An Introduction to Viscosity Solutions: theory, numerics and applications

M. Falcone
Dipartimento di Matematica

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OUTLINE OF THE COURSE:

- Lecture 1: Introduction to viscosity solutions
- Lecture 2: Approximation schemes for viscosity solutions
- Lecture 3: Approximation of optimal control problems via DP
- Lecture 4: Efficient methods and perspectives
OUTLINE OF THIS LECTURE:
Approximation schemes for viscosity solutions

- Approximation schemes for 1st order PDEs
- Finite difference
- A general convergence results for monotone schemes
- Semi-Lagrangian schemes
- Convergence for SL schemes
- HJPACK: a public domain library
Computing viscosity solutions

Viscosity solutions are typically uniformly continuous and bounded.

As we will see also discontinuous solution can be considered in the framework of this theory.

This means that the numerical methods should be able to re-construct kinks in the solution and, possibly, jumps.
Computing viscosity solutions

The main goals are:

- consistency
- stability
- convergence
- small "numerical viscosity"

Moreover, the schemes should also be able to compute solutions after the onset of singularities/jumps without producing spurious oscillations.
Let us consider the evolutive equation in $\mathbb{R}$.

\[
\begin{aligned}
&v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0, T), \\
v(x, 0) = v_0(x) & \text{in } \mathbb{R}
\end{aligned}
\]

(HJ)

where $H(\cdot)$ is convex.
We know the relation between (HJ) and the associated conservation law. Roughly speaking, the entropy solution of (CL) is the ”derivative” of the viscosity solution.
Typical approximation methods in this framework

Due to the hyperbolic nature of these equations and to their nonlinearity the most popular methods are

- Finite Differences schemes
- Semi-Lagrangian schemes
- Discontinuous Galerkin methods

We restrict our presentation to the first two.
Finite Difference schemes

An important source of FD approximation schemes is the huge collection of approximation schemes developed for conservation laws.
In fact, we can always integrate in space a scheme for (CL) and obtain a suitable scheme for (HJ).
The general discrete relation is

\[ U^n_i = \frac{V^n_{i+1} - V^n_i}{\Delta x}. \]

which implies

\[ V^n_{i+1} = V^n_i + U^n_i \Delta x \]

with the usual notation \( U^n_i \approx u(x_i, t_n) \).
HJ equation and Conservation Laws in \( \mathbb{R} \)

The link between the entropy solutions and viscosity solutions is only valid in \( \mathbb{R} \).

Consider the two problems

\[
\begin{align*}
\begin{cases}
  v_t + H(v_x) = 0 & \text{in } \mathbb{R} \times (0, T), \\
  v(x, 0) = v_0(x) & \text{in } \mathbb{R}
\end{cases}
\end{align*}
\]  

(HJ)

and the associated conservation law

\[
\begin{align*}
\begin{cases}
  u_t + H(u)_x = 0 & \text{in } \mathbb{R} \times (0, T), \\
  u(x, 0) = u_0(x) & \text{in } \mathbb{R}
\end{cases}
\end{align*}
\]  

(CL)
Assume that

\[ v_0(x) \equiv \int_{-\infty}^{x} u_0(\xi) d\xi \]

If \( u \) is the entropy solution of (CL), then

\[ v(x, t) = \int_{-\infty}^{x} u(\xi, t) d\xi \]

is the unique viscosity solution of (HJ).
Vice versa, let \( v \) be the viscosity solution of (HJ), then \( u = v_x \) is the unique entropy solution for (CL).
Note that \( v \) is typically Lipschitz continuous so it is a.e. differentiable.
Finite Difference Schemes

Using the standard notation, $V^n_i$ denotes the numerical approximation at $(x_i, t_n) = (i\Delta x, n\Delta t)$.

We use capital letters $U, V, ...$ for the solution on the $x$ lattice $L = \{(x_i : i \in \mathbb{Z}\}$ and their values at $x_i$ will be denoted by $U_i, V_i, ....$

Hence $V^n$ represents the numerical solution at the level time $n\Delta t$ as a function of the values $V^n_{i,j}$ on $L$.

NOTATIONS

$$\lambda = \frac{\Delta t}{\Delta x} \quad \text{and} \quad \Delta^\pm x V_{i,j} = V_{i\pm1,j} - V_{i,j}.$$
The numerical scheme for (HJ) we are interested in have the following explicit form

\[ V_{i,j}^{n+1} = G(V_{i-p,j-r}^n, \ldots, V_{i+q+1,j+s+1}^n), \quad \text{(FD)} \]

where \( p, q, r, s \) are fixed nonnegative integers determining the stencil of the scheme and \( G \) is a function of \((p+q+2)(r+s+2)\) variables.
DEFINITION
The scheme (FD) has **differenced form** if there exists a function $g$ such that

$$G(V_{i-p,j-r}, \ldots, V_{i+q+1,j+s+1}) =$$

$$V_{i,j} - \Delta t g \left( \frac{\Delta_x}{\Delta x} V_{i-p,j-r}, \ldots, \frac{\Delta_x}{\Delta x} V_{i+q,j+s+1}, \frac{\Delta_y}{\Delta y} V_{i-p,j-r}, \ldots, \frac{\Delta_y}{\Delta y} V_{i+q+1,j+s} \right)$$

$g$ is called the **numerical Hamiltonian** of the scheme (FD).
Note that a scheme in conservation form for (CL) produces a scheme in differenced form for (HJ).

**DEFINITION**

The scheme (FD) is said to be consistent with the equation

\[ v_t + H(v_x, v_y) = 0 \]

when

\[ g(a, \ldots, a; b, \ldots, b) = H(a, b) \quad \text{for } a, b \in \mathbb{R}; \]
DEFINITION

The scheme (FD) is said to be monotone on \([-R, R]\) if

\[G(V_{i-p,j-r}, \ldots, V_{i+q+1,j+s+1})\]

is a nondecreasing function of each argument as long as

\[|\Delta_x V_{l,m}|/\Delta x, |\Delta_y V_{l',m'}|/\Delta y \leq R\]

for \(i - p \leq l \leq i + q, \ j - r \leq m \leq j + s + 1, \ i - p \leq l' \leq i + q + 1, \ j - r \leq m' \leq j + s\)

Roughly speaking, \(R\) is an a priori bound on \(|v_x|, |v_y|\) for the solution of (HJ).
We now give some examples with $N = 1$ where it is shown the correspondence between this kind of schemes and some schemes for conservation laws.

Let us start by considering the scheme

$$V_{i}^{n+1} = V_{i}^{n} - \Delta t \left\{ H \left( \frac{V_{i+1}^{n} - V_{i-1}^{n}}{2\Delta x} \right) - \frac{\theta}{\lambda x} \frac{V_{i+1}^{n} - 2V_{i}^{n} + V_{i-1}^{n}}{\Delta x} \right\},$$

where $\theta > 0$ is given.

By using the notation $\Delta x_+$ we get its \textit{differenced form}:

$$V_{i}^{n+1} = V_{i}^{n} - \Delta t \left\{ H \left( \frac{\Delta x_+ V_{i}^{n} + \Delta x_+ V_{i-1}^{n}}{2\Delta x} \right) - \frac{\theta}{\lambda x} \left( \frac{\Delta x_+ V_{i}^{n} - \Delta x_+ V_{i-1}^{n}}{\Delta x} \right) \right\}.$$
The numerical Hamiltonian is given by

\[ g(\alpha, \beta) = H \left( \frac{\alpha + \beta}{2} \right) - (\beta - \alpha) \frac{\theta}{\lambda x} \quad \text{for } \alpha, \beta \in \mathbb{R}. \quad \text{(LF)} \]

As it is easy to check \( g(\alpha, \alpha) = H(\alpha) \), hence the scheme is consistent.

It is also monotone on \([-R, R]\) if \(1 - 2\theta \geq 0\) (monotonicity in \(V^n_i\)), and \(\theta - \lambda x |H'(\alpha)|/2 \geq 0\) for \(|\alpha| \leq R\) (monotonicity in \(V^n_{i+1, i-1}\)).

We obtain these two relations by first choosing \(0 < \theta < 1/2\) and then \(\lambda^x\) sufficiently small.

This scheme corresponds to the Lax-Friedrichs scheme for conservation laws.
Up-Wind scheme

The two “upwind” schemes

\[
V_{i+1}^{n+1} = V_i^n - \Delta t H \left( \frac{V_{i+1}^n - V_i^n}{\Delta x} \right), \quad (UW-)
\]

\[
V_{i-1}^{n+1} = V_i^n - \Delta t H \left( \frac{V_i^n - V_{i-1}^n}{\Delta x} \right), \quad (UW+)
\]

have the requested monotonicity properties provided

- \( H \) is nonincreasing for forward differences, i.e. \((UW-)
- \( H \) is nondecreasing for backward difference, i.e. \((UW+)

and \( 1 \geq \lambda^x |H'(\alpha)| \) for \( |\alpha| \leq R \).
A general convergence result for monotone schemes in $\mathbb{R}$ has been obtained by Crandall and Lions (1984).

Every FD scheme in differenced form for (EHJ) which is also monotone and consistent will converge in the $L^\infty$ norm to the viscosity solution.

Unfortunately, monotone schemes have at most rate of convergence 1!
References

FINITE DIFFERENCES


Semi-Lagrangian schemes

These schemes are based on a different idea which is to discretize directly the "directional derivative" which is hidden behind the nonlinearity of the hamiltonian $H(Du)$.

The goal is to mimic the method of characteristics by constructing the solution at each grid point integrating back along the characteristics passing through it and reconstructing the value at the foot of the characteristic line by interpolation.
Let us start, writing the equation

\[ u_t + c(x)|Du(x)| = 0 \quad (LS - HJ) \]

as

\[ u_t + \max_{a \in B(0,1)} [-c(x)a \cdot Du(x)] = 0 \]

Naturally, this is equivalent to our equation since plugging

\[ a^* = \frac{Du(x)}{|Du(x)|} \]

we get back to the first equation.
Semi-Lagrangian schemes

Let us examine now the typical SL-scheme for $N = 2$. Define the lattice $L(\Delta x, \Delta y, \Delta t)$ by

$$L \equiv \{(x_i, y_j, t_n) : x_i = i\Delta x, y_j = j\Delta y \text{ and } t_n = n\Delta t\}$$

where $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}$, $(x_i, y_j, t_n) \in \mathbb{R}^2 \times \mathbb{R}^+$, $\Delta x$ and $\Delta y$ are the space steps and $\Delta t$ is the time step.
Approximate Directional Derivatives

In order to obtain the SL-scheme let us consider the following approximation for $\delta > 0$

$$-a \cdot Dv(x_i, y_j, t_n) = \frac{v(x_i - a_1 \delta, y_j - a_2 \delta, t_n) - v(x_i, y_j, t_n)}{\delta} + O(\delta)$$

We will use the standard notation

$$V_{ij}^n \approx v(x_i, y_j, t_n), \quad i, j \in \mathbb{Z} \text{ and } n \in \mathbb{N}.$$

Since the point $(x_i - a_1 \delta, y_j - a_2 \delta)$ will not be nodes of the lattice $L$ we need to extend the solution everywhere in space by interpolation $V^n : \mathbb{R}^2 \rightarrow \mathbb{R}$.

This extension will allow to compute $V^n$ at any triple $(x, y, t_n)$. 
Semi-Lagrangian schemes for level set equation

Replacing in (LS-HJ) the term $v_t$ by forward finite differences (to have an explicit scheme) and the approximate directional derivative, we get

$$\frac{V_{ij}^{n+1} - V_{ij}^n}{\Delta t} = \min_{a \in \mathbb{R}^2} \left[ \frac{V^n(x_i - a_1 \delta c_{ij}, y_j - a_2 \delta c_{ij}) - V_{ij}^n}{\delta} \right]$$

For $\delta = \Delta t$ some terms cancel and we obtain

$$V_{ij}^{n+1} = \min_{a \in \mathbb{R}^2} \left[ V^n(x_i - a_1 \Delta t c_{ij}, y_j - a_2 \Delta t c_{ij}) \right]$$
Stability

By construction, the numeric dependence domain contains the continuous dependence domain without any additional conditions on $\Delta t$ and $\Delta x$. This allows larger time steps than FD where a stability condition (CFL condition) of the type

$$\frac{\Delta t}{\Delta x} C \leq 1$$

**HEURISTICS**

For FD scheme one has to be imposed because we FIRST fix the stencil so that we are not allowed to cross cells.

The SL scheme has a variable stencil which corresponds to the nodes of the cell containing the foot of the characteristics.
Note that for a general Hamiltonian $H(Du)$, replacing in (HJ) the term $v_t$ by forward finite differences and the directional derivative, we get

$$
\frac{V_{ij}^{n+1} - V_{ij}^n}{\Delta t} = \min_{a \in \mathbb{R}^2} \left[ \frac{V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) - V_{ij}^n}{\Delta t} + H^*(a) \right]
$$
Semi-Lagrangian schemes for general Hamiltonians

Finally, we can write the time explicit scheme

\[ V_{i,j}^{n+1} = \min_{a \in \mathbb{R}^2} \left[ V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \Delta t H^*(a) \right] \quad (SL) \]

Note that the SL-schemes has the same structure of the Hopf-Lax representation formula of the exact solution written for \( v_0 = V^n \) and \( t = \Delta t \).
In fact, taking the Hopf formula

\[ v(x, t) = \inf_{y \in \mathbb{R}^n} \left[ v_0(y) + tH^* \left( \frac{x - y}{t} \right) \right] \]

and taking

\[ a = \frac{x - y}{t} \text{ and } t = \Delta t \]

we immediately get

\[
V_{n+1}^{i,j} = \min_{a \in \mathbb{R}^2} \left[ V^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \Delta t H^*(a) \right] \quad \text{(SL)}
\]
Building blocks for Semi-Lagrangian schemes

Several steps are necessary in order to compute the solution.

**STEP 1: ODE INTEGRATION**
Compute the foot of the characteristic (low order/high order)

**STEP 2: INTERPOLATION**
Compute the value of $v$ on the right-hand side by an interpolation procedure based on the values on the nodes of the lattice $L$ (low order/high order);
Building blocks for Semi-Lagrangian schemes

STEP 3: LEGENDRE TRANSFORM
Then, one has to determine $H^*(a)$;

STEP 4: OPTIMIZATION
compute the minimum for $a \in \mathbb{R}^2$.

Note that the first two steps can be improved using high-order methods.
Fast Legendre Transform (FLT)

A major difficulty when applying (SL) is to compute $H^*$. Sometimes it is possible to determine its explicit expression (as we made for $H_1$ and $H_2$).

In general one has to rely on its approximation by the fast Legendre transform developed by Brenier and Corrias (1996). The FLT solution is feasible as far as the state space has 2 dimensions.

**WARNING:** this is NOT a major problem for control problems since they already have an Hamiltonian of the form

$$H(x, u, Du) = u + \max_a \{-b(x, a) \cdot Du(x) - f(x, a)\}.$$
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<th>$\Delta t$</th>
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<th>error $L^\infty$</th>
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</table>

Numerical errors for $u_t + 0.5|u_x| = 0$, $u_0(x) = 1 - x^2$
Convergence for first order SL schemes for stationary and evolu-tive HJ equations have been proved by F. (1987,...), F.-Giorgi (1998).

A convergence result for high-order schemes in $\mathbb{R}$ has been proved by Ferretti (2004) for the scheme

$$V_j^n = \min_{a \in \mathbb{R}} \{ I[V^{n-1}](x_j - a\Delta t) \} + \Delta t H^*(a)$$

where $I[\cdot]$ is a generic interpolation operator on the grid.
Convergence for SL schemes

THEOREM
Assume that
(A1) $H'' \geq m_H > 0$
(A2) $\Delta x = O(\Delta t^2)$

$$|I[v](x) - I_1[v](x)| \leq C \max |V_{j-1} - 2V_j + V_{j+1}| \text{ for } C < 1$$

where $I_1$ is linear interpolation operator.

Then, the scheme converges in $L^\infty(\Omega)$. 
The HJPACK Library

HJPACK is a public domain library for Hamilton-Jacobi equations. It includes

- finite difference schemes and semi-Lagrangian schemes in $\mathbb{R}$ and $\mathbb{R}^2$
- applications to control problems and front propagation
- fast-marching schemes
- a graphical interface and a user’s guide

You can get it at www.caspur.it/hjpack.
SEMI-LAGRANGIAN SCHEMES

Basic references


**FINITE ELEMENTS**

Very few papers available.
A Discontinuous Galerkin approach has been developed by B. Cockburn, C.W. Shu.
Recent contributions in the Ph.D. Thesis by Ch. Rasch (TU Munich).

**RATE OF CONVERGENCE**
