x) \equiv \liminf_{y \longrightarrow x} z(y)$$

Upper semi-continuous envelope

$$z^*(x) \equiv \limsup_{y \longrightarrow x} z(y)$$

DEFINITION

$u \in B(\Omega)$ is a viscosity solution of

$$H(x, u, Du) = 0 \text{ in } \Omega$$

if and only if, for any $\varphi \in C^1(\Omega)$ the following conditions hold:

i) at every local maximum point $x_0 \in \Omega$ for $u^* - \varphi$

$$H(x_0, u^*(x_0), D\varphi(x_0)) \leq 0$$

i.e. $u^*$is a viscosity sub-solution.

ii) at every local minimum point $x_0 \in \Omega$ for $u_* - \varphi$

$$H(x_0, u_*(x_0), D\varphi(x_0)) \geq 0$$

i.e. u_* is a viscosity super-solution.

Viscosity solutions for evolutive 2nd order PDEs

The two fully non linear Hamilton-Jacobi equations are in this case

$$H(x, u, Du, D^2u) = 0 \text{ in } \Omega$$

$$u_t + H(x, u, Du, D^2u) = 0 \text{ in } \Omega \times (O, T)$$

Naturally, one can deal with easier problems like

$$u_t - \Delta u + H(Du) = 0 \text{ in } \Omega \times (O, T)$$

$$u_t + \text{div}\left(\frac{Du}{|Du|}\right)|Du| = 0 \text{ in } \Omega \times (O, T)$$

Let us consider the Dirichlet problem

$$\begin{cases} u + H(Du, D^2u) = f & \text{in } \Omega, \\ u(x) = g(x) \text{ on } \partial\Omega \end{cases} \tag{HJ$_2$}$$

where $f, g \in C^0(\overline{\Omega})$.

THEOREM
Let $H(p, X)$ be continuous and (degenerate) elliptic.
Assume that :

1. u is an u.s.c. subsolution of (HJ$_2$)
2. v is a l.s.c. super-solution of (HJ$_2$)
3. $u \leq v$ on $\partial\Omega$.

Then, $u \leq v$ in Ω.

Note that the usual proof by variable doubling does NOT work.

In fact, if we define

$$\psi(x,y) \equiv u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}$$

and $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ for small ε. By the definitions of sub and super solutions (in the viscosity sense) we get

$$u(x_\varepsilon) + H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \frac{1}{\varepsilon}I\right) \le f(x_\varepsilon)$$

$$v(y_\varepsilon) + H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -\frac{1}{\varepsilon}I\right) \ge f(y_\varepsilon)$$

which implies

$$u(x_\varepsilon) - v(y_\varepsilon) \leq f(x_\varepsilon) - f(y_\varepsilon) + H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \frac{1}{\varepsilon}I\right) - H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -\frac{1}{\varepsilon}I\right)$$

and we cannot end the proof since $I \geq -I$ implies

$$H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, \frac{1}{\varepsilon}I\right) - H\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, -\frac{1}{\varepsilon}I\right) > 0.$$

so that in the limit we will not get

$$u(x) \leq v(x)$$

Let the comparison result (maximum principle) hold for (HJ_2). The following result gives the existence of viscosity solutions always in terms of "maximal subsolution".

THEOREM

Assume that there exist a sub-solution $\underline{u}$ and a super-solution $\overline{u}$ of (HJ_2) satisfying

$$\underline{u}_*(x) = \overline{u}^*(x) = g(x), \text{ for any } x \in \partial\Omega$$

Let S be the set of sub-solutions of (HJ_2).

Then,

$$W(x) = \sup_{w \in S}\{w(x) : \underline{u} \leq w(x) \leq \overline{u}\}$$

is a viscosity solution of (HJ_2).

The theory of viscosity solutions gives a good framework for the analysis of nonlinear PDEs of first and second order.

Existence and uniqueness results have been proved under very general hypotheses.

Applications of this theory range from control problems and games, to front propagation and image processing.

Basic references

GENERAL THEORY

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C. Evans, *Partial Differential Equations*, American Mathematical Society, 1999.

VISCOSITY SOLUTIONS AND CONTROL THEORY

M. Bardi, I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, 1997.

W.H. Fleming, H.M. Soner, *Controlled Markov processes and viscosity solutions*, Springer–Verlag, New York, 1993.

SOME RECENT DEVELOPMENTS

Recent papers by L. Caffarelli and X. Cabrè deal with fully nonlinear 2nd order equations.