An Introduction to Viscosity Solutions: theory, numerics and applications

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OUTLINE OF THE COURSE:

- Lecture 1: Introduction to viscosity solutions
- Lecture 2: Approximation schemes for viscosity solutions
- Lecture 3: Approximation of optimal control problems via DP
- Lecture 4: Efficient methods and perspectives

OUTLINE OF THIS LECTURE:
Approximation schemes for viscosity solutions

- Approximation schemes for 1st order PDEs
- Finite difference
- A general convergence results for monotone schemes
- Semi-Lagrangian schemes
- Convergence for SL schemes
- HJPACK: a public domain library

Computing viscosity solutions

Viscosity solutions are typically uniformly continuous and bounded.

As we will see also discontinuous solution can be considered in the framework of this theory.

This means that the numerical methods should be able to reconstruct kinks in the solution and, possibly, jumps.

The main goals are:

- consistency
- stability
- convergence
- small "numerical viscosity"

Moreover, the schemes should also be able to compute solutions after the onset of singularities/jumps without producing spurious oscillations.

Model problem: 1st order PDEs

Let us consider the evolutive equation in $\mathbb{R}$.

$$\begin{cases} v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0,T), \\ v(x,0) = v_0(x) & \text{in } \mathbb{R} \end{cases} \tag{HJ}$$

where $H(\cdot)$ is convex.
We know the relation between (HJ) and the associated conservation law. Roughly speaking, the entropy solution of (CL) is the "derivative" of the viscosity solution.

Typical approximation methods in this framework

Due to the hyperbolic nature of these equations and to their nonlinearity the most popular methods are

- Finite Differences schemes
- Semi-Lagrangian schemes
- Discontinuous Galerkin methods

We restrict our presentation to the first two.

An important source of FD approximation schemes is the huge collection of approximation schemes developed for conservation laws.

In fact, we can always integrate in space a scheme for (CL) and obtain a suitable scheme for (HJ).

The general discrete relation is

$$U_i^n = \frac{V_{i+1}^n - V_i^n}{\Delta x}.$$

which implies

$$V_{i+1}^n = V_i^n + U_i^n \Delta x$$

with the usual notation $U_i^n \approx u(x_i, t_n)$.

HJ equation and Conservation Laws in $\mathbb{R}$

The link between the entropy solutions and viscosity solutions is only valid in $\mathbb{R}$.
Consider the two problems

$$\begin{cases} v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0,T), \\ v(x,0) = v_0(x) & \text{in } \mathbb{R} \end{cases} \tag{HJ}$$

and the associated conservation law

$$\begin{cases} u_t + H(u)_x = 0 & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases} \tag{CL}$$

Assume that

$$v_0(x) \equiv \int_{-\infty}^{x} u_0(\xi)d\xi$$

If u is the entropy solution of (CL), then

$$v(x,t) = \int_{-\infty}^{x} u(\xi,t)d\xi$$

is the unique viscosity solution of (HJ).
Viceversa, let v be the viscosity solution of (HJ), then $u = v_x$ is the unique entropy solution for (CL).
Note that v is typically Lipschitz continuous so it is a.e. differentiable.

Using the standard notation, V_i^n denotes the numerical approximation at $(x_i, t_n) = (i\Delta x, n\Delta t)$.
We use capital letters $U, V, ...$ for the solution on the x lattice $L = \{(x_i : i \in \mathbb{Z}\}$ and their values at x_i will be denoted by $U_i, V_i, ...$
Hence V^n represents the numerical solution at the level time $n\Delta t$ as a function of the values $V_{i,j}^n$ on L.

NOTATIONS

$$\lambda = \frac{\Delta t}{\Delta x} \quad \text{and} \quad \Delta_x^{\pm} V_{i,j} = V_{i\pm 1,j} - V_{i,j}.$$

The numerical scheme for (HJ) we are interested in have the following explicit form

$$V_{i,j}^{n+1} = G(V_{i-p,j-r}^{n}, \dots, V_{i+q+1,j+s+1}^{n}), \qquad \text{(FD)}$$

where p, q, r, s are fixed nonnegative integers determining the stencil of the scheme and G is a function of $(p+q+2)(r+s+2)$ variables.

Scheme in differenced form

DEFINITION

The scheme (FD) has differenced form if there exists a function g such that

$$G(V_{i-p,j-r},\dots,V_{i+q+1,j+s+1}) =$$

$$V_{i,j} - \Delta t g\left(\frac{\Delta_+^x}{\Delta x}V_{i-p,j-r},..,\frac{\Delta_+^x}{\Delta x}V_{i+q,j+s+1};\frac{\Delta_+^y}{\Delta y}V_{i-p,j-r},..,\frac{\Delta_+^y}{\Delta y}V_{i+q+1,j+s}\right)$$

g is called the numerical Hamiltonian of the scheme (FD).

Note that a scheme in conservation form for (CL) produces a scheme in differenced form for (HJ).

DEFINITION

The scheme (FD) is said to be consistent with the the equation

$$v_t + H(v_x, v_y) = 0$$

when

$$g(a, \ldots, a; b, \ldots, b) = H(a, b) \quad \text{for } a, b \in \mathbb{R};$$

DEFINITION

The scheme (FD) is said to be monotone on $[-R, R]$ if

$$G(V_{i-p,j-r}, \ldots, V_{i+q+1,j+s+1})$$

is a nondecreasing function of each argument as long as

$$|\Delta_+^x V_{l,m}|/\Delta x,\ |\Delta_+^y V_{l',m'}|/\Delta y \leq R$$

for $i-p \leq l \leq i+q,\quad j-r \leq m \leq j+s+1,\quad i-p \leq l' \leq i+q+1,\ \ j-r \leq m' \leq j+s$

Roughly speaking, R is an a priori bound on $|v_x|, |v_y|$ for the solution of (HJ).

We now give some examples with $N = 1$ where it is shown the correspondence between this kind of schemes and some schemes for conservation laws.

Let us start by considering the scheme

$$V_i^{n+1} = V_i^n - \Delta t \left\{ H\left(\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x}\right) - \frac{\theta}{\lambda^x} \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x} \right\},$$

where $\theta > 0$ is given.

By using the notation Δ_+^x we get its *differenced form*:

$$V_i^{n+1} = V_i^n - \Delta t \left\{ H\left(\frac{\Delta_+^x V_i^n + \Delta_+^x V_{i-1}^n}{2\Delta x}\right) - \frac{\theta}{\lambda^x} \left(\frac{\Delta_+^x V_i^n - \Delta_+^x V_{i-1}^n}{\Delta x}\right) \right\}.$$

The numerical Hamiltonian is given by

$$g(\alpha,\beta) = H\left(\frac{\alpha+\beta}{2}\right) - (\beta-\alpha)\frac{\theta}{\lambda^x} \quad \text{for } \alpha,\beta \in \mathbb{R}. \tag{LF}$$

As it is easy to check $g(\alpha,\alpha) = H(\alpha)$, hence the scheme is consistent.

It is also monotone on $[-R,R]$ if $1-2\theta \geq 0$ (monotonicity in V_i^n), and $\theta - \lambda^x|H'(\alpha)|/2 \geq 0$ for $|\alpha| \leq R$ (monotonicity in V_{i+1}^n, V_{i-1}^n).

We obtain these two relations by first choosing $0 < \theta < 1/2$ and then λ^x sufficiently small.

This scheme corresponds to the Lax-Friedrichs scheme for conservation laws.

The two “upwind” schemes

$$V_i^{n+1} = V_i^n - \Delta t \, H\left(\frac{V_{i+1}^n - V_i^n}{\Delta x}\right), \qquad \text{(UW}-\text{)}$$

$$V_i^{n+1} = V_i^n - \Delta t \, H\left(\frac{V_i^n - V_{i-1}^n}{\Delta x}\right), \qquad \text{(UW}+\text{)}$$

have the requested monotonicity properties provided

H is nonincreasing for forward differences, i.e. (UW−)
H is nondecreasing for backward difference, i.e. (UW+)

and $1 \geq \lambda^x |H'(\alpha)|$ for $|\alpha| \leq R$.

Convergence for monotone FD schemes

A general convergence result for monotone schemes in $\mathbb{R}$ has been obtained by Crandall and Lions (1984).

Every FD scheme in differenced form for (EHJ) which is also monotone and consistent will converge in the L^∞ norm to the viscosity solution

Unfortunately, monotone schemes have at most rate of convergence 1 !

FINITE DIFFERENCES

M.G. Crandall, P.L. Lions, *Two approximations of solutions of Hamilton-Jacobi equations*, Mathematics of Computation **43** (1984), 1-19.

J.A. Sethian, Level Set Method. Evolving interfaces in geometry, fluid mechanics, computer vision, and materials science, Cambridge Monographs on Applied and Computational Mathematics, vol. 3, Cambridge University Press, Cambridge, 1996.

S. Osher, R.P. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, Springer–Verlag, New York, 2003.

Semi-Lagrangian schemes

These schemes are based on a different idea which is to discretize directly the ”directional derivative” which is hidden behind the nonlinearity of the hamiltonian $H(Du)$.

The goal is to mimic the method of characteristics by constructing the solution at each grid point integrating back along the characteristics passing through it and reconstructing the value at the foot of the characteristic line by interpolation.

Let us start, writing the equation

$$u_t + c(x)|Du(x)| = 0 \qquad (LS - HJ)$$

as

$$u_t + \max_{a \in B(0,1)} [-c(x)a \cdot Du(x)] = 0$$

Naturally, this is equivalent to our equation since plugging

$$a* = \frac{Du(x)}{|Du(x)|}$$

we get back to the first equation.

Let us examine now the typical SL-scheme for $N = 2$.
Define the lattice $L(\Delta x, \Delta y, \Delta t)$ by

$$L \equiv \{(x_i, y_j, t_n) : x_i = i\Delta x, y_j = j\Delta y \text{ and } t_n = n\Delta t\}$$

where $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}$, $(x_i, y_j, t_n) \in \mathbb{R}^2 \times \mathbb{R}^+$, Δx and Δy are the space steps and Δt is the time step.

Approximate Directional Derivatives

In order to obtain the SL-scheme let us consider the following approximation for $\delta > 0$

$$-a \cdot Dv(x_i, y_j, t_n) = \frac{v(x_i - a_1\delta, y_j - a_2\delta, t_n) - v(x_i, y_j, t_n)}{\delta} + O(\delta)$$

We will use the standard notation

$$V_{ij}^n \approx v(x_i, y_j, t_n), \quad i, j \in \mathbb{Z} \text{ and } n \in \mathbb{N}.$$

Since the the point $(x_i - a_1\delta, y_j - a_2\delta)$ will not be nodes of the lattice L we need to extend the solution everywhere in space by interpolation $V^n : \mathbb{R}^2 \to \mathbb{R}$.

This extension will allow to compute V^n at any triple (x, y, t_n).

Semi-Lagrangian schemes for level set equation

Replacing in (LS-HJ) the term v_t by forward finite differences (to have an explicit scheme) and the approximate directional derivative, we get

$$\frac{V_{ij}^{n+1} - V_{ij}^{n}}{\Delta t} = \min_{a \in \mathbb{R}^2} \left[\frac{V^n(x_i - a_1 \delta c_{ij}, y_j - a_2 \delta c_{ij}) - V_{ij}^n}{\delta} \right]$$

For $\delta = \Delta t$ some terms cancel and we obtain

$$V_{ij}^{n+1} = \min_{a \in \mathbb{R}^2} \left[V^n(x_i - a_1 \Delta t c_{ij}, y_j - a_2 \Delta t c_{ij}) \right]$$

By construction, the numeric dependence domain contains the continuous dependence domain without any additional conditions on Δt and Δx.
This allows larger time steps than FD where a stability condition (CFL condition) of the type

$$\frac{\Delta t}{\Delta x} C \leq 1$$

HEURISTICS
For FD scheme one has to be imposed because we FIRST fix the stencil so that we are not allowed to cross cells.

The SL scheme has a variable stencil which corresponds to the nodes of the cell containing the foot of the characteristics.

Semi-Lagrangian schemes for general Hamiltonians

Note that for a general Hamiltonian $H(Du)$, replacing in (HJ) the term v_t by forward finite differences and the directional derivative, we get

$$\frac{V_{ij}^{n+1} - V_{i,j}^{n}}{\Delta t} = \min_{a \in \mathbb{R}^2} \left[\frac{V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) - V_{ij}^{n}}{\Delta t} + H^*(a) \right]$$

Semi-Lagrangian schemes for general Hamiltonians

Finally, we can write the time explicit scheme

$$V_{i,j}^{n+1} = \min_{a \in \mathbb{R}^2} \left[V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \Delta t H^*(a) \right] \qquad \text{(SL)}$$

Note that the SL-schemes has the same structure of the Hopf-Lax representation formula of the exact solution written for $v_0 = V^n$ and $t = \Delta t$.

Semi-Lagrangian schemes

In fact, taking the Hopf formula

$$v(x,t) = \inf_{y \in \mathbb{R}^n} \left[v_0(y) + tH^* \left(\frac{x-y}{t} \right) \right]$$

and taking

$$a = \frac{x-y}{t} \text{ and } t = \Delta t$$

we immediately get

$$V_{i,j}^{n+1} = \min_{a \in \mathbb{R}^2} \left[V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \Delta t H^*(a) \right] \qquad \text{(SL)}$$

Building blocks for Semi-Lagrangian schemes

Several steps are necessary in order to compute the solution.

STEP 1: ODE INTEGRATION

Compute the foot of the characteric (low order/high order)

STEP 2: INTERPOLATION

Compute the value of v on the right-hand side by an interpolation procedure based on the values on the nodes of the lattice L (low order/high order);

STEP 3: LEGENDRE TRASFORM

Then, one has to determine $H^*(a)$;

STEP 4: OPTIMIZATION

compute the minimum for $a \in \mathbb{R}^2$.

Note that the first two steps can be improved using high-order methods.

A major difficulty when applying (SL) is to compute H^*.
Sometimes it is possible to determine its explicit expression (as we made for H_1 and H_2).

In general one has to rely on its approximation by the fast Legendre transform developed by Brenier and Corrias (1996).
The FLT solution is feasible as far as the state space has 2 dimensions.

WARNING: this is NOT a major problem for control problems since they already have an Hamiltonian of the form
$H(x, u, Du) = u + \max_a \{-b(x, a) \cdot Du(x) - f(x, a)\}$.

Δt	Δx	error L^∞	error L^1	error L^2
0.05	0.1	(SL) 7.50×10^{-2} (DU) 7.50×10^{-2} (LW) 4.78×10^{-2} (LF) 9.12×10^{-2}	(SL) 3.28×10^{-2} (DU) 3.28×10^{-2} (LW) 2.26×10^{-2} (LF) 4.36×10^{-2}	(SL) 3.56×10^{-2} (DU) 3.56×10^{-2} (LW) 2.38×10^{-2} (LF) 4.15×10^{-2}
0.025	0.05	(SL) 4.46×10^{-2} (DU) 4.46×10^{-2} (LW) 2.28×10^{-2} (LF) 5.44×10^{-2}	(SL) 1.69×10^{-2} (DU) 1.69×10^{-2} (LW) 1.15×10^{-2} (LF) 2.19×10^{-2}	(SL) 2.12×10^{-2} (DU) 2.12×10^{-2} (LW) 1.23×10^{-2} (LF) 2.10×10^{-2}
0.0125	0.025	(SL) 2.24×10^{-2} (DU) 2.24×10^{-2} (LW) 1.13×10^{-2} (LF) 2.67×10^{-2}	(SL) 9.56×10^{-3} (DU) 9.56×10^{-3} (LW) 6.21×10^{-3} (LF) 1.06×10^{-2}	(SL) 1.14×10^{-2} (DU) 1.14×10^{-2} (LW) 7.45×10^{-3} (LF) 1.12×10^{-2}
0.00625	0.0125	(SL) 1.12×10^{-2} (DU) 1.12×10^{-2} (LW) 8.35×10^{-3} (LF) 1.37×10^{-2}	(SL) 4.94×10^{-3} (DU) 4.94×10^{-3} (LW) 2.92×10^{-3} (LF) 5.49×10^{-3}	(SL) 7.44×10^{-3} (DU) 7.44×10^{-3} (LW) 3.55×10^{-3} (LF) 6.63×10^{-3}

Numerical errors for $u_t + 0.5|u_x| = 0,\ u_0(x) = 1 - x^2$

Convergence for first order SL schemes for stationary and evolutive HJ equations have been proved by F. (1987,...), F.-Giorgi (1998).

A convergence result for high-order schemes in $\mathbb{R}$ has been proved by Ferretti (2004) for the scheme

$$V_j^n = \min_{a\in\mathbb{R}}\{I[V^{n-1}](x_j - a\Delta t)\} + \Delta t H^*(a)]$$

where $I[\cdot]$ is a generic interpolation operator on the grid.

THEOREM

Assume that

(A1) $H'' \geq m_H > 0$

(A2) $\Delta x = O(\Delta t^2)$

$$|I[v](x) - I_1[v](x)| \leq C \max |V_{j-1} - 2V_j + V_{j+1}| \text{ for } C < 1$$

where I_1 is linear interpolation operator.

Then, the scheme converges in $L^\infty(\Omega)$.

The HJPACK Library

HJPACK is a public domain library for Hamilton-Jacobi equations. It includes

- finite difference schemes and semi-Lagrangian schemes in $\mathbb{R}$ and $\mathbb{R}^2$
- applications to control problems and front propagation
- fast-marching schemes
- a graphical interface and a user's guide

You can get it at www.caspur.it/hjpack.

SEMI-LAGRANGIAN SCHEMES

M. Falcone, T. Giorgi, *An approximation scheme for evolutive Hamilton–Jacobi equations*, in W.M. McEneaney, G. Yin, Q. Zhang (eds), "Stochastic analysis, Control, optimization and applications: a volume in honor of W.H. Fleming", Birkhäuser, 1998.

M. Falcone, R. Ferretti, *Semi-Lagrangian schemes for Hamilton-Jacobi equations discrete representation formulae and Godunov methods*, J. of Computational Physics, 175, (2002), 559-575.

Basic references

E. Carlini, R. Ferretti, G. Russo, *A Weno large time-step for Hamilton-Jacobi equation*, SIAM J. Sci. Comput.

M. Falcone, R. Ferretti, *Consistency of a large time-step scheme for mean curvature motion*, In F.Brezzi, A.Buffa, S.Corsaro and A.Murli (eds), Numerical Analysis and Advanced Applications-Proceedings on ENUMATH 2001, Ischia,(2001).

J. Strain, *Semi-Lagrangian methods for level set equations*, J. Comput. Phys. **151** (1999), 498-533.

M. Falcone, R. Ferretti, Semi-Lagrangian approximation schemes for linear and Hamilton-Jacobi equations, SIAM, to appear.

FINITE ELEMENTS

Very few papers available.
A Discontinuous Galerkin approach has been developed by B. Cockburn, C.W. Shu.
Recent contributions in the Ph.D. Thesis by Ch. Rasch (TU Munich).

RATE OF CONVERGENCE

G. Barles, E.R. Jakobsen, *On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations*, Math. Mod. Num. Anal. , **36** (2002), 33-54.

N. V. Krylov, *On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients*. Probab. Theory Related Fields 117 (2000), no. 1, 1–16.