Stabilization of non linear control systems

Jean-Michel Coron

Laboratory J.-L. Lions, University Pierre et Marie Curie (Paris 6)
BCAM OPTPDE summer school,, Bilbao, July 4-8 2011
Chapter 1: Stabilization: General results
Chapter 2: Design tools
Chapter I

Stabilization: General results
Outline: Stabilization: General results

1. Motivation and an example
2. Some abstract results in finite dimension
Motivation and an example

Some abstract results in finite dimension
The cart inverted pendulum control system
The cart inverted pendulum: the equilibrium

![Cart Inverted Pendulum Diagram]
Instability of the equilibrium
Instability of the equilibrium
Stabilization of the equilibrium

$F$ fonction de l’état

$F$ est un feedback
Outline: Stabilization: General results

1. Motivation and an example

2. Some abstract results in finite dimension
In the following we assume that the vector fields $X : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Then the Cauchy problem $\dot{x} = X(x)$ $x(0) = x_0$ have always classical solutions. However these solutions may not be unique.

**Example**

We take $n = 1$, $X(x) = |x|^{1/2}$ and $x_0 = 0$. One has infinitely many solutions: for every $a \in [0, +\infty)$, $x : \mathbb{R} \to \mathbb{R}$ defined by

(1) \hspace{1cm} x(t) = 0 \text{ for } t \text{ in } ]-\infty, a],

(2) \hspace{1cm} x(t) = \frac{(t-a)^2}{4} \text{ for } t \text{ in } (a, +\infty),

is solution to the Cauchy problem.

After important works by many mathematicians (e.g. J.-L. Lagrange, P.-S. Laplace, G. Dirichlet), A. Lyapunov gave in 1892 the “good” definition of “asymptotic stability”.
Stability
Stability
Stability
Attractor
Attractor
Attractor
Attractor
Asymptotically stable

Definition (Asymptotically stable)

Let $x_e$ be an equilibrium point of $X \in C^0(\mathbb{R}^n; \mathbb{R}^n)$, i.e. a point of $\mathbb{R}^n$ such that $X(x_e) = 0$. One says that $x_e$ is (locally) asymptotically stable for $\dot{x} = X(x)$ if it is stable and attractor for $\dot{x} = X(x)$. One says that $x_e$ is unstable for $\dot{x} = X(x)$ if it not stable.
Second Lyapunov’s theorem

Theorem

The point $x_e$ is asymptotically stable for $\dot{x} = X(x)$ if and only if there exists $\eta > 0$ and $V : B(x_e, \eta) : \{x \in \mathbb{R}^n; |x - x_e| < \eta\} \rightarrow \mathbb{R}$ of class $C^\infty$ such that

1. $V(x) > V(x_e), \forall x \in B(x_e, \eta) \setminus \{x_e\}$

2. $\nabla V(x) \cdot X(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i(x) < 0, \forall x \in B(x_e, \eta) \setminus \{x_e\}$.

Remark

- For the only if part: J. Massera and J. Kurzweil.
- This theorem explains why “asymptotic stability” is the good notion: asymptotic stabilization gives robustness with respect to small perturbation.
The stabilizability problem

We consider the control system $\dot{x} = f(x, u)$ where the state is $x$ in $\mathbb{R}^n$ and the control is $u$ in $\mathbb{R}^m$. We assume that $f(0,0) = 0$.

**Problem**

Does there exist $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ vanishing at 0 such that 0 $\in$ $\mathbb{R}^n$ is (locally) asymptotically stable for $\dot{x} = f(x, u(x))$? (If the answer is yes, one says that the control system is locally asymptotically stabilizable.)

**Remark**

The map $u : x \in \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a feedback (or feedback law). The dynamical system $\dot{x} = f(x, u(x))$ is called the closed loop system.
The regularity of $x \mapsto u(x)$ is an important point. With $u$ continuous, asymptotic stability implies the existence of a smooth strict Lyapunov function and one has robustness with respect to small actuator errors as well as small measurement errors. See above.

If $u$ is discontinuous, one needs to define the notion of solution of the closed loop system $\dot{x} = f(x, u(x))$ and study carefully the robustness of the closed loop system.
Discontinuous feedback laws

- **Add extra variables in order to have more robustness (hybrid feedback laws)**: Y. Ledyaev and E. Sontag (1997); C. Prieur (2005); R. Goebel, C. Prieur and A. Teel (2007, 2009).

Unless otherwise specified, from now on we assume that the feedback laws are continuous.
Controllability

Given two states $x^0$ and $x^1$, does there exist a control $t \in [0, T] \mapsto u(t)$ which steers the control system from $x^0$ to $x^1$, i.e. such that

$$\dot{x} = f(x, u(t)), \quad x(0) = x^0 \Rightarrow x(T) = x^1$$

(1)
Controllability of linear control systems

The control system is

\[ \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

**Theorem (Kalman’s rank condition)**

*The linear control system \( \dot{x} = Ax + Bu \) is controllable on \([0, T]\) if and only if*

\[ \text{Span} \{ A^i Bu; \ u \in \mathbb{R}^m, \ i \in \{0, 1, \ldots, n - 1\} \} = \mathbb{R}^n. \]

**Remark**

*This condition does not depend on \( T \). This is no longer true for nonlinear systems and for systems modeled by linear partial differential equations.*
We assume that \((x_e, u_e)\) is an equilibrium, i.e., \(f(x_e, u_e) = 0\). Many possible choices for natural definitions of local controllability. The most popular one is **Small-Time Local Controllability (STLC)**: the state remains close to \(x_e\), the control remains to \(u_e\) and the time is small.
$x_e$
$|u(t) - u_e| \leq \varepsilon$

$t = 0 \quad x_0$

$x_e$

$t = \varepsilon$

$x_1$

$\varepsilon$

$\eta$
The linear test

We consider the control system \( \dot{x} = f(x, u) \) where the state is \( x \in \mathbb{R}^n \) and the control is \( u \in \mathbb{R}^m \). Let us assume that \( f(x_e, u_e) = 0 \). The linearized control system at \((x_e, u_e)\) is the linear control system \( \dot{x} = Ax + Bu \) with

\[
(1) \quad A := \frac{\partial f}{\partial y}(x_e, u_e), \quad B := \frac{\partial f}{\partial u}(x_e, u_e).
\]

If the linearized control system \( \dot{x} = Ax + Bu \) is controllable, then \( \dot{x} = f(x, u) \) is small-time locally controllable at \((x_e, u_e)\).
Stabilizability of linear controllable systems

Notations. For a matrix $M \in \mathbb{R}^{n \times n}$, $P_M$ denotes the characteristic polynomial of $M$: $P_M(z) := \det(zI - M)$.

Let us denote by $\mathcal{P}_n$ the set of polynomials of degree $n$ in $z$ such that the coefficients are all real numbers and such that the coefficient of $z^n$ is 1. One has the following theorem:

**Theorem (Pole shifting theorem, M. Wonham (1967))**

Let us assume that the linear control system $\dot{x} = Ax + Bu$ is controllable. Then

$$\{P_{A+BK}; K \in \mathbb{R}^{m \times n}\} = \mathcal{P}_n.$$
Stabilizability of linear controllable systems

Notations. For a matrix $M \in \mathbb{R}^{n \times n}$, $P_M$ denotes the characteristic polynomial of $M$: $P_M(z) := \det(zI - M)$.

Let us denote by $\mathcal{P}_n$ the set of polynomials of degree $n$ in $z$ such that the coefficients are all real numbers and such that the coefficient of $z^n$ is 1. One has the following theorem

Theorem (Pole shifting theorem, M. Wonham (1967))

Let us assume that the linear control system $\dot{x} = Ax + Bu$ is controllable. Then

$$\{P_{A+BK}; K \in \mathbb{R}^{m \times n}\} = \mathcal{P}_n.$$

Corollary

If the linear control system $\dot{x} = Ax + Bu$ is controllable, there exists a linear feedback $x \mapsto u(x) = Kx$ such that $0 \in \mathbb{R}^n$ is (globally) asymptotically stable for the closed loop system $\dot{x} = Ax + Bu(x)$. 
Application to nonlinear controllable systems

We assume that $f(0, O) = 0$. Let us consider the linearized control system
\[
\dot{x} = Ax + Bu \quad \text{of} \quad \dot{x} = f(x, u) \quad \text{at} \quad (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m:
\]
\[
A := \frac{\partial f}{\partial x}(0, 0), \quad B := \frac{\partial f}{\partial u}(0, 0).
\]

Let us assume that the linearized control system $\dot{x} = Ax + Bu$ is controllable. Then, by the pole-shifting theorem, there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) = \{-1\}$. Let us consider the feedback $u(x) = Kx$. Then, if $X(x) := f(x, u(x))$, $X'(0) = A + BK$. Hence $0 \in \mathbb{R}^n$ is locally asymptotically stable for the closed loop system $\dot{x} = f(x, u(x))$. In conclusion, if the linearized control system is controllable, then

- The control system $\dot{x} = f(x, u)$ is small-time locally controllable at $(0, 0)$.
- The control system $\dot{x} = f(x, u)$ is locally asymptotically stabilizable (at the equilibrium $(0, 0)$).
An example: The cart-inverted pendulum
An example: The cart-inverted pendulum
The Cart-inverted pendulum: The equations

Let

$$x_1 := \xi, \ x_2 := \theta, \ x_3 := \dot{\xi}, \ x_4 := \dot{\theta}, \ u := F,$$

The dynamics of the cart-inverted pendulum system is $\dot{x} = f(x, u)$, with $x = (x_1, x_2, x_3, x_4)^{tr}$ and

$$f := \begin{pmatrix}
    x_3 \\
    x_4 \\
    \frac{mlx_4^2 \sin x_2 - mg \sin x_2 \cos x_2}{M + m \sin^2 x_2} + \frac{u}{M + m \sin^2 x_2} \\
    \frac{-mlx_4^2 \sin x_2 \cos x_2 + (M + m)g \sin x_2}{(M + m \sin^2 x_2)l} - \frac{u \cos x_2}{(M + m \sin^2 x_2)l}
\end{pmatrix}. $$
For the cart-inverted pendulum, the linearized control system around 
$(0, 0) \in \mathbb{R}^4 \times \mathbb{R}$ is $\dot{x} = Ax + Bu$ with

$$A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -\frac{mg}{M} & 0 & 0 \\
0 & \frac{(M+m)g}{Ml} & 0 & 0
\end{pmatrix}, \quad B = \frac{1}{Ml} \begin{pmatrix}
0 \\
0 \\
l \\
-1
\end{pmatrix}.$$  

(1)

One easily checks that this linearized control system satisfies the Kalman
rank condition and therefore is controllable. Hence the cart-inverted
pendulum is small-time locally controllable at $(0, 0) \in \mathbb{R}^4 \times \mathbb{R}$ and is locally
asymptotically stabilizable (at the equilibrium $(0, 0)$).
Obstruction to the stabilizability

**Theorem (R. Brockett (1983))**

*If the control system $\dot{x} = f(x, u)$ can be locally asymptotically stabilized then*

$(N)$ *the image by $f$ of every neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ is a neighborhood of $0 \in \mathbb{R}^n$.***
An example: The baby stroller
The baby stroller: The model

\[
\dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2, \quad n = 3, \quad m = 2.
\]
The baby stroller and the Brockett condition

The baby stroller control system

\[ \dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2 \]

is small-time locally controllable at \((0, 0)\). (This follows from the Chow-Rashevski theorem.) However \((N)\) does not hold for the baby stroller control system. Hence the baby stroller control system cannot be locally asymptotically stabilized.
Another example: The under-actuated satellite

\[ \dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i, \quad \dot{\eta} = A(\eta)\omega, \]

We consider again the case where \( m = 2 \). Using the fact that \( A(0) = \text{Id} \), one easily sees that \((N)\) never holds. However

\[ \text{Span} \{ b_1, b_2, S(\omega)J^{-1}\omega; \omega \in \text{Span} \{ b_1, b_2 \} \} = \mathbb{R}^3. \]

then the control system (1) is small-time locally controllable at \((0, 0) \in \mathbb{R}^6 \times \mathbb{R}^2\). (This follows from a sufficient condition for local controllability proved by H. Sussmann in 1987.)
A solution: Time-varying feedback laws

Instead of \( u(x) \), use \( u(t, x) \). Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function).
Theorem (JMC (1992))

Assume that

\[
\{g(x); g \in \text{Lie}\{f_1, \ldots, f_m\}\} = \mathbb{R}^n, \forall x \in \mathbb{R}^n \setminus \{0\}.
\]

Then, for every \(T > 0\), there exists \(u\) in \(C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)\) such that

\[
\begin{align*}
  u(t, 0) &= 0, \forall t \in \mathbb{R}, \\
  u(t + T, x) &= u(t, x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R},
\end{align*}
\]

0 is globally asymptotically stable for \(\dot{x} = \sum_{i=1}^{m} u_i(t, x) f_i(x)\).
Sketch of proof

Sketch of the proof of the theorem. Let $T > 0$. Assume that there exists $\bar{u}$ in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ $T$-periodic with time, vanishing for $x = 0$, and such that, if $\dot{x} = f(x, \bar{u}(t, x))$, then

(i) $x(T) = x(0)$,

(ii) If $x(0) \neq 0$, then the linearized control system around the trajectory $t \in [0, T] \mapsto (x(t), \bar{u}(t, x(t)))$ is controllable on $[0, T]$.

Using (i) and (ii) one easily sees that one can construct a "small" feedback $v$ in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ $T$-periodic with time and vanishing for $x = 0$ such that, if $\dot{x} = f(x, (\bar{u} + v)(t, x))$ and $x(0) \neq 0$, then $|x(T)| < |x(0)|$, which implies that 0 is globally asymptotically stable for $\dot{x} = f(x, (\bar{u} + v)(t, x))$. 
In order to get (i), one just imposes on $\bar{u}$ the condition

$$\bar{u}(t, x) = -\bar{u}(T - t, x), \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

which implies that $x(t) = x(T - t), \forall t \in [0, T]$, for every solution of $\dot{x} = f(x, u(t, x))$, and therefore gives $x(0) = x(T)$. Finally, one proves that (ii) holds for "many" $\bar{u}$'s (this is the difficult part of the proof).
\[ \dot{x} = f(x, \bar{u}(t, \bar{x})) \]
\[
\dot{x} = f(\bar{x}, \bar{u}(t, \bar{x}))
\]

\[
\dot{x} = f(x, \bar{u}(t, x) + v(t, x))
\]
\[
\dot{x} = f(x, \bar{u}(t, x) + v(t, x))
\]

\[
\dot{x} = f(\bar{x}, \bar{u}(t, \bar{x}))
\]
Definition

The origin (of $\mathbb{R}^n$) is *locally continuously reachable in small time* for the control system $\dot{x} = f(x, u)$ if, for every positive real number $T$, there exist a positive real number $\varepsilon$ and an element $u$ in $C^0(\bar{B}_\varepsilon; L^1((0,T); \mathbb{R}^m))$ such that

$$\operatorname{Sup}\{|u(a)(t)|; t \in (0, T)\} \to 0 \text{ as } a \to 0,$$

$$((\dot{x} = f(x, u(a)(t)), x(0) = a) \Rightarrow (x(T) = 0)), \forall a \in \bar{B}_\varepsilon.$$
General control systems

Definition

The origin (of $\mathbb{R}^n$) is *locally continuously reachable in small time* for the control system $\dot{x} = f(x, u)$ if, for every positive real number $T$, there exist a positive real number $\varepsilon$ and an element $u$ in $C^0 (\overline{B}_\varepsilon; L^1 ((0, T); \mathbb{R}^m))$ such that

$$\text{Sup}\{|u(a)(t)|; t \in (0, T)\} \to 0 \text{ as } a \to 0,$$

$$((\dot{x} = f(x, u(a)(t)), x(0) = a) \Rightarrow (x(T) = 0)), \forall a \in \overline{B}_\varepsilon.$$

Open problem

Assume that $f$ is analytic and that $\dot{x} = f(x, u)$ is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. Is the origin (of $\mathbb{R}^n$) locally continuously reachable in small time for the control system $\dot{x} = f(x, u)$?
Theorem

Let us assume that 0 is locally continuously reachable in small-time for the control system \( \dot{x} = f(x, u) \). Then the control system \( \dot{x} = f(x, u) \) is small-time locally controllable at \((0, 0) \in \mathbb{R}^n \times \mathbb{R}^m\).
Theorem

Let us assume that $0$ is locally continuously reachable in small-time for the control system $\dot{x} = f(x, u)$. Then the control system $\dot{x} = f(x, u)$ is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

Theorem (JMC (1995))

Assume that $0 \in \mathbb{R}^n$ is locally continuously reachable in small time for the control system $\dot{x} = f(x, u)$, that $f$ is analytic and that $n \not\in \{2, 3\}$. Then, for every positive real number $T$, there exist $\varepsilon$ in $(0, +\infty)$ and $u$ in $C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$, of class $C^\infty$ on $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$, $T$-periodic with respect to time, vanishing on $\mathbb{R} \times \{0\}$ and such that, for every $s \in \mathbb{R}$,

$$((\dot{x} = f(x, u(t, x)) \text{ and } x(s) = 0) \Rightarrow (x(\tau) = 0, \forall \tau \geq s)), $$

$$((\dot{x} = f(x, u(t, x)) \text{ and } |x(s)| \leq \varepsilon) \Rightarrow (x(\tau) = 0, \forall \tau \geq s + T)).$$
Stabilization of the under-actuated satellite

\[ \dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i, \quad \dot{\eta} = A(\eta)\omega, \]

We consider again the case where \( m = 2 \) and assume that \( \text{Span} \{ b_1, b_2, S(\omega)J^{-1}\omega; \omega \in \text{Span} \{ b_1, b_2 \} \} = \mathbb{R}^3 \).

Then \( 0 \in \mathbb{R}^6 \) is locally continuously reachable in small-time for the control system the control system (1) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws.
\[
\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i, \quad \dot{\eta} = A(\eta)\omega,
\]

We consider again the case where \( m = 2 \) and assume that

\[
\text{Span} \{b_1, b_2, S(\omega)J^{-1}\omega; \omega \in \text{Span} \{b_1, b_2\}\} = \mathbb{R}^3.
\]

Then \( 0 \in \mathbb{R}^6 \) is locally continuously reachable in small-time for the control system the control system (1) and therefore can be locally asymptotically stabilized by means of periodic time-varying feedback laws. Construction of explicit time-varying stabilizing feedback laws:

In most practical situations, only part of the state (called the observation $y = h(x) \in \mathbb{R}^p$) is measured. Hence, one cannot use $u(x)$ or $u(t, x)$. At a first glance, one would like to use $u(h(x))$ or $u(t, h(x))$. 
In most practical situations, only part of the state (called the observation $y = h(x) \in \mathbb{R}^p$) is measured. Hence, one cannot use $u(x)$ or $u(t, x)$. At a first glance, one would like to use $u(h(x))$ or $u(t, h(x))$.

**Question (separation principle):** Do controllability and a good observability condition imply stabilizability by means of output feedback laws?
As least two possible definitions are possible (even for small time)

(i) For every $T > 0$ and for every $u : [0, T] \to \mathbb{R}^m$,
\[
\begin{align*}
\dot{x}_1 &= f(x, u(t)), \\
\dot{x}_2 &= f(x, u(t)), \\
h(x_1) &= h(x_2) \\
\Rightarrow (x_1(0) = x_2(0)).
\end{align*}
\]

(ii) For every $T > 0$, for every $(a_1, a_2) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $u : [0, T] \to \mathbb{R}^m$ such that
\[
\begin{align*}
\dot{x}_1 &= f(x, u(t)), \\
\dot{x}_2 &= f(x, u(t)), \\
x_1(0) &= a_1, \\
x_2(0) &= a_2 \\
\Rightarrow (\exists \tau \in [0, T] \text{ such that } h(x_1(\tau)) \neq h(x_1(\tau))).
\end{align*}
\]

Of course (ii) is weaker than (i). We shall use (ii). Moreover, since we want to use a control which vanishes if the output is 0, it is natural to require that, for every $T > 0$

(iii) $(\dot{x} = f(x, 0), \ |x(0)| \text{ small and } h(x(t)) = 0, \ \forall t \in [0, T]) \Rightarrow (x(0) = 0)$.

From now on, by “observable” we mean (ii) and (iii).
$u(h(x))$ and $u(t, h(x))$ give poor results

Example (A 1-D car)

Let us consider the control system

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad y = x_1. 
\]

Note that (C) is controllable and observable. However there is no $y \mapsto u(y)$ or more generally no $(t, y) \mapsto u(t, y)$ such that $(0, 0)$ is asymptotically stable for the closed loop system $\dot{x}_1 = x_2, \dot{x}_2 = u(t, x_1)$. (Proof: Take the divergence of $(x_2, u(t, x_1))^\text{tr} \ldots$)
Dynamic output feedback laws

Definition

The control system

\[
\dot{x} = f(x, u), \quad y = h(x), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^p
\]

is locally stabilizable by means of dynamic output feedback laws if there exists \( k \in \mathbb{N} \) such that the control system

\[
\dot{x} = f(x, u), \ \dot{z} = v, \ \tilde{y} = (h(x); z), \ x \in \mathbb{R}^n, \ z \in \mathbb{R}^k,
\]

where the state is \((x; z) \in \mathbb{R}^{n+k}\), the control is \((u; v) \in \mathbb{R}^{n+m}\) and the observation is \(\tilde{y} \in \mathbb{R}^{p+k}\).
The control system is

$$(C) \quad \dot{x}_1 = x_2, \dot{x}_2 = u, \ y = x_1,$$

where the state is $(x_1, x_2)^{tr} \in \mathbb{R}^2$, the control is $u \in \mathbb{R}$ and the observation is $y \in \mathbb{R}$. Let us take $k = 1$. Let us choose the following general dynamic linear output feedback law

$$(*) \quad \dot{x} = x_2, \ \dot{x}_2 = \alpha x_1 + \beta z, \ \dot{z} = \gamma x_1 + \delta z.$$

The point $0 \in \mathbb{R}^3$ is asymptotically stable if and only if the zeroes of the polynomial

$$P(\lambda) := \lambda^3 - \delta \lambda^2 - \alpha \lambda + \alpha \delta - \gamma \beta.$$

One can choose $(\alpha, \beta, \gamma, \delta)^{tr} \in \mathbb{R}^4$ such that $P(\lambda) = (\lambda + 1)^3$. 
Interest of time-varying output feedback laws

We go back to the control system

\[ \dot{x} = u, \ y = x^2. \] (*)

This control system is controllable and observable. However one has the following proposition.

Proposition

Let \( k \in \mathbb{N} \). There are no \( u \in C^{0}(\mathbb{R}^{1+k}; \mathbb{R}), (y; z) \mapsto u(y; z), \) and \( v \in C^{0}(\mathbb{R}^{1+k}; \mathbb{R}^k), (y; z) \mapsto v(y; z), \) such that \((0; 0) \in \mathbb{R}^{1+k}\) is locally asymptotically stable for the closed loop system

\[ \dot{x} = u(x^2; z), \ \dot{z} = v(x^2; z), x \in \mathbb{R}, z \in \mathbb{R}^k. \]

(One uses the convention that, if \( k = 0 \), the closed system is just \( \dot{x} = u(x^2) \) and \((0; 0) \in \mathbb{R}^{1+k}\) is just \( 0 \in \mathbb{R} \).)
Sketch of the proof of the proposition

Let \( X \in C^0(\mathbb{R}^{1+k}; \mathbb{R}^{1+k}) \) be defined by

\[
X(x; z) := (u(x^2; z); v(x^2; z)), \quad x \in \mathbb{R}, \; z \in \mathbb{R}^k.
\]

(2)

By a theorem due to Krasnosel’skii, the fact that 0 is locally asymptotically stable for \( \dot{x} = X(x) \) implies the existence of \( \varepsilon > 0 \) such that, with

\[
B_\varepsilon := \{(x; z) \in \mathbb{R}^{1+k}; \; x^2 + |z|^2 < \varepsilon^2\},
\]

(3) \( X(x; z) \neq 0, \; \forall (x, z) \in \mathbb{R} \times \mathbb{R}^k \) such that \( x^2 + |z|^2 = \varepsilon^2 \),

(4)

\[
\text{degree} (X, B_\varepsilon, 0) = (-1)^{k+1}.
\]

Note that, by (2),

(5) \( X(x; z) = X(-x; z), \; \forall (x; z) \in B_\varepsilon \),

from which we get that \( \text{degree} (X, B_\varepsilon, 0) = 0 \), a contradiction with (4).
Output stabilization of $\dot{x} = u, \ y = x^2$ by means of time-varying feedback laws
Output stabilization of $\dot{x} = u$, $y = x^2$ by means of time-varying feedback laws
Output stabilization of \( \dot{x} = u, y = x^2 \) by means of time-varying feedback laws

\[ x(t) = \begin{cases} x_+ & \text{for } t = 0 \\ x_- & \text{for } t = T' \\ x_- & \text{for } t = 2T' \end{cases} \]
Theorem (JMC (1994))

Assume that $f$ and $h$ are analytic. Assume that $0 \in \mathbb{R}^n$ is locally continuously reachable in small time for $\dot{x} = f(x, u)$. Assume that the observability properties (ii) and (iii) hold. Then, there exist $k \in \mathbb{N}^*$ such that, for every $T > 0$, there exist $\varepsilon > 0$, $u \in C^0(\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^k; \mathbb{R}^m)$ and $v \in C^0(\mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^k; \mathbb{R}^k)$, of class $C^\infty$ on $\mathbb{R} \times (\mathbb{R}^p \times \mathbb{R}^k \setminus \{0\})$, $T$-periodic with respect to time, vanishing on $\mathbb{R} \times \{0\}$ such that, $\forall s \in \mathbb{R}$,

\[
\begin{align*}
\left(\dot{x} = f(x, u(t, (h(x), z))), \dot{z} = v(t, (h(x), z)), (x(s), z(s)) = 0\right) & \Rightarrow \left((x(\tau), z(\tau)) = 0, \forall \tau \geq s\right), \\
\left(\dot{x} = f(x, u(t, (h(x), z))), \dot{z} = v(t, (h(x), z)), |x(s)| + |z(s)| \leq \varepsilon\right) & \Rightarrow \left((x(\tau), z(\tau)) = 0, \forall \tau \geq s + T\right).
\end{align*}
\]
Time-varying feedback laws and measurement: An experiment

Material:
- A jigsaw
- A vise
- Two Meccano© strips (length ≃ 30 and 4 cm)
- A nut and 3 bolts
- A plastic tube
Time-varying feedback laws and measurement: An experiment

Material:
A jigsaw
A vise
Two Meccano© strips (length $\approx$ 30 and 4 cm)
A nut and 3 bolts
A plastic tube
Chapter II

Design tools
Outline: Design tools

3. Control Lyapunov function
4. Damping
5. Phantom tracking
6. Averaging
7. Backstepping
Design tools: Commercial break

Control and Nonlinearity
Jean-Michel Coron

Outline: Design tools

3 Control Lyapunov function
4 Damping
5 Phantom tracking
6 Averaging
7 Backstepping
A basic tool to study the asymptotic stability of an equilibrium point is the Lyapunov function. In the case of a control system, the control is at our disposal, so there are more “chances” that a given function could be a Lyapunov function for a suitable choice of feedback laws. Hence Lyapunov functions are even more useful for the stabilization of control systems than for dynamical systems without control.
A function \( V \in C^1(\mathbb{R}^n; \mathbb{R}) \) is a control Lyapunov function for the control system \( (C) \) if

\[
V(x) \to +\infty, \text{ as } |x| \to +\infty,
\]

\[
V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\},
\]

\[
\forall x \in \mathbb{R}^n \setminus \{0\}, \exists u \in \mathbb{R}^m \text{ s.t. } f(x, u) \cdot \nabla V(x) < 0.
\]

Moreover, \( V \) satisfies the \textit{small control property} if, for every strictly positive real number \( \varepsilon \), there exists a strictly positive real number \( \eta \) such that, for every \( x \in \mathbb{R}^n \) with \( 0 < |x| < \eta \), there exists \( u \in \mathbb{R}^m \) satisfying \( |u| < \varepsilon \) and \( f(x, u) \cdot \nabla V(x) < 0 \).
If the control system \((C)\) is globally asymptotically stabilizable by means of continuous stationary feedback laws, then it admits a control Lyapunov function satisfying the small control property. If the control system \((C)\) admits a control Lyapunov function satisfying the small control property, then it can be globally asymptotically stabilized by means of

1. Continuous stationary feedback laws if the control system \((C)\) is control affine \((f(x,u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x))\) (Z. Artstein (1983)),
2. Time-varying feedback laws for general \(f\) (JMC-L. Rosier (1994)).
Outline: Design tools

3 Control Lyapunov function

4 Damping

5 Phantom tracking

6 Averaging

7 Backstepping
Damping

For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies.
For mechanical systems at least, a natural candidate for a control Lyapunov function is given by the total energy, i.e., the sum of potential and kinetic energies. Consider the classical spring-mass control system.
The control system is

(Spring-mass) \[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{m}x_1 + \frac{u}{m},
\]

where \( m \) is the mass of the point attached to the spring, \( x_1 \) is the displacement of the mass (on a line), \( x_2 \) is the speed of the mass, \( k \) is the spring constant, and \( u \) is the external force applied to the mass. The state is \((x_1, x_2)^{tr} \in \mathbb{R}^2\) and the control is \( u \in \mathbb{R}\).
The total energy $E$ of the system is

$$E = \frac{1}{2}(kx_1^2 + mx_2^2).$$

One has

$$\dot{E} = ux_2.$$

Hence if $x_2 = 0$, one cannot have $\dot{E} < 0$. However it tempting to consider the following feedback laws

$$u := -\nu x_2,$$

where $\nu > 0$. Using the LaSalle invariance principle, one gets that these feedback laws globally asymptotically stabilize the spring-mass control system.
Electric propulsion is characterized by a low-thrust acceleration level but a high specific impulse. They can be used for large amplitude orbit transfers if one is not in a hurry.

The state of the control system is the position of the satellite (here identified to a point: we are not considering the attitude of the satellite) and the speed of the satellite. Instead of using Cartesian coordinates, one prefers to use the “orbital” coordinates. The advantage of this set of coordinates is that, in this set, the first five coordinates remain unchanged if the thrust vanishes: these coordinates characterize the Keplerian elliptic orbit. When thrust is applied, they characterize the Keplerian elliptic osculating orbit of the satellite. The last component is an angle which gives the position of the satellite on the Keplerian elliptic osculating orbit of the satellite.
A usual set of orbital coordinates is

\[ p := a(1 - e^2), \]
\[ e_x := e \cos \tilde{\omega}, \quad \text{with} \quad \tilde{\omega} = \omega + \Omega, \]
\[ e_y := e \sin \tilde{\omega}, \]
\[ h_x := \tan \frac{i}{2} \cos \Omega, \]
\[ h_y := \tan \frac{i}{2} \sin \Omega, \]
\[ L := \tilde{\omega} + v, \]

where \( a, e, \omega, \Omega, i \) characterize the Keplerian osculating orbit:

1. \( a \) is the semi-major axis,
2. \( e \) is the eccentricity,
3. \( i \) is the inclination with respect to the equator,
4. \( \Omega \) is the right ascension of the ascending node,
5. \( \omega \) is the angle between the ascending node and the perigee,

and where \( v \) is the true anomaly.
\[ \dot{p} = 2\sqrt{\frac{p^3}{\mu Z}} S, \]

\[ \dot{e}_x = \sqrt{\frac{p}{\mu Z}} \left[ Z (\sin L) Q + A S - e_y (h_x \sin L - h_y \cos L) W \right], \]

\[ \dot{e}_y = \sqrt{\frac{p}{\mu Z}} \left[ -Z (\cos L) Q + B S - e_x (h_x \sin L - h_y \cos L) W \right], \]

\[ \dot{h}_x = \frac{1}{2} \sqrt{\frac{p}{\mu Z}} X (\cos L) W, \quad \dot{h}_y = \frac{1}{2} \sqrt{\frac{p}{\mu Z}} X (\sin L) W, \]

\[ \dot{L} = \sqrt{\frac{\mu}{p^3}} Z^2 + \sqrt{\frac{p}{\mu Z}} (h_x \sin L - h_y \cos L) W, \]

where \( \mu > 0 \) is a gravitational coefficient depending on the central gravitational field, \( Q, S, W \), are the radial, orthoradial, and normal components of the thrust and where

\[ Z := 1 + e_x \cos L + e_y \sin L, \quad A := e_x + (1 + Z) \cos L, \]

\[ B := e_y + (1 + Z) \sin L, \quad X := 1 + h_x^2 + h_y^2. \]
We study the case, useful in applications, where

\[ Q = 0, \]

and, for some \( \varepsilon > 0, \)

\[ |S| \leq \varepsilon \text{ and } |W| \leq \varepsilon. \]

Note that \( \varepsilon \) is small, since the thrust acceleration level is low.

The goal: give feedback laws, which (globally) asymptotically stabilize a given Keplerian elliptic orbit characterized by the coordinates \( \bar{p}, \bar{e}_x, \bar{e}_y, \bar{h}_x, \bar{h}_y. \)

In order to simplify the notations (this is not essential for the method), we restrict our attention to the case where the desired final orbit is geostationary, that is,

\[ \bar{e}_x = \bar{e}_y = \bar{h}_x = \bar{h}_y = 0. \]
We start with a change of “time”. One describes the evolution of 

\( (p, e_x, e_y, h_x, h_y) \)

as a function of \( L \) instead of \( t \). Then our system reads

\[
\begin{aligned}
\frac{dp}{dL} &= 2KpS, \\
\frac{de_x}{dL} &= K[AS - e_y(h_x \sin L - h_y \cos L)W], \\
\frac{de_y}{dL} &= K[BS - e_x(h_x \sin L - h_y \cos L)W], \\
\frac{dh_x}{dL} &= \frac{K}{2}X(\cos L)W, \\
\frac{dh_y}{dL} &= \frac{K}{2}X(\sin L)W, \\
\frac{dt}{dL} &= K\sqrt{\frac{\mu}{p}}Z, \\
\end{aligned}
\]

with

\[
K = \left[ \frac{\mu}{p^2}Z^3 + (h_x \sin L - h_y \cos L)W \right]^{-1}.
\]
Typically, one consider the following control Lyapunov function

\[
V(p, e_x, e_y, h_x, h_y) = \frac{1}{2} \left( \frac{(p - \bar{p})^2}{p} + \frac{e^2}{1 - e^2 + h^2} \right),
\]

with \( e^2 = e_x^2 + e_y^2 < 1 \) and \( h^2 = h_x^2 + h_y^2 \). The time derivative of \( V \) along a trajectory of our control system is is given by

\[
\dot{V} = K (\alpha S + \beta W),
\]

with

\[
\begin{align*}
\alpha & := 2p \frac{\partial V}{\partial p} + A \frac{\partial V}{\partial e_x} + B \frac{\partial V}{\partial e_y}, \\
\beta & := (h_y \cos L - h_x \sin L) \left( e_y \frac{\partial V}{\partial e_x} + e_x \frac{\partial V}{\partial e_y} \right) \\
& \quad + \frac{1}{2} X \left( (\cos L) \frac{\partial V}{\partial h_x} + (\sin L) \frac{\partial V}{\partial h_y} \right).
\end{align*}
\]
Following the damping method, one defines

\[
S := -\sigma_1(\alpha),
\]
\[
W := -\sigma_2(\beta)\sigma_3(p, e_x, e_y, h_x, h_y),
\]

where \( \sigma_1 : \mathbb{R} \to \mathbb{R}, \sigma_2 : \mathbb{R} \to \mathbb{R} \) and \( \sigma_3 : (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \to (0, 1] \) are such that

\[
\sigma_1(s)s > 0, \quad \sigma_2(s)s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\},
\]
\[
\| \sigma_1 \|_{L^\infty(\mathbb{R})} < \varepsilon, \quad \| \sigma_2 \|_{L^\infty(\mathbb{R})} < \varepsilon,
\]
\[
\sigma_3(p, e_x, e_y, h_x, h_y) \leq \frac{1}{1 + \varepsilon} \frac{\mu (1 - |e|)^3}{p^2 |h|}.
\]
Following the damping method, one defines

\[
S := -\sigma_1(\alpha),
\]

\[
W := -\sigma_2(\beta)\sigma_3(p, e_x, e_y, h_x, h_y),
\]

where \(\sigma_1 : \mathbb{R} \to \mathbb{R}, \sigma_2 : \mathbb{R} \to \mathbb{R}\) and \(\sigma_3 : (0, +\infty) \times \mathcal{B}_1 \times \mathbb{R}^2 \to (0, 1]\) are such that

\[
\sigma_1(s)s > 0, \sigma_2(s)s > 0, \quad \forall s \in \mathbb{R} \setminus \{0\},
\]

\[
\|\sigma_1\|_{L^\infty(\mathbb{R})} < \varepsilon, \quad \|\sigma_2\|_{L^\infty(\mathbb{R})} < \varepsilon,
\]

\[
\sigma_3(p, e_x, e_y, h_x, h_y) \leq \frac{1}{1 + \varepsilon} \frac{\mu (1 - |e|)^3}{|h|}. \]

It works!
Comparison with optimal control

It is interesting to compare the feedback constructed here to the open-loop optimal control for the minimal time problem (reach \((\bar{p}, 0, 0, 0, 0)\) in a minimal time with the constraint \(|u(t)| \leq M\)). Numerical experiments show that the use of the previous feedback laws (with suitable saturations \(\sigma_i, i \in \{1, 2, 3\}\)) gives trajectories which are nearly optimal if the state is not too close to \((\bar{p}, 0, 0, 0, 0)\). Note that our feedback laws are quite easy to compute compared to the optimal trajectory and provide already good robustness properties compared to the open-loop optimal trajectory (the optimal trajectory in a closed-loop form being, at least for the moment, out of reach numerically).
It is interesting to compare the feedback constructed here to the open-loop optimal control for the minimal time problem (reach $(\bar{p}, 0, 0, 0, 0)$ in a minimal time with the constraint $|u(t)| \leq M$). Numerical experiments show that the use of the previous feedback laws (with suitable saturations $\sigma_i, i \in \{1, 2, 3\}$) gives trajectories which are nearly optimal if the state is not too close to $(\bar{p}, 0, 0, 0, 0)$. Note that our feedback laws are quite easy to compute compared to the optimal trajectory and provide already good robustness properties compared to the open-loop optimal trajectory (the optimal trajectory in a closed-loop form being, at least for the moment, out of reach numerically). However, when one is close to the desired target, our feedback laws are very far from being optimal. When one is close to the desired target, it is much better to linearize around the desired target and apply a standard Linear-Quadratic strategy.
\[ \dot{x}_1 = x_2, \dot{x}_2 = -x_1 + u, \ |u| \leq 2 \]
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad |u| \leq 1 \]
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad |u| \leq \frac{1}{2}
\]
\dot{x}_1 = x_2, \dot{x}_2 = -x_1 + u, \ |u| \leq 1/4
An important limitation of the damping method

Let us come back to the spring-mass control system (with normalized physical constants)

\[ \dot{x}_1 = x_2, \dot{x}_2 = -x_1 + u. \]

With the Lyapunov strategy used above, let \( V : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[ V(x) = x_1^2 + x_2^2, \forall x = (x_1, x_2)^{\text{tr}} \in \mathbb{R}^2. \]

As we have seen above \( \dot{V} = 2x_2 u \). and it is tempting to take, at least if we remain in the class of linear feedback laws, \( u := -\nu x_2 \), where \( \nu \) is some fixed positive real number. An a priori guess would be that, if we let \( \nu \) be quite large, then we get a quite good convergence, as fast as we want.
An important limitation of the damping method

Let us come back to the spring-mass control system (with normalized physical constants)

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u. \]

With the Lyapunov strategy used above, let \( V: \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[ V(x) = x_1^2 + x_2^2, \quad \forall x = (x_1, x_2)^{\text{tr}} \in \mathbb{R}^2. \]

As we have seen above \( \dot{V} = 2x_2u \). and it is tempting to take, at least if we remain in the class of linear feedback laws, \( u := -\nu x_2 \), where \( \nu \) is some fixed positive real number. An a priori guess would be that, if we let \( \nu \) be quite large, then we get a quite good convergence, as fast as we want. But this is completely wrong. On a given \([0, T]\) time-interval, as \( \nu \to +\infty \), \( x_2 \) goes very quickly to 0 and \( x_1 \) does not change.
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - \left(\frac{1}{10}\right)x_2 \]
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - \frac{1}{2}x_2 \]
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 2x_2 \]
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 3x_2 \]
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 4x_2 \]
$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 5x_2$
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 6x_2
\]
\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 10x_2 \]
\begin{align*}
    \dot{x}_1 &= x_2, \\
    \dot{x}_2 &= -x_1 - 20x_2
\end{align*}
The quantum control system considered is

\[ \dot{\psi} = H_0 \psi + uH_1 \psi + \omega \psi, \]

where \( H_0 \) and \( H_1 \) are \( N \times N \) Hermitian matrices. The state is \( \psi \in S^{2N-1} \), the unit sphere of \( R^{2N} \cong C^N \), the control is \((u, \omega)^{tr} \in \mathbb{R}^2\). The control \( \omega \) is a fictitious phase control which allows to take care of the fact that global phase of the state is physically meaningless. Let \( \psi_e \in S^{2N-1} \) and \( \lambda_e \in \mathbb{R} \) be such that \( H_0 \psi_e = \lambda \psi_e \). Replacing \( \psi \) by \( \psi e^{-i\lambda_e} \) and \( \omega \) by \( \omega - \lambda_e \), we may assume that \( \lambda_e = 0 \). Then \((\psi, (u, \omega)) := (\psi_e, (0, 0)^{tr})\) is an equilibrium of our quantum control system. The goal is to stabilize asymptotically this equilibrium.
Remark

Since \( S^{2N-1} \) is not contractible, one cannot achieve global stabilizability. However, one can try to get global stabilizability on \( S^{2N-1} \setminus \{-\psi_e\} \).

A natural control Lyapunov function to consider is

\[
V := |\psi - \psi_e|^2.
\]

Indeed, the time-derivative of \( V \) along the trajectory of our control system is

\[
\dot{V} = 2u \Im(\langle H_1 \psi, \psi_e \rangle) + 2\omega \Im(\langle \psi, \psi_e \rangle).
\]

This leads to choose the following feedback laws

\[
u := -\nu_1 \Im(\langle H_1 \psi, \psi_e \rangle), \quad \omega := -\nu_2 \Im(\langle \psi, \psi_e \rangle),
\]

where \( \nu_1 \) and \( \nu_2 \) are two strictly positive real numbers. With these feedback laws one has

\[
\dot{V} = -2\nu_1 \Im(\langle H_1 \psi, \psi_e \rangle)^2 - 2\nu_2 \omega \Im(\langle \psi, \psi_e \rangle)^2 \leq 0.
\]
Theorem (M. Mirrahimi, P. Rouchon, G. Turinici (2005))

The above feedback law insures global asymptotic stabilization on $\mathbb{S}^{2N-1} \setminus \{-\psi_e\}$ if and only the two following properties hold

(i) If $(\alpha, \beta) \in \sigma(H_0)^2$ and $|\alpha| = |\beta|$, then $\alpha = \beta$,

(ii) If $\phi$ in an eigenvector of $H_0$ which is not colinear to $\Psi_e$, then $\langle \phi, H_1\psi_e \rangle \neq 0$.

Remark

The properties (i) and (ii) hold together if and only if the linearized control system around $(\psi_e, 0) \in \mathbb{S}^{2N-1}$ is controllable.
Theorem (M. Mirrahimi, P. Rouchon, G. Turinici (2005))

The above feedback law insures global asymptotic stabilization on $\mathbb{S}^{2N-1} \setminus \{-\psi_e\}$ if and only the two following properties hold

(i) If $(\alpha, \beta) \in \sigma(H_0)^2$ and $|\alpha| = |\beta|$, then $\alpha = \beta$,

(ii) If $\phi$ in an eigenvector of $H_0$ which is not colinear to $\Psi_e$, then $\langle \phi, H_1 \psi_e \rangle \neq 0$.

Remark

The properties (i) and (ii) hold together if and only if the linearized control system around $(\psi_e, 0) \in \mathbb{S}^{2N-1}$ is controllable.

Question: What to do if (i) or (ii) do not hold but the quantum control system is controllable? Partial solution: use the “phantom tracking” method (see below).
We are now in infinite dimension. In order to apply LaSalle invariance principle one needs to have the precompactness of the trajectories. This is an open problem. However with clever and important modifications of the method one can get global approximate controllability results. Let us mention in particular

- M. Mirrahimi (2006),
- V. Nersesyan (2009,2010),
- K. Beauchard and M. Mirrahimi (2009).
LaSalle invariance principle/Strict Lyapunov function

Another possibility to overcome the problem of the precompactness of the trajectories is to try to modify the control Lyapunov function in order to get a strict Lyapunov function. Some recent examples of this possibility:


\[ y_t + \sum_{j=1}^{n} (F^j(y)) x_j = (0, B(y))^{tr}, \quad x \in \mathbb{R}^n. \]


\[ y_t + A(y)y_x = 0, \quad x \in (0, L), \]

incoming Riemann invariants = \( F \)(outgoing Riemann invariants).
Our hyperbolic control system is

\[ y_t + A(y)y_x = 0, \quad (t, x) \in [0, T] \times [0, L], \]

where, at time \( t \in [0, T] \), the state is \( x \in [0, L] \mapsto y(t, x) \in \mathbb{R}^n \). Let \( y^* \in \mathbb{R}^n \) be fixed. Assume that \( A(y^*) \) has \( n \) distinct real distinct eigenvalues: \( \lambda_1(y^*) < \cdots < \lambda_k(y^*) < \cdots < \lambda_n(y^*) \). We assume that

\[ \lambda_1(y^*) < \cdots < \lambda_{m-1}(y^*) < 0 < \lambda_{m+1}(y^*) < \cdots < \lambda_n(y^*), \]

for some \( m \in \{1, \cdots, n\} \).
Motivated by 1994 Li Tatsien’s book

\( y_t + A(y)y_x = 0, \ y \in \mathbb{R}^n, \ x \in [0, 1], \ t \in [0, +\infty), \)

**Assumptions on** \( A \)

\( A(0) = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n), \)

\( \lambda_i > 0, \ \forall i \in \{1, \ldots, m\}, \ \lambda_i < 0, \ \forall i \in \{m + 1, \ldots, n\}, \)

\( \lambda_i \neq \lambda_j, \ \forall (i, j) \in \{1, \ldots, n\}^2 \text{ such that } i \neq j. \)
• Boundary conditions on $y$:

\[
\begin{bmatrix}
    y_+(t, 0) \\
    y_-(t, 1)
\end{bmatrix}
= G
\begin{bmatrix}
    y_+(t, 1) \\
    y_-(t, 0)
\end{bmatrix},
\quad t \in [0, +\infty),
\]

where

(i) $y_+ \in \mathbb{R}^m$ and $y_- \in \mathbb{R}^{n-m}$ are defined by

\[
y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix},
\]

(ii) the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at 0.
For $K \in M_{n,m}(\mathbb{R})$,

$$\|K\| := \max\{|Kx|; x \in \mathbb{R}^n, |x| = 1\}.$$ 

If $n = m$,

$$\rho_1(K) := \inf \{\|\Delta K \Delta^{-1}\|; \Delta \in D_{n,+}\},$$

where $D_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements.
Theorem (JMC-G. Bastin-B. d’Andréa-Novel (2008))

If $\rho_1(G'(0)) < 1$, then the equilibrium $\bar{y} \equiv 0$ of the quasi-linear hyperbolic system

$$y_t + A(y)y_x = 0,$$

with the above boundary conditions, is exponentially stable for the Sobolev $H^2$-norm.

Complements:

- $y_t + Ay_x + By = 0$: G. Bastin and JMC (2010), A. Diagne, G. Bastin and JMC (2010), R. Vazquez, M. Krstic and JMC (2011),
- $y_t + A(x,y)y_x + B(x,y)y = 0$: A. Diagne and A. Drici (2011), R. Vazquez, JMC, M. Krstic and G. Bastin (2011),
Estimate on the exponential decay rate

For every \( \nu \in (0, -\min\{|\lambda_1|, \ldots, |\lambda_n|\} \ln(\rho_1(G'(0)))) \), there exist \( \varepsilon > 0 \) and \( C > 0 \) such that, for every \( y_0 \in H^2((0, 1), \mathbb{R}^n) \) satisfying \( |y_0|_{H^2((0,1),\mathbb{R}^n)} < \varepsilon \) (and the usual compatibility conditions) the classical solution \( y \) to the Cauchy problem

\[
y_t + A(y)y_x = 0, \quad y(0, x) = y_0(x) + \text{boundary conditions}
\]

is defined on \([0, +\infty)\) and satisfies

\[
|y(t, \cdot)|_{H^2((0,1),\mathbb{R}^n)} \leq Ce^{-\nu t}|y_0|_{H^2((0,1),\mathbb{R}^n)}, \quad \forall t \in [0, +\infty).
\]
The Li Tatsien condition

\[ R_2(K) := \text{Max} \left\{ \sum_{j=1}^{n} |K_{ij}|; \ i \in \{1, \ldots, n\} \right\}, \]

\[ \rho_2(K) := \text{Inf} \left\{ R_2(\Delta K \Delta^{-1}); \ \Delta \in D_{n,+} \right\}. \]

**Theorem (Li Tatsien, 1994)**

*If \( \rho_2(G'(0)) < 1 \), then the equilibrium \( \bar{y} \equiv 0 \) of the quasi-linear hyperbolic system

\[ y_t + A(y)y_x = 0, \]

with the above boundary conditions, is exponentially stable for the \( C^1 \)-norm.*

The Li Tatsien proof relies mainly on the use of direct estimates of the solutions and their derivatives along the characteristic curves.
Open problem: Does there exist $K$ such that one has exponential stability for the $C^1$-norm but not for the $H^2$-norm?

Open problem: Does there exist $K$ such that one has exponential stability for the $H^2$-norm but not for the $C^1$-norm?
Comparison of $\rho_2$ and $\rho_1$

**Proposition**

For every $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

(1) \hspace{1cm} \rho_1(K) \leq \rho_2(K).

Example where (1) is strict: for $a > 0$, let

$$K_a := \begin{pmatrix} a & a \\ -a & a \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}).$$

Then

$$\rho_1(K_a) = \sqrt{2}a < 2a = \rho_2(K_a).$$

Open problem: Does $\rho_1(K) < 1$ implies the exponential stability for the $C^1$-norm?
Comparison with stability conditions for linear hyperbolic systems

For simplicity we assume that $\lambda_i$ are all positive and consider the special case of linear hyperbolic systems

$$y_t + \Lambda y_x = 0, \ y(t,0) = K y(t,1),$$

where

$$\Lambda := \text{diag} \ (\lambda_1, \ldots, \lambda_n), \ \text{with} \ \lambda_i > 0, \ \forall i \in \{1, \ldots, n\}.$$

**Theorem**

*Exponential stability for the $C^1$-norm is equivalent to the exponential stability in the $H^2$-norm.*
A Necessary and sufficient condition for exponential stability

Notation:

\[ r_i = \frac{1}{\lambda_i}, \quad \forall i \in \{1, \ldots, n\}. \]

Theorem

\( \tilde{y} \equiv \) is exponentially stable for the system

\[ y_t + \Lambda y_x = 0, \quad y(t, 0) = K y(t, 1) \]

if and only if there exists \( \delta > 0 \) such that

\[ \left( \det \left( I_{d_n} - \left( \text{diag} \left( e^{-r_1 z}, \ldots, e^{-r_n z} \right) \right) K \right) = 0, \quad z \in \mathbb{C} \right) \Rightarrow (\Re(z) \leq -\delta). \]
An example

This example is borrowed from the book Hale-Lunel (1993). Let us choose $\lambda_1 := 1$, $\lambda_2 := 2$ (hence $r_1 = 1$ and $r_2 = 1/2$) and

$$K_a := \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \ a \in \mathbb{R}.$$  

Then $\rho_1(K) = 2|a|$. Hence $\rho_1(K_a) < 1$ is equivalent to $a \in (-1/2, 1/2)$. However exponential stability is equivalent to $a \in (-1, 1/2)$.  


Robustness issues

For a positive integer $n$, let

$$\lambda_1 := \frac{4n}{4n + 1}, \quad \lambda_2 = \frac{4n}{2n + 1}. $$

Then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} \sin \left( 4n\pi \left( t - \frac{x}{\lambda_1} \right) \right) \\ \sin \left( 4n\pi \left( t - \frac{x}{\lambda_2} \right) \right) \end{pmatrix}$$

is a solution of $y_t + \Lambda y_x$, $y(t, 0) = K\frac{1}{2} y(t, 1)$ which does not tends to 0 as $t \to +\infty$. Hence one does not have exponential stability. However $\lim_{n \to +\infty} \lambda_1 = 1$ and $\lim_{n \to +\infty} \lambda_2 = 2$. The exponential stability is not robust with respect to $\Lambda$: small perturbations of $\Lambda$ can destroy the exponential stability.
Robust exponential stability

Notations:

\[ \rho_0(K) := \max \{ \rho(\text{diag} (e^{t\theta_1}, \ldots, e^{t\theta_n})K); (\theta_1, \ldots, \theta_n)^{\text{tr}} \in \mathbb{R}^n \} \]

Theorem (R. Silkowski, 1993)

If the \((r_1, \ldots, r_n)\) are rationally independent, the linear system

\[ y_t + \Lambda y_x = 0, \; y(t, 0) = Ky(t, 1) \]

is exponentially stable if and only if \(\rho_0(K) < 1\).

Note that \(\rho_0(K)\) depends continuously on \(K\) and that “\((r_1, \ldots, r_n)\) are rationally independent” is a generic condition. Therefore, if one wants to have a natural robustness property with respect to the \(r_i\)’s, the condition for exponential stability is

\[ \rho_0(K) < 1. \]

This condition does not depend on the \(\lambda_i\)’s!
Comparison of $\rho_0$ and $\rho_1$

Proposition (JMC-G. Bastin-B. d’Andréa-Novel, 2008)

For every $n \in \mathbb{N}$ and for every $K \in \mathcal{M}_{n,n}(\mathbb{R})$, 

$$\rho_0(K) \leq \rho_1(K).$$

For every $n \in \{1, 2, 3, 4, 5\}$ and for every $K \in \mathcal{M}_{n,n}(\mathbb{R})$, 

$$\rho_0(K) = \rho_1(K).$$

For every $n \in \mathbb{N} \setminus \{1, 2, 3, 4, 5\}$, there exists $K \in \mathcal{M}_{n,n}(\mathbb{R})$ such that 

$$\rho_0(K) < \rho_1(K).$$

Open problem: Is $\rho_0(G'(0)) < 1$ a sufficient condition for exponential stability (for the $H^2$-norm) in the nonlinear case?
Proof of the exponential stability if $A$ is constant and $G$ is linear

Main tool: a Lyapunov approach. $A(y) = \Lambda$, $G(y) = Ky$. For simplicity, all the $\lambda_i$’s are positive. Lyapunov function candidate:

$$V(y) := \int_0^1 y^{\text{tr}} Q ye^{-\mu x} dx$$

where $Q$ is positive symmetric.

$$\dot{V} = -\int_0^1 (y_x^{\text{tr}} \Lambda Q y + y^{\text{tr}} Q y_x) \Lambda e^{-\mu x} dx$$

$$= -\mu \int_0^1 y^{\text{tr}} \Lambda Q y e^{-\mu x} dx - B,$$

with

$$B := [y^{\text{tr}} \Lambda Q ye^{-\mu x}]_{x=0}^{x=1} = y(1)^{\text{tr}} (\Lambda Q e^{-\mu} - K^{\text{tr}} \Lambda Q K) y(1)$$
Let $D \in D_{n,+}$ be such that $\|DKD^{-1}\| < 1$ and let $\xi := Dy(1)$. We take $Q = D^2 \Lambda^{-1}$. Then

$$B = e^{-\mu} |\xi|^2 - |DKD^{-1}\xi|^2.$$ 

Therefore it suffices to take $\mu > 0$ small enough.

**Remark**

*Introduction of $\mu$:*  
- JMC (1998) for the stabilization of the Euler equations.  
New difficulties if \( A(y) \) depends on \( y \)

We try with the same \( V \):

\[
\dot{V} = -\int_0^1 (y^\text{tr} A(y)^\text{tr} Q y + y^\text{tr} QA(y)y_x) e^{-\mu x} \, dx
\]

\[
= -\mu \int_0^1 y^\text{tr} A(y)Qy e^{-\mu x} \, dx - B + N_1 + N_2
\]

with

\[
N_1 := \int_0^1 y^\text{tr} (QA(y) - A(y)Q)y_x e^{-\mu x} \, dx,
\]

\[
N_2 := \int_0^1 y^\text{tr} (A'(y)y_x)^\text{tr} Qy e^{-\mu x} \, dx
\]
Take $Q$ depending on $y$ such that $A(y)Q(y) = Q(y)A(y)$, $Q(0) = D^2F(0)^{-1}$. (This is possible since the eigenvalues of $F(0)$ are distinct.) Now

$$\dot{V} = -\mu \int_0^1 y^\text{tr} A(y)Q(y)ye^{-\mu x} dx - B + N_2$$

with

$$N_2 := \int_0^1 y^\text{tr} (A'(y)y_x Q(y) + A(y)Q'(y)y_x)^\text{tr} ye^{-\mu x} dx$$

What to do with $N_2$?
Solution for $N_2$

New Lyapunov function:

$$V(y) = V_1(y) + V_2(y) + V_3(y)$$

with

$$V_1(y) = \int_0^1 y^{\text{tr}} Q(y) y e^{-\mu x} dx,$$

$$V_2(y) = \int_0^1 y_x^{\text{tr}} R(y) y_x e^{-\mu x} dx,$$

$$V_3(y) = \int_0^1 y_{xx}^{\text{tr}} S(y) y_{xx} e^{-\mu x} dx,$$

where $\mu > 0$, $Q(y)$, $R(y)$ and $S(y)$ are symmetric positive definite matrices.
Commutations:

\[ A(y)Q(y) - Q(y)A(y) = 0, \]
\[ A(y)R(y) - R(y)A(y) = 0, \]
\[ A(y)S(y) - S(y)A(y) = 0, \]

\[ Q(0) = D^2 A(0)^{-1}, \quad R(0) = D^2 A(0), \quad S(0) = D^2 A(0)^3. \]
Estimates on $\dot{V}$

Lemma

If $\mu > 0$ is small enough, there exist positive real constants $\alpha, \beta, \delta$ such that, for every $y : [0, 1] \rightarrow \mathbb{R}^n$ such that $|y|_{C^0([0,1])} + |y_x|_{C^0([0,1])} \leq \delta$, we have

$$\frac{1}{\beta} \int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2) dx \leq V(y) \leq \beta \int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2) dx,$$

$$\dot{V} \leq -\alpha V.$$
La Sambre (The same + Luc Moens)
The Saint-Venant equations

The index $i$ is for the $i$-th reach.
Conservation of mass:

$$H_{it} + (H_i V_i)_x = 0,$$

Conservation of momentum:

$$V_{it} + \left(g H_i + \frac{V_i^2}{2}\right)_x = 0.$$

Flow rate: $Q_i = H_i V_i$. 
Barré de Saint-Venant
(Adhémar-Jean-Claude)
1797-1886
Boundary conditions

Underflow (sluice)

\[ Q = K \sqrt{u(H_{up} - H_{down})} \]

Overflow (spillway)

\[ Q = K(H_{up} - u)^{3/2} \]
Work in progress: La Meuse
Closed loop versus open loop
Closed loop versus open loop
Outline: Design tools

3. Control Lyapunov function
4. Damping
5. Phantom tracking
6. Averaging
7. Backstepping
Phantom tracking method: An example

Let us consider the following control system.

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = -x_3 + 2x_1x_2,
\]

where the state is \((x_1, x_2, x_3, x_4)^\text{tr} \in \mathbb{R}^4\) and the control is \(u \in \mathbb{R}\).
Phantom tracking method: An example

Let us consider the following control system.

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = -x_3 + 2x_1x_2,
\]

where the state is \((x_1, x_2, x_3, x_4)^\text{tr} \in \mathbb{R}^4\) and the control is \(u \in \mathbb{R}\). Roughly, we have two oscillators which are coupled by means of a quadratic term. The control is acting only on the first oscillator. The point \((x^\gamma, u^\gamma) := ((\gamma, 0, 0, 0)^\text{tr}, \gamma)\) is an equilibrium of the control system. The linearized control system at this equilibrium is the linear control system

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = -x_3 + 2\gamma x_2,
\]

where the state is \((x_1, x_2, x_3, x_4)^\text{tr} \in \mathbb{R}^4\) and the control is \(u \in \mathbb{R}\). This linear control system is controllable if (and only if) \(\gamma \neq 0\). Therefore, if \(\gamma \neq 0\) the equilibrium can be asymptotically stabilized for the nonlinear control system.
Stabilization of $x^\gamma$

One considers the following control Lyapunov function $V^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$V^\gamma(x) := (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2, \quad \forall x = (x_1, x_2, x_3, x_4)^{\text{tr}} \in \mathbb{R}^4.$$ 

The time derivative of $V^\gamma$ along the trajectory of our control system is

$$\dot{V}^\gamma = 2x_2(u - \gamma + 2x_1x_4).$$

Hence, in order to asymptotically stabilize $x^\gamma$ for our control system, it is natural to consider the feedback law $u^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$u^\gamma := \gamma - 2x_1x_4 - x_2.$$ 

One gets $\dot{V}^\gamma = -2x_2^2$. Using the LaSalle invariance principle, one gets that this feedback law globally asymptotically stabilizes $x^\gamma$. 
Let us now follow the phantom tracking strategy. In fact, instead of using $u\tilde{\gamma}$ with a suitable $\tilde{\gamma} : \mathbb{R}^4 \to \mathbb{R}$ it is better to use directly a control Lyapunov of the type $V\tilde{\gamma}$. Theoretically, the best way to choose $\tilde{\gamma}$ is to define it implicitly by proceeding in the following way. There exits an open neighborhood $\Omega$ of $0 \in \mathbb{R}^4$ and $V \in C^\infty(\Omega; \mathbb{R})$ such that

$$V(0) = 0, \forall x \in \Omega \setminus \{0\},$$
$$V(x) = (x_1 - V(x))^2 + x_2^2 + x_3^2 + x_4^2, \forall x = (x_1, x_2, x_3, x_4)^{\text{tr}} \in \Omega.$$

Therefore our choice of $\tilde{\gamma} = V(x)$, i.e. is such that $\tilde{\gamma}(x) = V\tilde{\gamma}(x)$, $\tilde{\gamma}(0) = 0$. For the existence of $V$: use the implicit function theorem. In this simple case, $V$ can be computed explicitly. One has

$$\dot{V} = 2(x_1 - V)(x_2 - \dot{V}) + 2x_2(-x_1 + u) + x_3x_4 + x_4(-x_3 + 2x_1x_2),$$

i.e., $(1 + 2x_1 - 2V)\dot{V} = 2x_2(u - V + x_1x_4)$. We define a feedback law $u : \Omega \to \mathbb{R}$ by $u := V - x_1x_4 - x_2$, which leads to

$$(1 + 2x_1 - 2V)\dot{V} = -2x_2^2 \leq 0.$$

One concludes that the feedback law $u$ locally asymptotically our control system.
Two possible improvements:

(i) One can get global asymptotic stability. It suffices to modify $V$ by requiring $V = V(x) = (x_1 - \theta(V(x)))^2 + x_2^2 + x_3^2 + x_4^2$, with a well chosen function $\theta : [0, +\infty) \rightarrow [0, +\infty)$.

(ii) One can get explicit feedback laws by using a dynamic extension: Replace the initial control system by the following one

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + u, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -x_3 + 2x_1x_2, \\
\dot{\gamma} &= v
\end{align*}
$$

where the state is $(x_1, x_2, x_3, x_4, \gamma)^{\text{tr}} \in \mathbb{R}^5$ and the control is $(u, v)^{\text{tr}} \in \mathbb{R}^2$. For $z = (x_1, x_2, x_3, x_4, \gamma)^{\text{tr}} \in \mathbb{R}^5$, one defines

$$
\begin{align*}
\varphi(z) &:= (x_1 - \gamma)^2 + x_2^2 + x_3^2 + x_4^2, \\
W(z) &:= \varphi(z) + (\gamma - \varphi(z))^2.
\end{align*}
$$

Compute $\dot{W}$ etc.
Phantom tracking and stabilization of the Euler equations of incompressible fluids

$\Omega$ is assumed to be connected and simply connected.
The Euler control system

We denote by $\nu : \partial \Omega \to \mathbb{R}^2$ the outward unit normal vector field to $\Omega$. Euler equations:

$$
y_t + (y \cdot \nabla)y + \nabla p = 0, \text{ div } y = 0,
$$

$$
y \cdot \nu = 0 \text{ on } [0, T] \times (\partial \Omega \setminus \Gamma_0).
$$

This system is under-determined. In order to have a determined system, one has to specify what is the control. There are at least two natural possibilities:

(a) The control is $y(t, x) \cdot n(x)$ on $\Gamma_0$ and the time derivative $\partial \omega / \partial t(t, x)$ of the vorticity at the points $x$ of $\Gamma_0$ where $y(t, x) \cdot n(x) < 0$, i.e., at the points where the fluid enters into the domain $\Omega$.

(b) The control is $y(t, x) \cdot n(x)$ on $\Gamma_0$ and the vorticity $\omega$ at the points $x$ of $\Gamma_0$ where $y(t, x) \cdot n(x) < 0$.

To fix ideas, we deal only with the first case.
The linearized control is

\[ y_t + \nabla p = 0, \ \text{div} \ y = 0, \ y \cdot \nu = 0 \ \text{on} \ [0, T] \times (\partial \Omega \setminus \Gamma_0). \]

Taking the curl of the first equation, one gets, with \( \omega := \text{curl} \ y \)

\[ \omega_t = 0. \]

One cannot change \( \omega \)! This linear control system is not stabilizable.
Definition of $y^\gamma$

Take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta \theta = 0 \text{ in } \Omega, \quad \frac{\partial \theta}{\partial \nu} = 0 \text{ on } \partial \Omega \setminus \Gamma_0.$$ 

Let us define $(y^\gamma, p^\gamma) : [0, T] \times \Omega \to \mathbb{R}^2 \times \mathbb{R}$ by

$$y^\gamma(x) = \gamma \nabla \theta(x), \quad p^\gamma(x) = -\frac{\gamma^2}{2} |\nabla \theta(x)|^2.$$ 

Then $(y^\gamma, p^\gamma)$ is an equilibrium point of our Euler control system. The corresponding control is $\gamma \partial \theta / \partial n$ on $\Gamma_0$ and 0 for the vorticity at the points $x$ of $\Gamma_0$ where $\partial \theta / \partial n < 0.$
The linearized control system around \( y^\gamma \) is

\[
\begin{cases}
y_t + (y^\gamma \cdot \nabla)y + (y \cdot \nabla)y^\gamma + \nabla p = 0, & \text{div } y = 0 \text{ in } [0, T] \times \Omega, \\
y \cdot \nu = 0 \text{ on } [0, T] \times (\partial \Omega \setminus \Gamma_0).
\end{cases}
\]

Taking once more the curl of the first equation, one gets

\[
(*) \quad \omega_t + (y^\gamma \cdot \nabla)\omega = 0.
\]

This is a simple transport equation on \( \omega \). If there exists \( a \in \overline{\Omega} \) such that \( \nabla \theta(a) = 0 \), then \( y^\gamma(a) = 0 \) and \( \omega_t(t, a) = 0 \) showing that \( (*) \) is not stabilizable. This is the only obstruction: If \( \nabla \theta \) does not vanish in \( \overline{\Omega} \), one can easily stabilize \( (*) \): just take \( C > 0 \) and use the control \( \omega_t(t, x) = -C\omega(t, x) \) on the set \( \{ x \in \Gamma_0; \partial \theta / \partial \nu(x) < 0 \} \).
Construction of a good $\theta$
Construction of a good $\theta$
Construction of a good $\theta$
Construction of a good $\theta$

$$g : \partial \Omega \to \mathbb{R}$$

$$\int_{\partial \Omega} g ds = 0,$$

$$\{ g > 0 \} = \Gamma_+, \{ g < 0 \} = \Gamma_-$$
Construction of a good $\theta$

$\Delta \theta = 0,$

$\frac{\partial \theta}{\partial \nu} = g$ on $\partial \Omega$

$\int_{\partial \Omega} g ds = 0,$

$\{g > 0\} = \Gamma_+, \{g < 0\} = \Gamma_-$
Construction of a good \( \theta \)

\[
\Delta \theta = 0, \\
\frac{\partial \theta}{\partial \nu} = g \text{ on } \partial \Omega
\]

\[
\int_{\partial \Omega} g \, ds = 0,
\]

\[
\{ g > 0 \} = \Gamma_+, \quad \{ g < 0 \} = \Gamma_-
\]
Asymptotic stabilization of the Euler equations

Our stabilizing feedback law is

\[ y \cdot \nu := M |\omega \omega|_0 \frac{\partial \theta}{\partial \nu} \text{ on } \Gamma_-, \quad \frac{\partial \omega}{\partial t} := -M |\omega|_{C^0(\overline{\Omega})} \omega \text{ on } \Gamma_- \]

Theorem (JMC (1999))

There exists a positive constant \( M_0 \) such that, for every \( \varepsilon \in (0, 1] \), every \( M \geq M_0 / \varepsilon \) and every solution \( \omega \) of the closed loop system,

\[ |\omega(t)|_0 \leq \text{Min} \left\{ |\omega(0)|_{C^0(\overline{\Omega})}, \frac{\varepsilon}{t} \right\}, \forall t > 0. \]

Remark

Applications to quantum control systems

1. K. Beauchard, JMC, M. Mirrahimi and P. Rouchon (2007) for

\[ \dot{\psi} = H_0 \psi + u H_1 \psi, \quad \psi(t, \cdot) \in S^{2N-1}. \]

2. K. Beauchard and M. Mirrahimi (2009) for a quantum particle in a one-dimensional infinite square potential well:

\[ \begin{cases} \ \dot{\psi}_t = -\psi_{xx} + u(t)x \psi, \quad x \in (0,1), \ \psi \in L^2((0,1); \mathbb{C}) \\ \int_0^1 |\psi(t, x)|^2 \, dx = 1. \end{cases} \]

This control is in infinite dimension and there is a problem to use the LaSalle invariance principle. This leads to important difficulties.

3. JMC, A. Grigoriu (in progress) for

\[ \dot{\psi} = H_0 \psi + u H_1 \psi + u^2 H_2 \psi, \quad \psi(t, \cdot) \in S^{2N-1}. \]
Outline: Design tools

3 Control Lyapunov function
4 Damping
5 Phantom tracking
6 Averaging
7 Backstepping
The control system is

$$\dot{\psi} = H_0\psi + uH_1\psi + u^2H_2 + \omega\psi,$$

where $H_0$, $H_1$ and $H_2$ are $N \times N$ Hermitian matrices. The state is $\psi \in S^{2N-1}$, the unit sphere of $\mathbb{R}^{2N} \simeq \mathbb{C}^N$, the control is $(u, \omega)^{tr} \in \mathbb{R}^2$. Again we assume that 0 is an eigenvalue of $H_0$ and consider a corresponding eigenvector $\psi_e \in S^{2N-1}$ We consider the following time dependent feedback:

$$u(t, \psi) = \alpha(\psi) + \beta(\psi)\sin(t/\varepsilon).$$

The closed loop system is

$$\begin{cases}
\dot{\psi} = (H_0 + \alpha(\psi)H_1 + \beta(\psi)\sin(t/\varepsilon)H_1 \\
+ \alpha^2(\psi)H_2 + 2\alpha(\psi)\beta(\psi)\sin(t/\varepsilon)H_2 \\
+ \beta^2(\psi)\sin^2(t/\varepsilon)H_2 + \omega(t))\psi(t).
\end{cases}$$
Averaged system

For a differential system $\dot{x} = f(t, x)$, with $f$ a $T$-periodic function: $f(T + t, x) = f(t, x)$, the averaged system is defined by $\dot{x}_{av} = f_{av}(x)$ where $f_{av}(x) = \frac{1}{T} \int_0^T f(t, x) dt$.

In our case the averaged system corresponding to the closed loop system is:

$$(C_{av}) \quad i\dot{\psi} = \left( H_0 + \alpha H_1 + \left( \alpha^2 + \frac{1}{2} \beta^2 \right) H_2 + \omega \right) \psi.$$

In some sense we have now three independent controls, namely $\alpha$, $\beta$ and $\omega$, instead of two, namely $u$ and $\omega$. Moreover, one knows that, if $\varepsilon > 0$ is small enough, the trajectories of $(C_\varepsilon)$ are close to the trajectory of $(C_{av})$.

The strategy is now simple. Using the damping method, one gets feedback laws $\psi \mapsto (\alpha(\psi); \beta(\psi); \omega(\psi))$ leading to global stabilizability of $\psi_e$ on $S^{2N-1} \setminus \{-\psi_e\}$ for the averaged system. Then, taking $\varepsilon > 0$ small enough, one gets a “practical” global stabilizability of $\psi_e$ on $S^{2N-1} \setminus \mathcal{V}$ for the closed loop system $(C_\varepsilon)$, where $\mathcal{V}$ is a given neighborhood of $-\psi_e$. 
Backstepping

For the backstepping method, we are interested in a control system \((C)\) having the following structure:

\[
\Sigma : \dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = u,
\]

where the state is \(x = (x_1; x_2) := (x_1^{\text{tr}}, x_2^{\text{tr}})^{\text{tr}} \in \mathbb{R}^{n_1+m} = \mathbb{R}^n\) with \((x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{m}\) and the control is \(u \in \mathbb{R}^{m}\). The key theorem for backstepping is the following one.

**Theorem**

Assume that \(f_1 \in C^1(\mathbb{R}^{n_1} \times \mathbb{R}^{m}; \mathbb{R}^{n_1})\) and that the control system

\[
\Sigma_1 : \dot{x}_1 = f_1(x_1, v),
\]

where the state is \(x_1 \in \mathbb{R}^{n_1}\) and the control \(v \in \mathbb{R}^{m}\), can be globally asymptotically stabilized by means of a feedback law of class \(C^1\). Then \(\Sigma\) can be globally asymptotically stabilized by means of a continuous feedback law.
Local version: very old, precise father(s)/mother(s) unknown,

Global version:
- D. Koditschek (1987),
- C. Byrnes and A. Isidori (1989),
- J. Tsinias (1989),
- L. Praly, B. d’Andréa-Novel and JMC (1991) for low regularity for the feedback stabilizing $\dot{x}_1 = f_1(x_1, v)$. 
Proof of the theorem

Let \( v \in C^1(\mathbb{R}^{n_1}; \mathbb{R}^m) \) be a feedback law which globally asymptotically stabilizes \( 0 \in \mathbb{R}^{n_1} \) for the control system \( \Sigma_1 \). Then, by the converse of the second Lyapunov theorem, there exists a Lyapunov function of class \( C^\infty \) for the closed-loop system \( \dot{x}_1 = f_1(x_1, v(x_1)) \), that is, there exists a function \( V \in C^\infty(\mathbb{R}^{n_1}; \mathbb{R}) \) such that

\[
 f_1(x_1, v(x_1)) \cdot \nabla V(x_1) < 0, \quad \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\},
\]

\[
 V(x_1) \to +\infty \text{ as } |x_1| \to +\infty,
\]

\[
 V(x_1) > V(0), \quad \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\}.
\]

A natural candidate for a control Lyapunov function for \( \Sigma \) is

\[
 W(x_1; x_2) := V(x_1) + \frac{1}{2}|x_2 - v(x_1)|^2, \quad \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^m.
\]

Indeed, one has, for such a \( W \),

\[
 W(x_1; x_2) \to +\infty \text{ as } |x_1| + |x_2| \to +\infty,
\]

\[
 W(x_1; x_2) > W(0, 0), \quad \forall (x_1, x_2) \in (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.
\]
The time-derivative $\dot{W}$ of $W$ along the trajectories of $\Sigma$ is

$$\dot{W} = f_1(x_1, x_2) \cdot \nabla V(x_1) - (x_2 - v(x_1)) \cdot (v'(x_1)f_1(x_1, x_2) - u).$$

There exists $G \in C^0(\mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^m; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{n_1}))$ such that

$$f_1(x_1, x_2) - f_1(x_1, y) = G(x_1, x_2, y)(x_2 - y).$$

Therefore

$$\dot{W} = f_1(x_1, v(x_1)) \cdot \nabla V(x_1) + [u^\text{tr} - (v'(x_1)f_1(x_1, x_2))^\text{tr} + (\nabla V(x_1))^\text{tr} G(x_1, x_2, v(x_1))] (x_2 - v(x_1)).$$

Hence, if one takes as a feedback law for the control system $\Sigma$

$$u := v'(x_1)f_1(x_1, x_2) - G(x_1, x_2, v(x_1))^\text{tr} \nabla V(x_1) - (x_2 - v(x_1)),$$

one gets $\dot{W} = f_1(x_1, v(x_1)) \cdot \nabla V(x_1) - |x_2 - v(x_1)|^2$. Hence

$$\dot{W}(x_1; x_2) < 0, \forall (x_1, x_2) \in (\mathbb{R}^{n_1} \times \mathbb{R}^m) \setminus \{(0, 0)\}.$$ 

In conclusion, $u$ globally asymptotically stabilizes $\Sigma$. 
Various generalizations

Various (straightforward) adaptations are possible. For example

1. Adaptation to the framework of time-varying feedback laws.
2. It is possible to add an “integrator” to only part of the components of $v$.
3. In the above construction, instead of using a strict Lyapunov function one can use a Lyapunov function satisfying the assumptions of the LaSalle invariance principle.

In the next slides we show how to use these three adaptations together in order to asymptotically stabilize the baby stroller control system.
Stabilization of the baby stroller

The baby stroller control system is

\[ \Sigma \]
\[ \dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2, \]

where the state is \( x := (x_1; x_2; x_3) \in \mathbb{R}^3 \) and the control is \( (u_1; u_2) \in \mathbb{R}^2 \). Following the backstepping approach, we first deal with the stabilization of the control system

\[ \Sigma_1 \]
\[ \dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \]

where the state is \( (x_1; x_2) \in \mathbb{R}^2 \) and the control is \( (u_1; x_3) \in \mathbb{R}^2 \) and then we add an “integration \( x_3 \)”. As a potential control Lyapunov for \( \Sigma_1 \), we consider

\[ V(x_1; x_2) := \frac{1}{2} (x_1^2 + x_2^2), \quad \forall (x_1; x_2) \in \mathbb{R}^2. \]

Note that

\[ V(x') > V(0), \quad \forall x' \in \mathbb{R}^2, \quad \lim_{|x'| \to +\infty} V(x') = +\infty. \]
Along the trajectory of $\Sigma_1$, we have
\[
\dot{V} = (x_1 \cos x_3 + x_2 \sin x_3)u_1.
\]

Let $T > 0$. Let $f \in C^1(\mathbb{R})$ be a $T$-periodic function which is not constant. We define $\bar{x}_3 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $\bar{u}_1 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by
\[
\bar{x}_3(t, x') := f(t)|x'|^2,
\]
\[
\bar{u}_1(t, x') := -(x_1 \cos(\bar{x}_3(t, x')) + x_2 \sin(\bar{x}_3(t, x'))).
\]

The functions $\bar{x}_3$ and $\bar{u}_1$ are $T$-periodic to time and vanish on $\mathbb{R} \times \{0\}$. One has $\dot{V} = -\bar{u}_1^2 \leq 0$. Using the LaSalle invariance principle one checks that $0 \in \mathbb{R}^2$ is globally asymptotically stable for the closed loop system
\[
\dot{x}_1 = \bar{u}_1 \cos \bar{x}_3, \quad \dot{x}_2 = \bar{u}_1 \sin \bar{x}_3.
\]
Following the backstepping strategy, we consider the potential time-varying control Lyapunov function $W$ for the baby stroller control system $\Sigma$ (which is obtained by “adding an integration” on the variable $x_3$ to the control system $\Sigma_1$)

$$W(t, x) := V(x') + \frac{1}{2}(x_3 - \bar{x}_3(t, x'))^2, \forall t \in \mathbb{R}, \forall x = (x_1; x_2; x_3) \in \mathbb{R}^3,$$

with $x' := (x_1; x_2)$. Note that $W$ is T-periodic with respect to time and that

$$W(t, x) > W(t, 0) = 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^3 \setminus \{0\},$$

$$\lim_{|x| \to +\infty} \min\{W(t, x); t \in [0, T]\} = +\infty.$$

The time derivative of $W$ along the trajectory of $\Sigma$ is

$$\dot{W} = (x_1 \cos x_3 + x_2 \sin x_3)u_1 + (x_3 - \bar{x}_3)(u_2 - \xi),$$

with $\xi(t, (x; u_1)) := f'(t)|x'|^2 + 2f(t)u_1(x_1 \cos x_3 + x_2 \sin x_3)$. 
We define our time-varying stabilizing feedback law by

\[
\begin{align*}
    u_1(t, (x_1; x_2; x_3)) &:= -(x_1 \cos x_3 + x_2 \sin x_3), \\
    u_2(t, (x_1; x_2; x_3)) &:= \xi(t, (x; -(x_1 \cos x_3 + x_2 \sin x_3))) \\
    &\quad - (x_3 - \bar{x}_3(t, x')), 
\end{align*}
\]

so that

\[
\dot{W} = -(x_1 \cos x_3 + x_2 \sin x_3)^2 + (x_3 - \bar{x}_3)^2 \leq 0.
\]

The functions \(u_1\) and \(u_2\) are \(T\)-periodic to time and vanish on \(\mathbb{R} \times \{0\}\). Using once more the LaSalle invariance principle, one checks that \(0 \in \mathbb{R}^3\) is globally asymptotically stable for the closed loop system

\[
\dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2.
\]