THE PARABOLIC FRACTIONAL OBSTACLE PROBLEM

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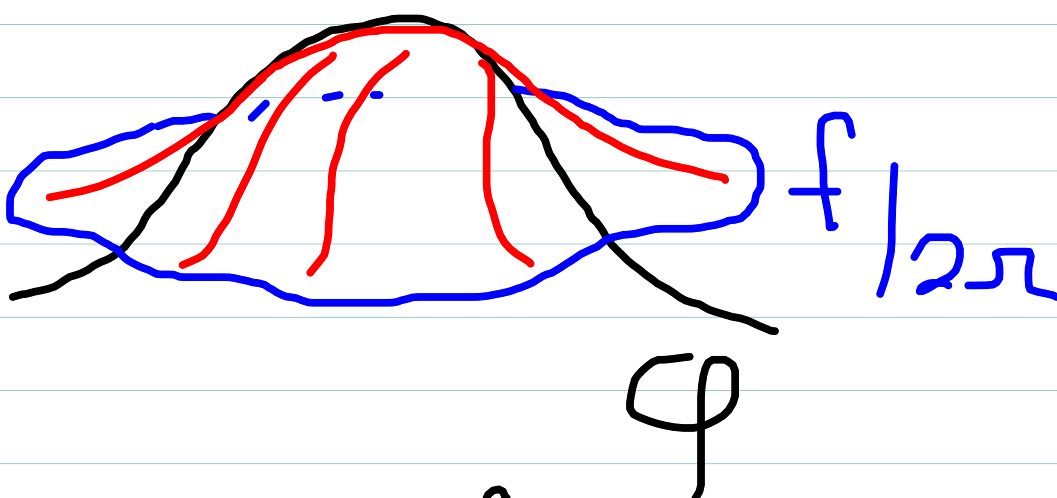
CLASSICAL OBSTACLE PB.

$$\begin{cases} \Delta u = 0 \\ u = f \text{ on } \partial\Omega \end{cases} \longleftrightarrow \min_{u|_{\partial\Omega}=f} \int_\Omega |\nabla u|^2$$



$$\min_{u \geq \varphi,\, u|_{\partial\Omega}=f} \int_\Omega |\nabla u|^2$$

Properties of sols :

u min $\Rightarrow$ $u+\varepsilon v$ adm

if $\varepsilon>0$,

$v \geq 0$, $v \in C_c^\infty(\Omega)$

$\Downarrow$

$\Delta u \leq 0$.

Moreover, $\Delta u = 0$ on $\{u>\varphi\}$.

Q : (1) Regularity of u?

(2) Regularity of $\{u=\varphi\}$

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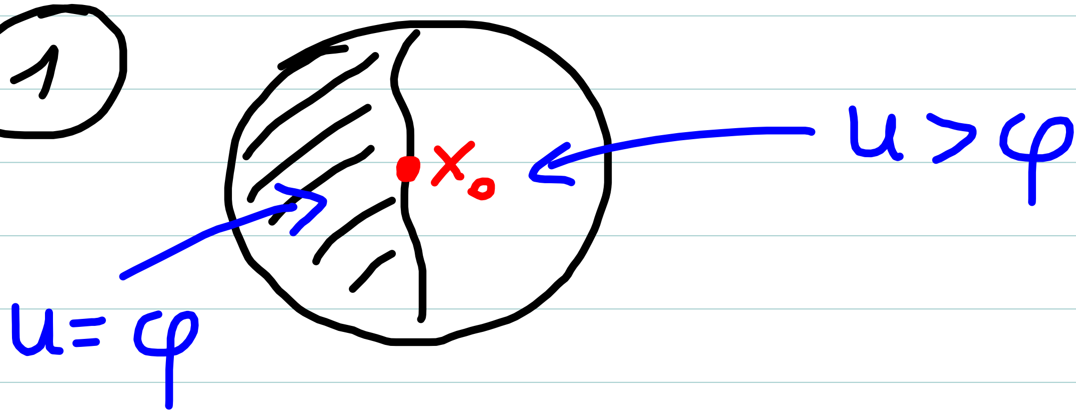
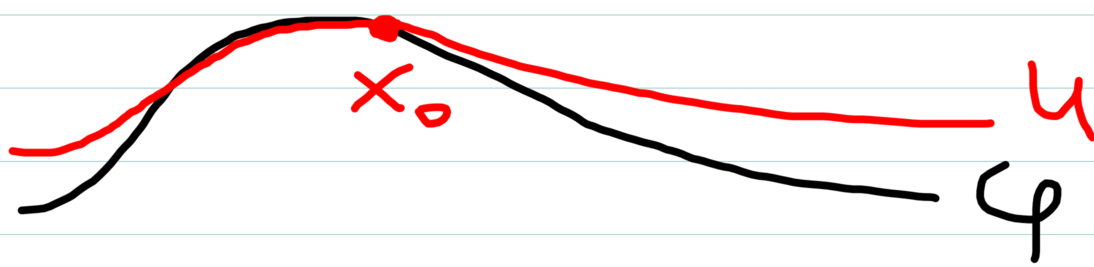


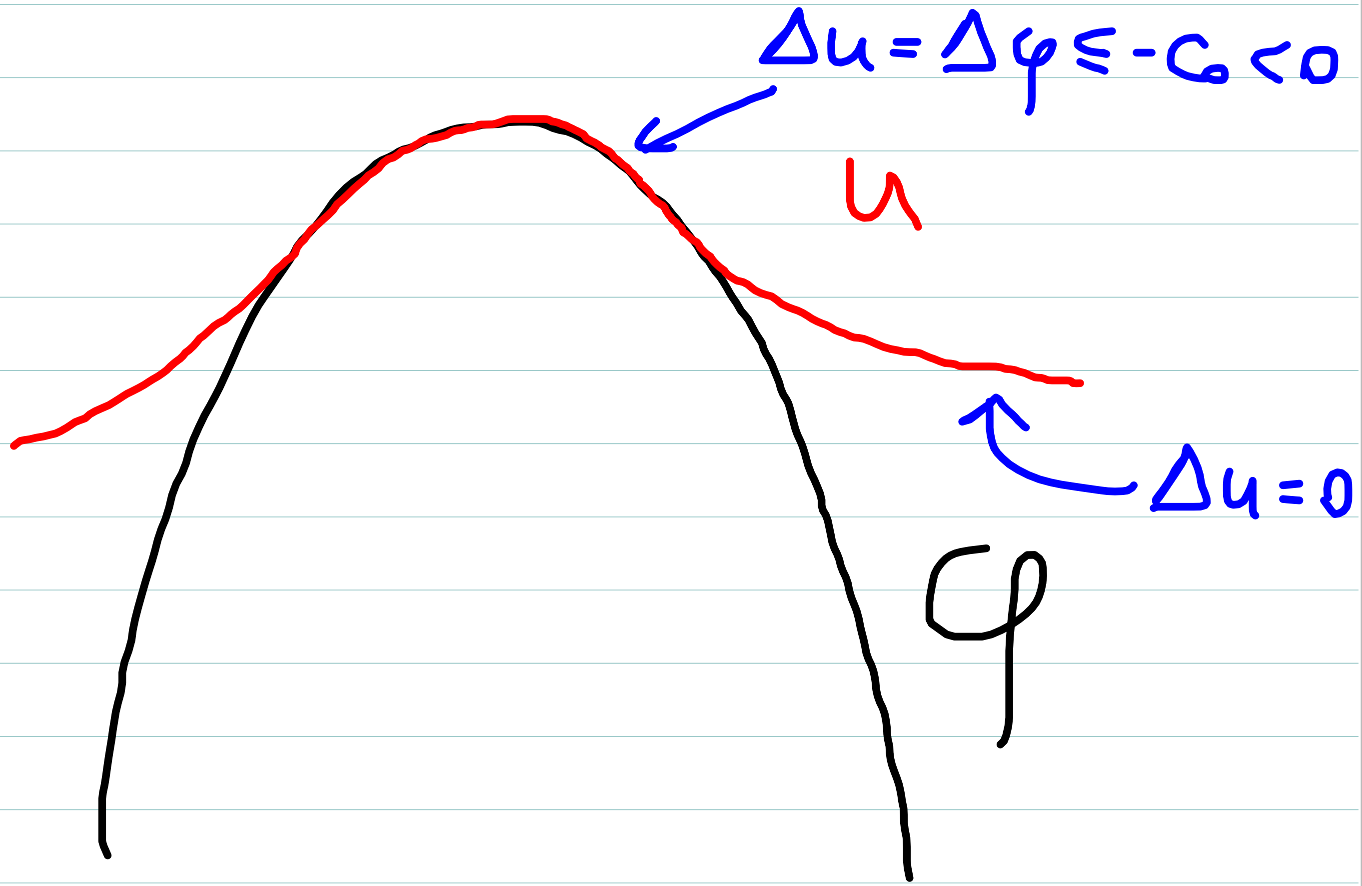
IDEA:

$$\Delta u \le 0 \Rightarrow u(x) \ge \fint_{B_r(x)} u \quad \forall x, r$$

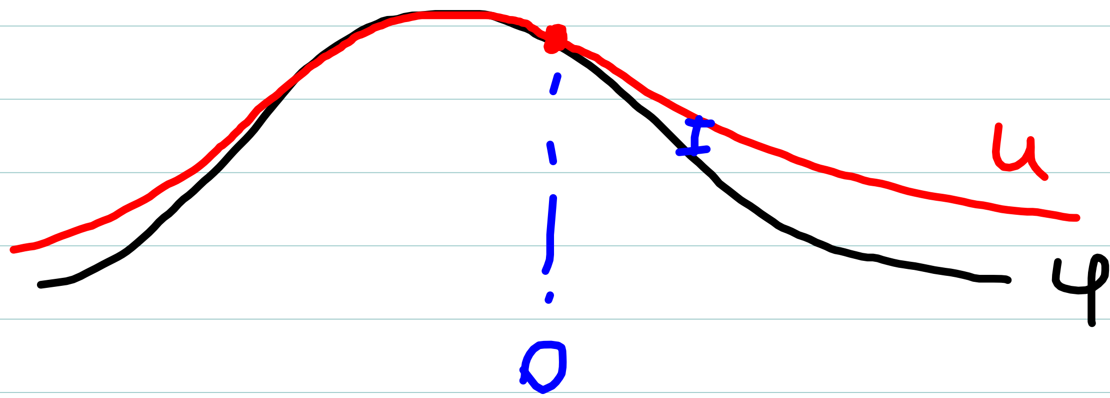


- $u \ge \varphi \longrightarrow$ bd from below
- $u(x_0)$ controls u in average
- u harmonic if $\{u > \varphi\}$

THM $\varphi \in C^2 \Rightarrow u \in C^{1,1}$



② Key fact:



$$\sup_{B_r}(u-\varphi) \sim r^2 \quad \text{(NONDEGENERACY)}$$

$\Downarrow$

$\partial\{u>\varphi\}$ is $C^{1,\alpha}$

MATH. FINANCE

$v(\tau, x)$ = price of an option x at time $\tau \le T$

$\psi(x)$ = payoff

$$\begin{cases} v \geq \psi \\ v > \psi \Rightarrow \mathcal{L} v = 0 \\ v(T) = \psi \end{cases}$$

T ← expiration time

$$\mathcal{L}v = v_\tau + r v$$
$$+ \sum_i (r - d_i) x_i v_{x_i}$$
$$+ \frac{1}{2} \sum_{ij} x_i x_j v_{x_i x_j}$$
$$+ \int \Big[v(\tau, x_1 e^{y_1}, \dots) - v(\tau, x)$$
$$- \sum_i \left[e^{y_i} - 1 \right] v_{x_i} \Big] d\mu$$

BACKWARD PARABOLIC

$$x_i \longleftrightarrow \log x_i$$

$$v_\tau + r v + b \cdot \nabla v + a_{ij} v_{x_i x_j}$$
$$+ K[v]$$

- $\left.\begin{array}{l} K \equiv 0 \\ \varphi \in C^\infty \\ a_{ij} \geq \lambda I \end{array}\right] \Longrightarrow v \in C^1_t C^{1,1}_x$

Q: What happen if $a \equiv 0$?

We need to gain regularity from K.

Hyp: $K[v] = \Delta^s v + K'[v]$

l.o.t.

$$v_\tau + r v + b\cdot\nabla v + \Delta^s v + K'[v] = 0$$

- $s > \frac{1}{2}$: rv, $b\cdot\nabla v$, $K'[v]$ are l.o.t.

- $s \le \frac{1}{2}$: $b\cdot\nabla v$ may create problems

Do the change of variables: $\tau = T - t$, $t \geq 0$

Set $u(t,x) = v(\tau, x)$.

We consider the model equation:

$$\begin{cases} u \geq \psi \\ u > \psi \Rightarrow u_t = \Delta^s u \\ u(0) = \psi \end{cases}$$

<u>GOAL</u>: study the regularity of u.

Preliminaries on Δ^s

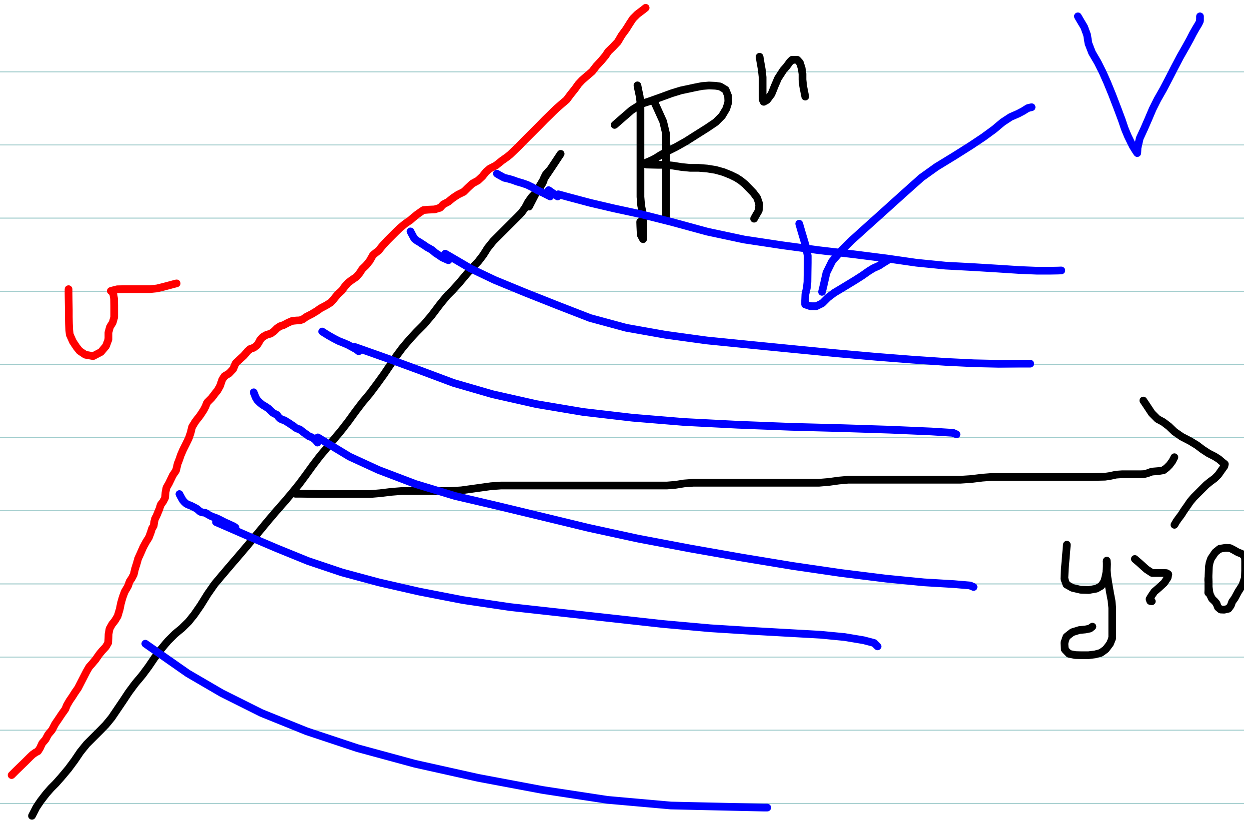
$v : \mathbb{R}^n \to \mathbb{R}$, $v \in C_c^\infty$

$$\Delta^s v(x) = \left(-|\xi|^{2s} \hat{v}\right)^\vee (x)$$

$$= \text{p.v.} \int \frac{v(z) - v(x)}{|z-x|^{n+2s}} dz$$

Other def:

Dirichlet-to-Neumann

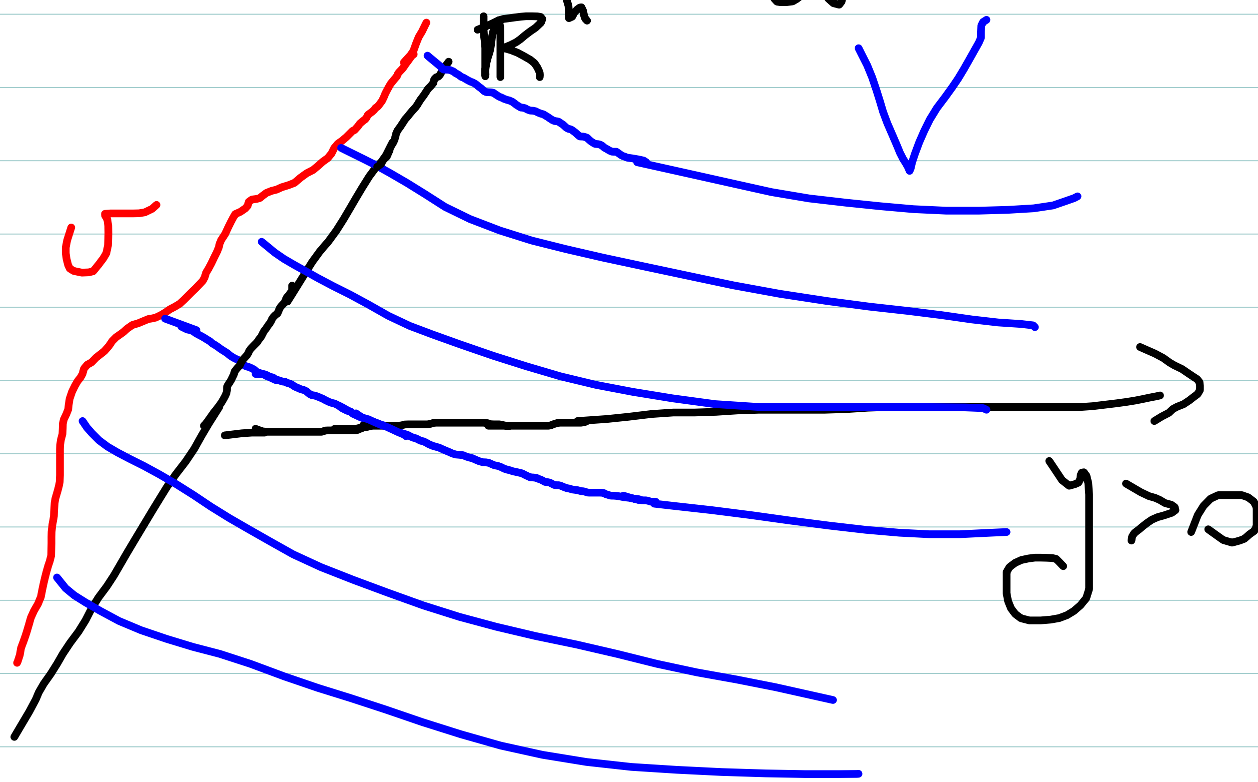


$$\begin{cases} \Delta_{x,y} V = 0 \\ V(x,0) = v(x) \end{cases}$$

Then

$$\Delta^{\frac{1}{2}} v(x) = V_y(x,0)$$

In general (Caff-Silv, '07)

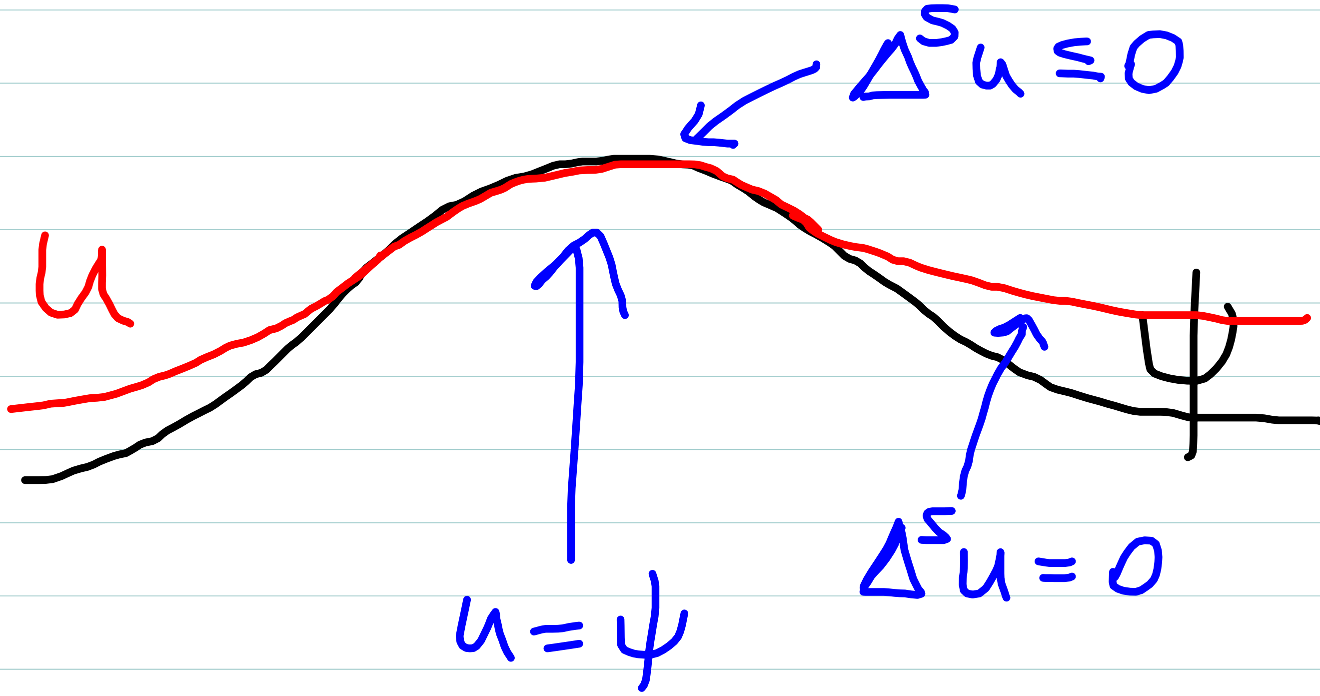


$$\begin{cases} L_a V := \operatorname{div}_{x,y}\left(y^a \nabla_{x,y} V\right) = 0 \\ V(x,0) = v(x) \end{cases}$$

$a = 1-2s$

$$\Delta^s v(x) = \lim_{y\to 0^+} y^a V_y(x,y)$$

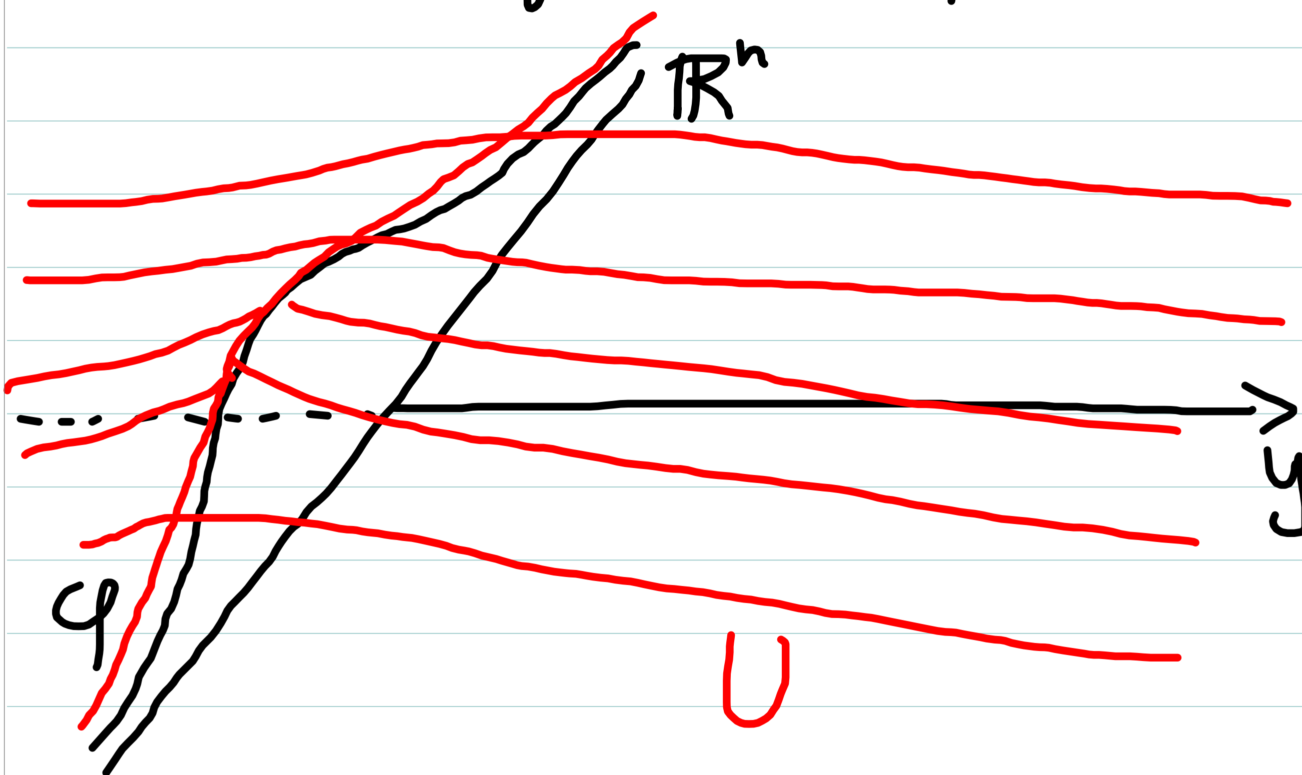
STATIONARY CASE



THM (Caff-Salsa-Silv, '09)
$u \in C^{1,s}$ (equiv, $\Delta^s u \in C^{0,1-s}$).

RMK This result is optimal

RMK: Signorini pb



$$\min_{U(x,0)\geq\varphi(x)} \int |\nabla_{x,y} U|^2$$

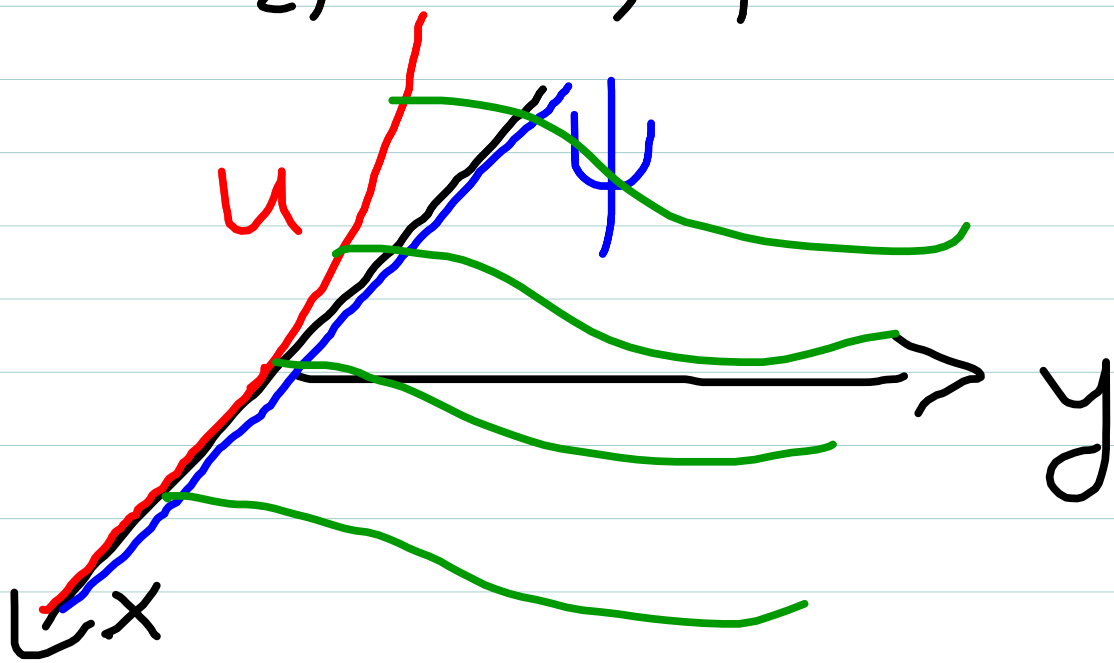
$$\Downarrow$$

$$\begin{cases} \Delta_{x,y} U = 0 \\ U_y(x,0) = 0 \text{ if } \{U>\varphi\} \end{cases}$$

$\Delta^{1/2}\, U(\cdot,0)$

Q: What regularity can we expect?

$s = 1/2, \quad n = 1, \quad \psi = 0$



$$\begin{cases} \Delta_{x,y} U = 0 \\ U_t(t,x,0) = U_y(t,x,0) \quad \text{if } U > 0 \end{cases}$$

$u_t = u_y$ has hyperbolic scaling $\rightsquigarrow$ look for travelling waves:

$$U(t,x,y) = W(at+x,y)$$

$$\begin{cases} W_y = a W_x, & \{y=0\} \cap \{W>0\} \\ \Delta_{x,y} W = 0 \end{cases}$$

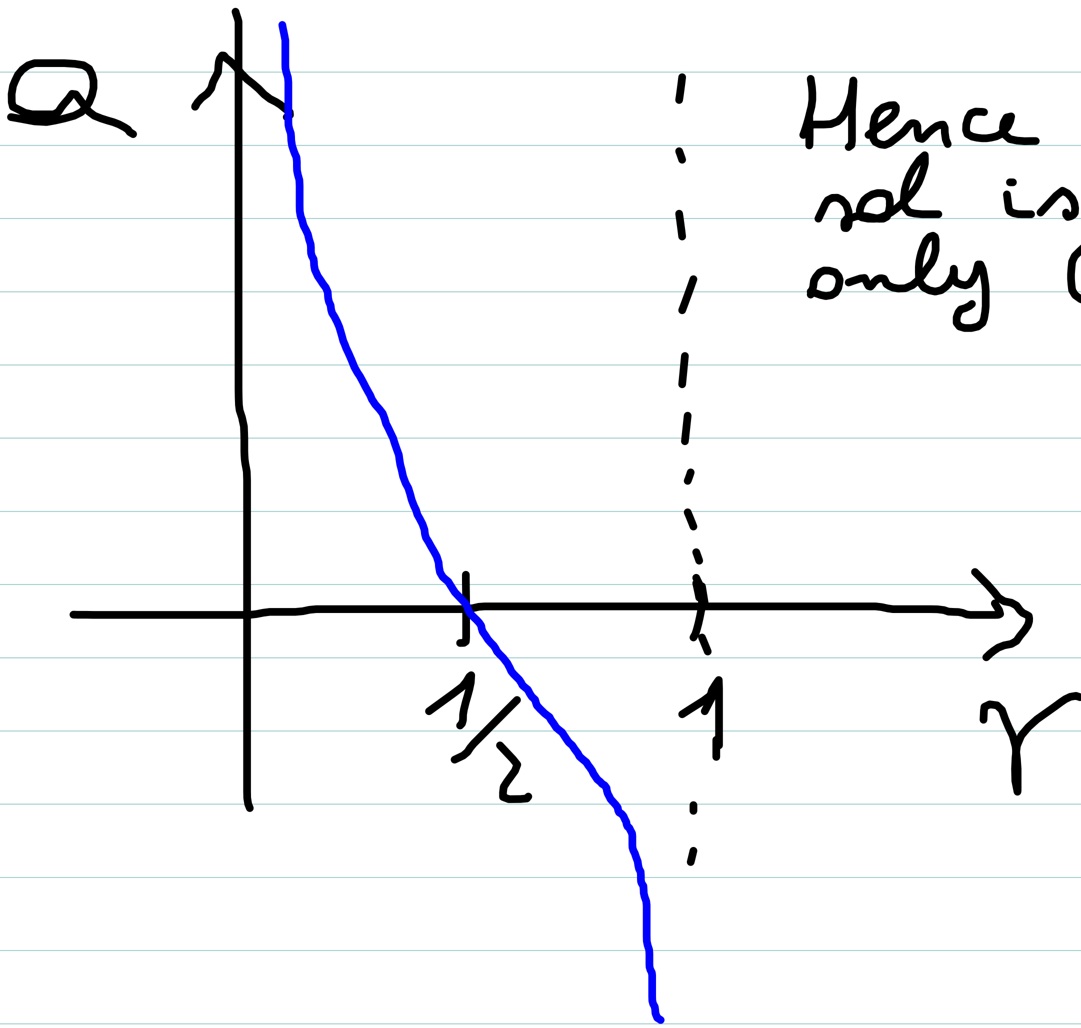
Try $\mathrm{Re}(z^\beta)$, $\mathrm{Im}(z^\beta)$,

$z = x + iy$.

$W = -\operatorname{Im}(z^{1+\gamma}), \quad \gamma \in (0,1)$

solve the eq for

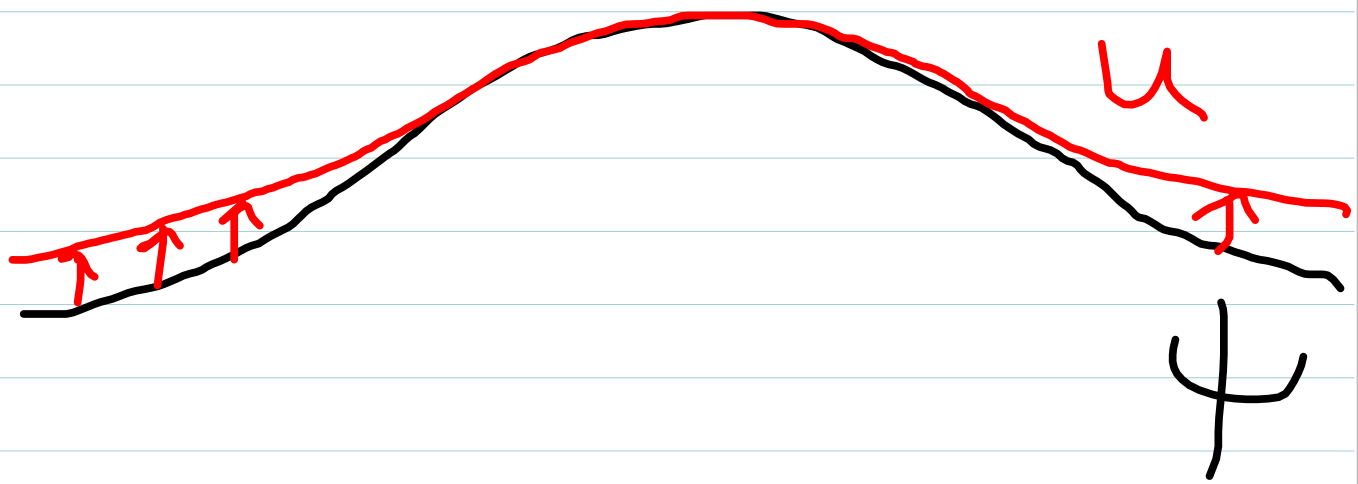
$a = 1 / tg(\gamma\pi)$



Hence sol is only $C^{1,\gamma}$!!

KEY OBSERVATION

$u(0)=\psi, \; u \geq \psi \rightsquigarrow u_t \geq 0$



Hence only $a \leq 0$ is admissible $\rightsquigarrow \gamma \geq 1/2$

$\Downarrow$

sols should be $C^{1,\frac{1}{2}}$

(RMK $a=0 \Leftrightarrow$ stat. case)

THM (Caff-F., '10)

ψ nice. Then

- $s > 1/3 \longrightarrow u_t, \Delta^s u \in C_t^{\frac{1-s}{2s}} C_x^{1-s}$
- $s \leq 1/3 \longrightarrow u_t, \Delta^s u \in \log lip_t\, C_x^{1-s}$

RMK $s = \frac{1}{2} \longrightarrow C_t^{\frac{1}{2}} C_x^{1/2}$ "OPTIMAL"

RMK Why $s = 1/3$?

$u_t = \Delta^s u \longleftrightarrow (\lambda^{2s} t, \lambda x)$

$C_x^{1-s} \longleftrightarrow C_t^{\frac{1-s}{2s}}$ if $\frac{1-s}{2s} < 1$

PROOF

1) Basic properties:

- $u \in Lip_x$
- u semiconvex
- $u_t \geq 0$
- $u_t \in L^\infty$
- $\Delta^s u \in L^\infty$

- $u \in Lip_x$:

$u(t)$ sol for ψ

$\Downarrow$

$u(t, x+v) + C|v|$

sol for $\psi(x+v) + C|v|$

$\forall v \in \mathbb{R}^n, C \in \mathbb{R}$

Choose $C = Lip(\psi)$. Then

$$\psi(x+v) + C|v| \geq \psi(x)$$

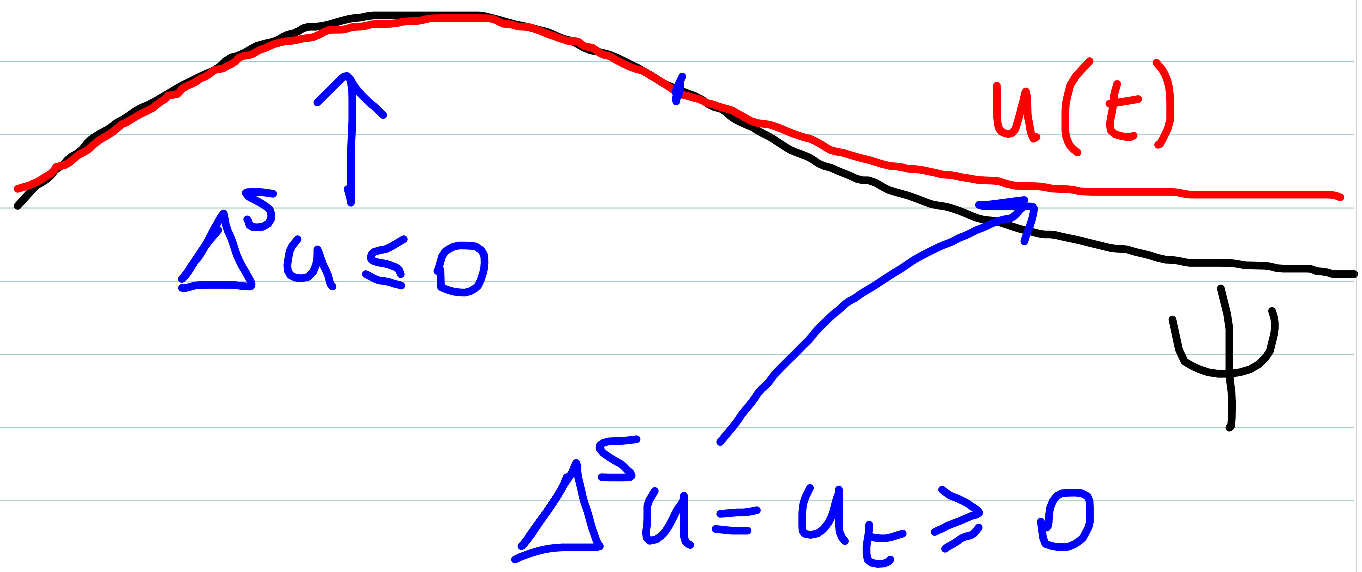
$\Downarrow$ comparison

$$u(t,x+v) + C|v| \geq u(t,x)$$

$\Downarrow$

$$Lip(u(t)) \leq Lip(\psi)$$

(2)



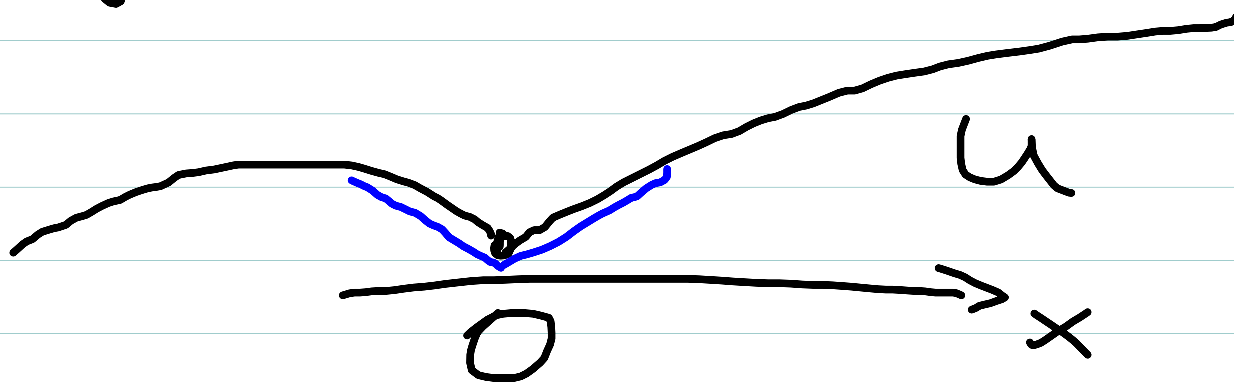
IDEA: these informations + semiconvexity should give regularity

Example

$s = 1/2$, $n = 1$

CLAIM: $u \in C^1_x$

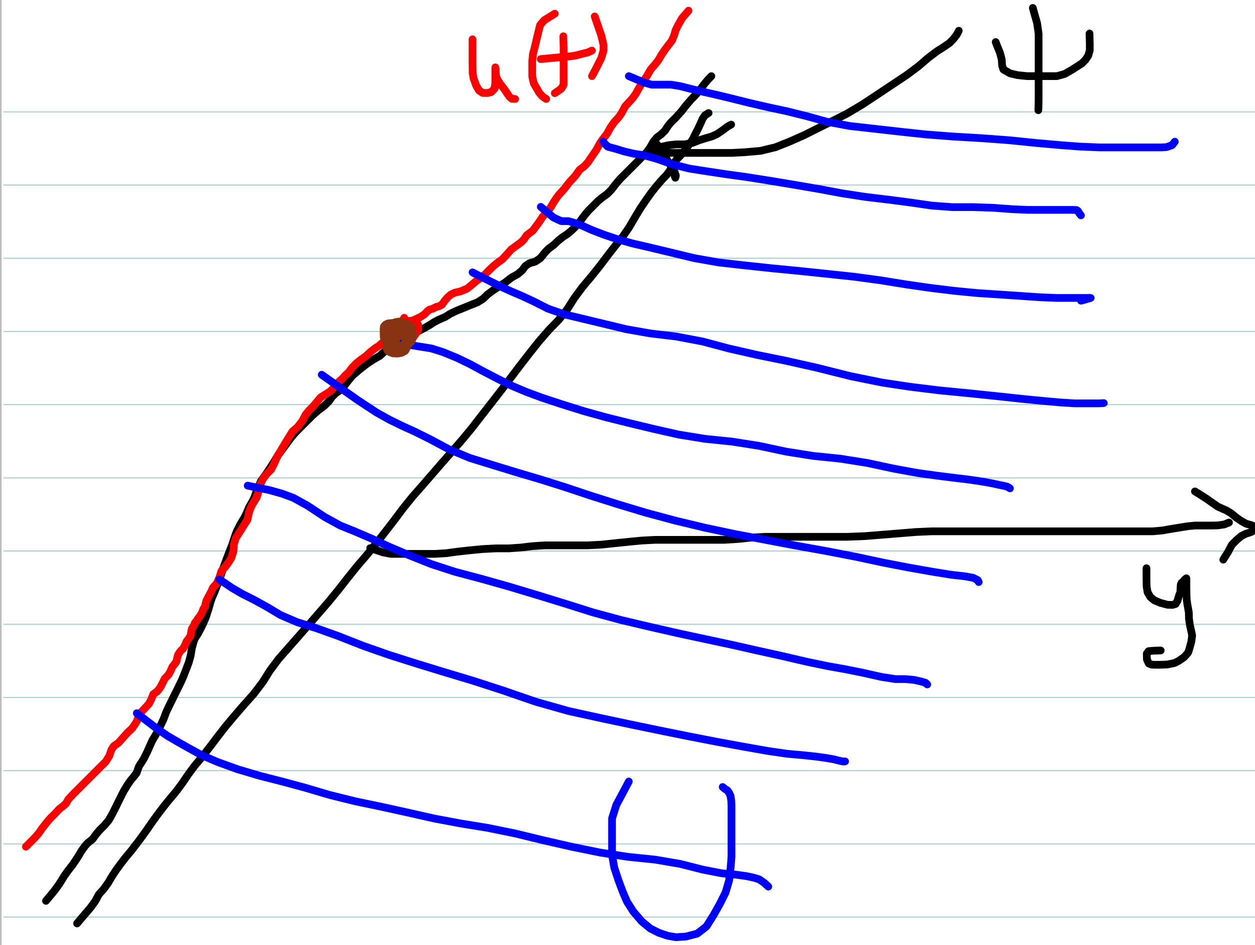
If not



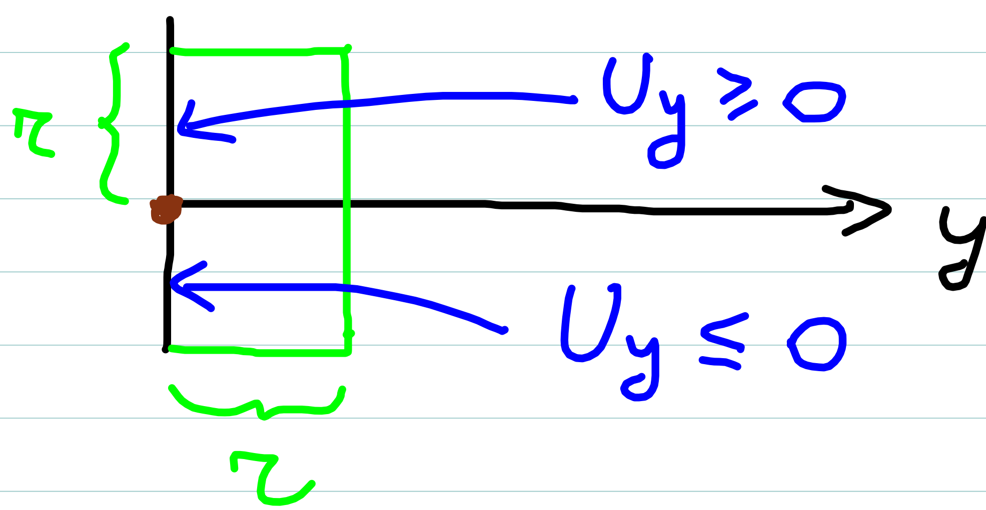
$u(x) \geq \alpha |x|$ for $|x| \leq 1$, $\alpha > 0$

$$\Delta^{\frac{1}{2}} u(0) = \int \frac{u(z) - u(0)}{|z|^{n+1}} \, dz$$

$$= C + \underbrace{\int_{B_1} \frac{u(z)}{|z|^{n+1}} \, dz}_{= +\infty}$$



For simplicity, $s = 1/2$.



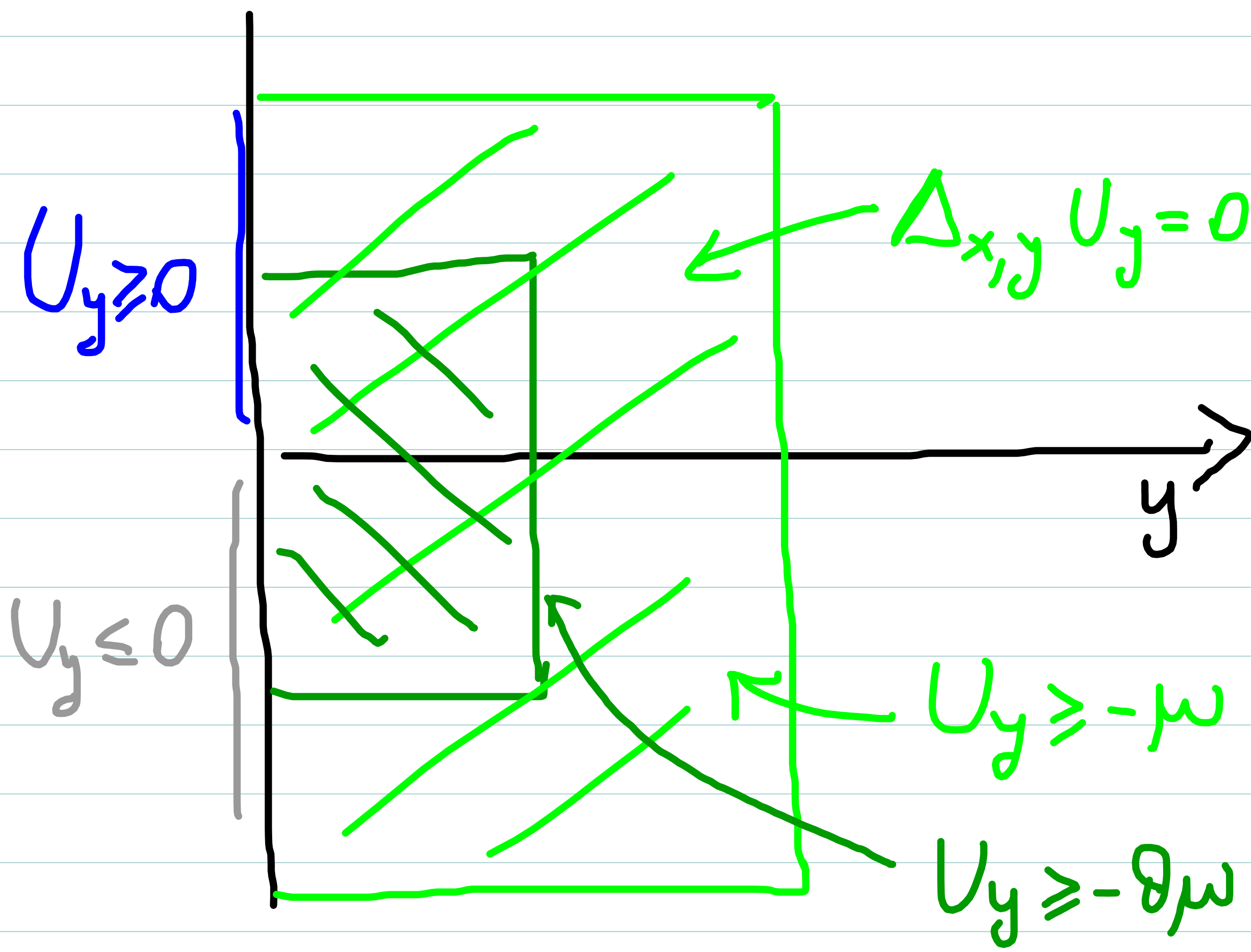
u semiconvex

$$\Downarrow$$

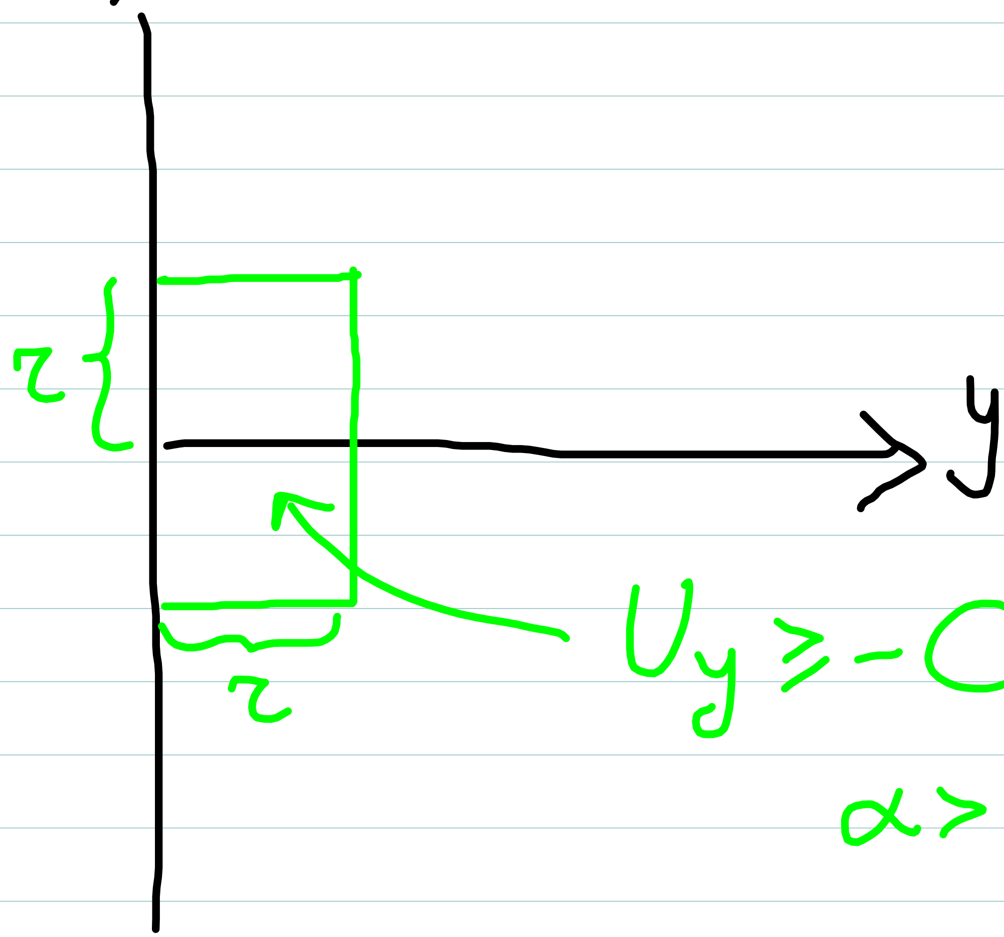
$$D_x^2 U \geq -C \Rightarrow U_{yy} \leq nC$$

$$\uparrow$$

$$\Delta_x U + U_{yy} = 0$$



The above strategy proves that



$$u_y \geq -C\tau^{\alpha},$$
$$\alpha > 0$$

COR $\quad \Delta^s u(t)\,\chi_{\{u(t)=\psi\}} \in C^\alpha_x$

$\forall\, t>0.$

By a monotonicity formula: $\Delta^s u(t)\,\chi_{\{u(t)=\psi\}} \in C^{1-s}_x$

Back to the equation:

$$u_t = \Delta^s u - \underbrace{\Delta^s u\,\chi_{\{u=\psi\}}}_{\in L^\infty_t C^{1-s}_x}$$

By parabolic regularity: regularity on u_t

Reg on u_t

$\Downarrow$

time regularity on $\Delta^s u \chi_{\{u=\varphi\}}$

$\downarrow$

Bootstrap to get the desired result. $\square$

FREE BOUNDARY

- Stationary case:

$n=1, \psi=0$: $-\operatorname{Im}(z^{k+1/2})$

sol $\forall k \geq 1$

a $u-\psi \sim r^{1+s}$

$\Rightarrow$ free bdry is $C^{1,\alpha}$

b $|u-\psi| \leq C r^2$

$\leadsto$ very delicate!

- Parabolic case: open pb.

THAT'S ALL!

THANKS FOR YOUR ATTENTION!