Numerical Methods for Differential Games based on Partial Differential Equations *

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Abstract

In this paper we present some numerical methods for the solution of two-persons zero-sum deterministic differential games. The methods are based on the dynamic programming approach. We first solve the Isaacs equation associated to the game to get an approximate value function and then we use it to reconstruct approximate optimal feedback controls and optimal trajectories. The approximation schemes also have an interesting control interpretation since the time-discrete scheme stems from a dynamic programming principle for the associated discrete time dynamical system. The general framework for convergence results to the value function is the theory of viscosity solutions. Numerical experiments are presented solving some classical pursuit-evasion games.

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1 Introduction

In this paper we present a class of numerical methods for two-persons zero-sum deterministic differential games. These methods are strongly connected to Dynamic Programming (DP in the sequel) for two main reasons. The first is that we solve the Isaacs equation related to the game and compute an approximate value function from which we derive all the informations and the approximations for optimal feedbacks and optimal trajectories. The second is that the schemes are derived from a discrete version of the Dynamic Programming Principle which gives a nice control interpretation and helps in the analysis of the properties of the schemes. This paper is intended to be a tutorial on the subject so we will present the main ideas and results trying to avoid technicalities. The interested reader can find all the details and recent developments in the list of references.

It is worth to note that DP is one of the most important tools in the analysis of two-persons zero-sum differential games. It has its roots in the classical work on calculus of variations and was applied extensively to optimal control problems by Bellman.

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The central object of investigation for this method is the *value function* $v(x)$ of the problem, which is, roughly speaking, the outcome of the game with initial state x if the players behave optimally. In the '60s many control problems in discrete-time were investigated via the DP method showing that v satisfies a functional equation and that one can derive from its knowledge optimal controls in feedback form. For discrete-time systems, DP leads to a difference equation for v, we refer the interested reader to the classical books [Ber] and [BeS] for an extended presentation of these results. For continuous-time systems, the situation is more delicate since the DP equation is a nonlinear partial differential equation (PDE in the sequel) and one has to show first that the equation has a solution and that this solution is unique, so it coincides with the value function. The main difficulty is the fact that, in general, the value functions of deterministic optimal control problems and games are not differentiable (they are not even continuous for some problems) so they are not classical solutions of the DP equation. Moreover, for a deterministic optimal control/game problem the DP equation is of first order and it is well-known that such equations do not have global classical solutions. In general, one does not even know how to interpret the DP equation at points where v is not differentiable. One way to circumvent the problem is to try to solve the equation explicitly but this can be done only for systems with low state-space dimensions (typically 1 or 2 dimensions). Isaacs used extensively the DP principle and the first order PDE which is now associated to his name in his book [I] on two-persons zero-sum differential games, but he worked mainly on the explicit solution of several examples where the value function is regular except some smooth surfaces. Only at the beginning of the eighties M.C. Crandall and P.L. Lions [CL] introduced a new notion of weak solution for a class of first-order PDEs including Isaacs' equations, and proved their existence, uniqueness and stability for the main boundary value problems. These solutions are called *viscosity solutions* because they coincide with the limits of the approximations obtained adding a vanishing second order term to the equation (a vanishing artificial viscosity in physical terms, which explains the name). The theory was reformulated by Crandall, L.C. Evans and Lions [CEL] and P.L. Lions proved in [L] that the value functions of optimal control problems are viscosity solutions of the corresponding DP equations as soon as they are continuous. The same type of results was obtained for the value function of some zero-sum differential games by Barron, Evans, Jensen and Souganidis in [BEJ], [ES], [So] for various definitions of upper and lower value. Independently, Subbotin [Su 1, Su 2] found that the value functions in the Krassovski-Subbotin [KS] sense of some differential games satisfy certain inequalities for the directional derivatives which reduce to the Isaacs' equation at points of differentiability. Moreover, he introduced a different notion of weak solution for first order nonlinear PDEs, the minmax solution. The book [Su 6] presents this theory with a special emphasis on the solution of differential games which motivated his study and the name for these solutions (since in games the DP equation has a min-max operator, see Section 2). It is important to note that several proofs were given of the equivalence of this notion of weak solution with viscosity solutions (see [EI, LS, SuT] so that nowadays the two theories are essentially unified, *cfr.*[Su 3, Su 4, Su 6]. The theory of viscosity solutions has received many contributions in the last twenty years so that it now covers several boundary and Cauchy problems for general first and second order nonlinear PDEs. Moreover, the theory has been developed to deal with discontinuous solutions at least for some classes of Hamilton-Jacobi equations, which include DP equations for control problems (see [Ba 1], [BJ 2], [S 3]). An extensive presentation of the theory of viscosity solutions can be found in the survey paper [CIL] and in the book [Ba 2]. For the applications to control problems and games the interested reader should refer to the books [FS] and [BCD].

The numerical approximation of viscosity solutions has grown in parallel with the above mentioned theoretical results and in [CL 2] Crandall and Lions have investigated a number of monotone schemes proving a general convergence result. As we said, the theory has shown to be successful in many fields of application so that numerical schemes are now available for nonlinear equations arising *e.g.* in control, games, image processing, phase-transitions, economics. The theory of numerical approximation offers some general convergence results mainly for first order schemes and for convex Hamiltonians, see [BSo], [LT]. A general results for high-order semi-Lagrangian approximation schemes has been recently proved by Ferretti [Fe]. The approximation of the value function in the framework of viscosity solutions has been investigated by several authors starting from Capuzzo Dolcetta [CD] and [F 1]. Several convergence results as well as *a-priori* error estimates are now available for the approximation of classical control problems and games, see *e.g.* [BF 1], [BF 2], [CDF], [A 1], [A 2], [BS 3]. The interested reader will find in the survey papers [F 1] and [BFS2] a comprehensive presentation of this theory respectively for control problems and games. We will present in the sequel the main results of this approach which is based on a discretization in time of the original control/game problem followed by a discretization in space which result in a fixed point problem. This approach is natural for control problems since at every discretization we keep the meaning of the approximate solutions in terms of the control problem and we have *a-priori* error estimates which just depend on the data of the problem and on the discretization steps. Thus the algorithms based on this approach produce approximate solutions which are close to the exact solution within a given tolerance. Moreover, by the approximate value function one can easily compute approximate feedback controls and optimal trajectories. For the synthesis of feedback controls we have some error estimates in the case of control problems [F 2] but the problem is still open for games. Before starting our presentation let us quote other numerical approaches related to the approximation of games. The theory of minmax solutions has also a numerical counterpart which is based on the construction of generalized gradients adapted to finite difference operators which approximate the value function. This approach has been developed by the russian school (see [TUU], [PT]) and has also produced an approximation of optimal feedback controls [T]. Another approximation for the value and for the optimal policies of dynamic zero-sum stochastic games has been proposed in [TA], [TPA] and it is based on the approximation of the game by a finite state approximation (see also [TG 1] for a numerical approximation of zero-sum differential games with stopping time). The theory of viability [A] gives a different characterization of the value function of control/game problems: the value function is the boundary of the viability kernel. This approach is based on set-valued analysis and allows to deal easily with lower semicontinuous solutions of the DP equation [F]. The numerical counterpart of this approach is based on the approximation of the viability kernel and can be found in [CQS 1] and [CQS 2]. Finally, let us mention that other numerical methods based on the approximation of open-loop control have been proposed. The advantage is of course to replace the approximation of the DP equation (which can be difficult or impossible to solve for high-dimensional problems) by a large system of ordinary differential equations exploiting the necessary conditions for the optimal policy and trajectory. The interested reader can find in [P] a general presentation and in [LBP 1], [LBP 2] some examples of the effectiveness of the method.

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The outline of the paper is the following.
In Section 1 we briefly present the general framework of the theory of viscosity solutions which allows to characterize the value function of control/game problems as the unique solution of the DP equation, at the end we give some informations about the extension to discontinuous viscosity solutions. The first part of this section partly follows the presentation in [B 2]. Section 3 is devoted to the numerical approximation, here we analyze the time and the space discretization giving the basic results on convergence and error estimates. A reader already skilled on viscosity solutions can start from there. In Section 4 we shortly present the algorithm to compute a numerical synthesis of optimal controls and give some informations on recent developments. Finally, Section 5 is devoted to the numerical solution of some classical pursuit-evasion games by the methods presented in this paper.

2 Dynamic Programming for games and viscosity solutions

Let us consider the nonlinear system

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & t > 0, \\ y(0) = x \end{cases} \tag{D}$$

where
$y(t) \in \mathbb{R}^N$ is the state
$a(\cdot) \in \mathcal{A}$ is the control of player 1 (player a)

$$\mathcal{A}\{\, a : [0, +\infty[\, \to A, \text{measurable}\,\} \tag{1}$$

$b(\cdot) \in \mathcal{B}$ is the control of player 2 (player b),

$$\mathcal{B} = \{\, b : [0, +\infty[\, \to B, \ \text{measurable}\,\}, \tag{2}$$

$A, B \subset \mathbb{R}^M$ are given compact sets. A typical choice is to take as admissible control function for the two players piecewise constant functions respectively with values in A or B. Assume f is continuous and

$$|f(x, a, b) - f(y, a, b)| \le L\, |x - y| \quad \forall x, y \in \mathbb{R}^N, \ a \in A, \ b \in B.$$

By Caratheodory's theorem the choice of measurable controls guaratees that for any given $a(\cdot) \in \mathcal{A}$ and $b(\cdot) \in \mathcal{B}$, there is a unique trajectory of (D) which we will denote by $y_x(t; a, b)$. The *payoff* of the game is

$$t_x(a(\cdot), b(\cdot)) = \min\{\, t : y_x(t; a, b) \in \mathcal{T}\,\} \le +\infty, \tag{3}$$

where $\mathcal{T} \subseteq \mathbb{R}^N$ is a given closed target. Naturally, $t_x\{a(\cdot), b(\cdot)\}$ will be finite only under additional assumptions on the target and on the dynamics. The two players are opponents since player a wants to minimize the payoff (he is called the *pursuer*) whereas player b wants to maximize the payoff (he is called the *evader*). Let us give some examples.

Example 1: Minimum time problem
This is a classical control problem, here we have just one player:

$$\begin{cases} \dot{y} = a, & A = \{\, a \in \mathbb{R}^N : |a| = 1 \,\}, \\ y(0) = x. \end{cases}$$

Since the maximum speed is 1, $t_x(a^*)$ is equal to the length of the optimal trajectory joining x and the point $y_x(t_x(a^*))$, thus

$$t_x(a^*) = \min_{a \in \mathcal{A}} t_x(a) = \mathrm{dist}(x, \mathcal{T}).$$

Note that any optimal trajectory is a straight line.

Example 2: Pursuit-Evasion games
We have two players, each one controlling its own dynamics

$$\begin{cases} \dot{y}_1 = f_1(y_1, a), & y_i \in \mathbb{R}^{N/2},\ i = 1, 2 \\ \dot{y}_2 = f_2(y_2, b) \end{cases} \tag{PEG}$$

The target is

$$\mathcal{T}_\epsilon \equiv \{\, |y_1 - y_2| \le \epsilon \,\}, \quad \text{for } \epsilon > 0, \text{ or } \mathcal{T}_0 \equiv \{\, (y_1, y_2) : y_1 = y_2 \,\} \,.$$

Then, $t_x(a(\,\cdot\,), b(\,\cdot\,))$ is the capture time corresponding to the strategies $a(\cdot)$ and $b(\cdot)$.

2.1 Dynamic Programming for a single player

Let us consider first the case of a single player. So in this section we assume $B = \{\, \bar{b} \,\}$ which allow us to write the dynamics (D) as

$$\begin{cases} \dot{y} = f(y, a), & t > 0, \\ y(0) = x. \end{cases}$$

Define the *value function*

$$T(x) \equiv \inf_{a(\,\cdot\,) \in \mathcal{A}} t_x(a) \,.$$

$T(\,\cdot\,)$ is the minimum-time function, it is the best possible outcome of the game for player a, as a function of the initial position x of the system.

Definition 2.1 *The reachable set is $\mathcal{R} \equiv \{\, x \in \mathbb{R}^N : T(x) < +\infty \,\}$,* i.e.*it is the set of starting points from which it is possible to reach the target.*

The reachable set depends on the target, on the dynamics and on the set of admissible controls in a rather complicated way, it is *not* a datum in our problem.

Lemma 1 (Dynamic Programming Principle) *For all $x \in \mathcal{R}$, $0 \le t < T(x)$ (so that $x \notin \mathcal{T}$),*

$$T(x) = \inf_{a(\,\cdot\,) \in \mathcal{A}} \{\, t + T(y_x(t; a)) \,\} \,. \tag{DPP}$$

"Proof"
The inequality "$\leq$" follows from the intuitive fact that $\forall a(\,\cdot\,)$

$$T(x) \leq t + T(y_x(t;a)).$$

The proof of the opposite inequality "$\geq$" is based on the fact that the equality holds if $a(\,\cdot\,)$ is optimal for x.

To prove rigorously the above inequalities the following two properties of $\mathcal{A}$ are crucial:

1. $a(\,\cdot\,) \in \mathcal{A} \Rightarrow \forall s \in \mathbb{R}$ the function $t \mapsto a(t+s)$ is in $\mathcal{A}$;
2. $a_1, a_2 \in \mathcal{A}$ and
$$a(t) \equiv \begin{cases} a_1(t) & t \leq s, \\ a_2(t) & t > s. \end{cases}$$
then $a(\,\cdot\,) \in \mathcal{A}$, $\forall s > 0$.

Note that the DPP works for $\mathcal{A} = \{\text{piecewise constants functions into } A\}$ but not for $\mathcal{A} = \{\text{continuous functions into } A\}$ because joining together two continuous controls we are not guaranteed that the resulting control is continuous.

Let us derive the Hamilton-Jacobi-Bellman equation from the DPP. Rewrite (DPP) as

$$T(x) - \inf_{a(\,\cdot\,)} T(y_x(t;a)) = t$$

and divide by $t > 0$,

$$\sup_{a(\,\cdot\,)} \left\{ \frac{T(x) - T(y_x(t;a))}{t} \right\} = 1 \qquad \forall t < T(x)\ .$$

We want to pass to the limit as $t \to 0^+$.

Assume T is differentiable at x and $\lim_{t\to 0^+}$ commute with $\sup_{a(\,\cdot\,)}$. Then, if $\dot{y}_x(0;a)$ exists,

$$\sup_{a(\,\cdot\,)\in\mathcal{A}} \{ -\nabla T(x) \cdot \dot{y}_x(0,a) \} = 1,$$

so that, if $\lim\limits_{t\to 0^+} a(t) = a_0$, we get

$$\sup_{a_0 \in A} \{ -\nabla T(x) \cdot f(x,a_0) \} = 1\ . \tag{HJB}$$

This is the Hamilton-Jacobi-Bellman partial differential equation associated to the minimum time problem. It is a first order, fully nonlinear PDE. Note that in (HJB) the supremum is taken over the A and not on the set of measurable controls $\mathcal{A}$.

Let us define the Hamiltonian,

$$H_1(x,p) \equiv \sup_{a\in A}\{ -p \cdot f(x,a) \} - 1,$$

we can rewrite (HJB) in short as

$$H_1(x, \nabla T(x)) = 0 \text{ in } \mathcal{R} \setminus \mathcal{T}.$$

Note that $H_1(x, \cdot)$ is convex since is the sup of linear operators. A natural *boundary condition* on $\partial\mathcal{T}$ is

$$T(x) = 0 \qquad \text{for } x \in \partial\mathcal{T}$$

Let us prove that if T is regular then it is a classical solution of (HJB) ($i.e. T \in C^1(\mathbb{R} \setminus \mathcal{T})$) and (HJB) is satisfied pointwise).

Proposition 2 *If $T(\cdot)$ is C^1 in a neighborhood of $x \in \mathcal{R} \setminus \mathcal{T}$, then $T(\cdot)$ satisfies (HJB) at x.*

Proof
We first prove the inequality "$\leq$".
Fix $\bar{a}(t) \equiv a_0\ \forall t$, and set $y_x(t) = y_x(t; \bar{a})$. (DPP) gives

$$T(x) - T(y_x(t)) \leq t \qquad \forall\ 0 \leq t < T(x).$$

We divide by $t > 0$ and let $t \to 0^+$ to get

$$-\nabla T(x) \cdot \dot{y}_x(0) \leq 1,$$

where $\dot{y}_x(0) = f(x, a_0)$ (since $\bar{a}(t) \equiv a_0$). Then,

$$-\nabla T(x) \cdot f(x, a_0) \leq 1 \qquad \forall a_0 \in A$$

and we get

$$\sup_{a \in A} \{ -\nabla T(x) \cdot f(x, a) \} \leq 1 \,.$$

Next we prove the inequality "$\geq$".
Fix $\varepsilon > 0$. For all $t \in]0, T(x)[$, by (DPP) there exists $\alpha_\varepsilon \in \mathcal{A}$ such that

$$T(x) \geq t + T(y_x(t; \alpha_\varepsilon)) - \varepsilon t \,,$$

which implies

$$1 - \varepsilon \leq \frac{T(x) - T(y_x(t; \alpha_\varepsilon))}{t} = -\frac{1}{t} \int_0^t \frac{\partial}{\partial s} T(y_x(s; \alpha_\varepsilon))\, ds = -\frac{1}{t} \int_0^t \nabla T(y_x(s)) \cdot \dot{y}_x(s; \alpha_\varepsilon)\, ds \,. \tag{4}$$

Adding and subtracting the term $\nabla T(x) \cdot f(y_x(s; \alpha_\varepsilon))$ by the Lipschitz continuity of f we get from (4)

$$1 - \varepsilon \leq -\frac{1}{t} \int_0^t \nabla T(x) \cdot f(x, \alpha(s))\, ds + o(1) \tag{5}$$

so for $t \to 0^+$, $\varepsilon \to 0^+$ we finally get

$$\sup_{a \in A} \{ -\nabla T(x) \cdot f(x, a) \} \geq 1 \,.$$

□

Unfortunately T is not regular even for simple dynamics as the following example shows. Consider Example 1 where $T(x) = \text{dist}(x, \mathcal{T})$ it is easy to see that T is not differentiable at x if there exist two distinct points of minimal distance. In fact, for $N = 1$, $f(x,a) = a$, $A = B(0,1)$ and

$$\mathcal{T} =]-\infty, -1] \cup [1, +\infty[\ .$$

we have

$$T(x) = 1 - |x|$$

which is *not* differentiable at $x = 0$. We must observe also that in this example the Bellman equation is the eikonal equation

$$|Du(x)| = 1 \tag{6}$$

which has infinitely many a.e. solutions also when we fix the values on the boundary $\partial\mathcal{T}$,

$$u(-1) = u(1) = 0$$

Also the continuity of T is, in general, not guarateed.
Take the previous example and set $A = [-1, 0]$, then we have

$$T(1) = 0 \qquad \lim_{x\to 1} T(x) = 2 \tag{7}$$

However, the continuity of $T(\,\cdot\,)$ is equivalent to the property of Small-Time Local Controllability (STLC) around $\mathcal{T}$.

Definition 2.2 *Assume $\partial\mathcal{T}$ smooth. We say that the STLC is satisfied if*

$$\forall x \in \partial\mathcal{T}\ \exists \hat{a} \in A : \quad f(x, \hat{a}) \cdot \eta(x) < 0. \tag{STLC}$$

where $\eta(x)$ is the exteriour normal to $\mathcal{T}$ at x.

The STLC guarantees that $\mathcal{R}$ is an open subset of $\mathbb{R}^N$ and that

$$\lim_{x\to x_0} T(x) = +\infty, \qquad \forall x_0 \in \partial\mathcal{R}$$

We want to interpret the HJB equation in a "weak sense" so that $T(\,\cdot\,)$ is a "solution" (non-classical), unique under suitable boundary conditions.

Let's go back to the proof of Proposition 2.
We showed that

1. $T(x) - T(y_x(t)) \le t$, $\forall t$ small and $T \in C^1$ implies $H(x, \nabla T(x)) \le 0$;
2. $T(x) - T(y_x(t)) \ge t(1-\varepsilon)$, $\forall t, \varepsilon$ small and $T \in C^1$ implies $H(x, \nabla T(x)) \ge 0$.

Idea: If $\phi \in C^1$ and $T - \phi$ has a maximum at x then

$$T(x) - \phi(x) \ge T(y_x(t)) - \phi(y_x(t)) \qquad \forall t,$$

thus

$$\phi(x) - \phi(y_x(t)) \le T(x) - T(y_x(t)) \le t,$$

so we can replace T by ϕ in the proof of Proposition 2 and get

$$H(x, \nabla\phi(x)) \le 0\ .$$

Similarly, if $\phi \in C^1$ and $T - \phi$ has a minimum at x, then

$$T(x) - \phi(x) \leq T(y_x(t)) - \phi(y_x(t)), \qquad \forall t.$$

thus

$$\phi(x) - \phi(y_x(t)) \geq T(x) - T(y_x(t)) \geq t(1-\varepsilon)$$

and, by the proof of Proposition 2,

$$H(x, \nabla\phi(x)) \geq 0 \; .$$

Thus, the classical proof can be fixed when T is not C^1 replacing T with a "test function" $\phi \in C^1$.

Definition 2.3 (Crandall-Evans-Lions [CEL]) *Let $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous, $\Omega \subseteq \mathbb{R}^N$ open. We say that $u \in C(\Omega)$ is a viscosity subsolution of*

$$F(x, u, \nabla u) = 0 \qquad \textit{in } \Omega$$

if $\forall \phi \in C^1$, $\forall x_0$ local maximum point of $u - \phi$,

$$F(x_0, u(x_0), \nabla\phi(x_0)) \leq 0.$$

It is a viscosity supersolution if $\forall \phi \in C^1$, $\forall x_0$ local minimum point of $u - \phi$,

$$F(x_0, u(x_0), \nabla\phi(x_0)) \geq 0.$$

A viscosity solution is a sub- and supersolution.

Theorem 3 *If $\mathcal{R} \setminus \mathcal{T}$ is open and $T(\,\cdot\,)$ is continuous, then $T(\,\cdot\,)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB).*

Proof. The proof is the argument before the definition. □

The following result (see *e.g.* [BCD] for the proof) shows the link between viscosity and classical solutions.

Corollary 4

1. *If u is a classical solution of $F(x, u, \nabla u) = 0$ in Ω then u is a viscosity solution;*
2. *if u is a viscosity solution of $F(x, u, \nabla u) = 0$ in Ω and if u is differentiable at x_0 then the equation is satisfied in the classical sense at x_0,* i.e.

$$F(x_0, u(x_0), \nabla u(x_0)) = 0 \; .$$

It is clear that the set of viscosity solutions contains that of classical solutions. The main issue in this theory of weak solutions is to prove uniqueness results. This point is very important also for numerical purposes since the fact that we have a unique solution allow to prove convergence results for the approximation schemes. To this end, let us consider the Dirichlet boundary value problem

$$\begin{cases} u + H(x, \nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{BVP}$$

and prove a uniqueness result under assumptions on H including Bellman's Hamiltonian H_1. This new boundary value problem is connected to (HJB) because the new solution of (BVP) is a rescaling of T. In fact, introducing the new variable

$$V(x) \equiv \begin{cases} 1 - e^{-T(x)} & \text{if } T(x) < +\infty, \text{ i.e. } x \in \mathcal{R} \\ 1 & \text{if } T(x) = +\infty, \ (x \notin \mathcal{R}) \end{cases} \tag{8}$$

it is easy to check that, by the DPP,

$$V(x) = \inf_{a(\,\cdot\,)\in\mathcal{A}} J(x, a)$$

where

$$J(x, a) \equiv \int_0^{t_x(a)} e^{-t}\, dt\ .$$

Moreover, V is a solution of

$$\begin{cases} V + \max_{a\in A}\{ -\nabla V \cdot f(x, a) - 1 \} = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{T} \\ V = 0 & \text{on } \partial\mathcal{T}, \end{cases} \tag{BVP-B}$$

which is a special case of (BVP), with $H(x, p) = H_1(x, p) \equiv \max_{a\in A}\{ -p\cdot f(x, a) - 1 \}$ and $\Omega = \mathcal{T}^c \equiv \mathbb{R}^N \setminus \mathcal{T}$.

The change of variable (8) is called Kružkov transformation and has several advantages. First of all V takes values in $[0, 1]$ whereas T is generally unbounded and this helps in the numerical approximation. Moreover, one can always reconstruct T and $\mathcal{R}$ from V by the relations

$$T(x) = -\log(1 - V(x)), \qquad \mathcal{R} = \{ x : V(x) < 1 \}.$$

Lemma 5 *The Mininimum Time Hamiltonian H_1 satisfies the "structural condition"*

$$|H(x, p) - H(y, q)| \le K(1 + |x|)|p - q| + |q|\, L\, |x - y| \quad \forall x, y, p, q, \tag{SH}$$

where K and L are two positive constants.

Theorem 6 (Crandall-Lions [CL 1]) *Assume H satisfies (SH), $u, w \in BUC(\bar\Omega)$, u subsolution, w supersolution of $v + H(x, \nabla v) = 0$ in Ω (open), $u \le w$ on $\partial\Omega$. Then, $u \le w$ in Ω.*

Definition 2.4 *We call subsolution (respectively supersolution) of (BVP-B) a subsolution (respectively supersolution) u of the differential equation such that $u \le 0$ on $\partial\Omega$ (respectively ≥ 0 on $\partial\Omega$).*

Corollary 7 *If the value function $V(\cdot) \in BUC(\mathcal{T}^c)$, then V is the maximal subsolution and the minimal supersolution of (BVP-B) (we say it is the complete solution). Thus V is the unique viscosity solution.*

If the system is STLC around $\mathcal{T}$ (*i.e.* $T(\cdot)$ is continuous at each point of $\partial\mathcal{T}$) then $V \in \mathrm{BUC}(\mathcal{T}^c)$ and we can apply the Corollary.

2.2 Dynamic Programming for games

We now go back to our original problem to develop the same approach. The first question is: how can we define the value function for the 2-players game ? Certainly it is *not*

$$\inf_{a \in \mathcal{A}} \sup_{b \in \mathcal{B}} J(x, a, b)$$

because a would choose his control function with the information of the whole future response of player b to any control function $a(\cdot)$ and this will give him a big advantage.

A more unbiased information pattern can be modeled by means of the notion of *nonanticipating strategies* (see [EK] and the references therein),

$$\Delta \equiv \{ \alpha : \mathcal{B} \to \mathcal{A} : \ b(t) = \widetilde{b}(t) \ \forall t \le t' \Rightarrow \ \alpha[b](t) = \alpha[\widetilde{b}](t) \ \forall t \le t' \}, \tag{9}$$

$$\Gamma \equiv \{ \beta : \mathcal{A} \to \mathcal{B} : \ a(t) = \widetilde{a}(t) \ \forall t \le t' \Rightarrow \qquad \beta[a](t) = \beta[\widetilde{a}](t) \ \forall t \le t' \} . \tag{10}$$

The above definition is fair with respect to the two players. In fact, if player a chooses his control in Δ he will not be influenced by the future choices of player b (Γ has the same role for player b). Now we can define the *lower value* of the game

$$T(x) \equiv \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} t_x(\alpha[b], b),$$

or

$$V(x) \equiv \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} J(x, \alpha[b], b)$$

when the payoff is $J(x, a, b) = \int_0^{t_x(a,b)} e^{-t}\, dt$.

Similarly the *upper value* of the game is

$$\widetilde{T}(x) \equiv \sup_{\beta \in \Gamma} \inf_{a \in \mathcal{A}} t_x(a, \beta[a]),$$

or

$$\widetilde{V}(x) \equiv \sup_{\beta \in \Gamma} \inf_{a \in \mathcal{A}} J(x, a, \beta[a]) \ .$$

We say that the game has a value if the upper and lower values coincide, *i.e.*if $T = \widetilde{T}$ or $V = \widetilde{V}$.

Lemma 8 (DPP for games) *For all* $0 \le t < T(x)$

$$T(x) = \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} \{ t + T(y_x(t; \alpha[b], b)) \}, \quad \forall x \in \mathcal{R} \setminus \mathcal{T},$$

and

$$V(x) = \inf_{\alpha \in \Delta} \sup_{b \in \mathcal{B}} \Big\{ \int_0^t e^{-s}\, ds + e^{-t} V(y_x(t; \alpha[b], b)) \Big\}, \quad \forall x \in \mathcal{T}^c.$$

The proof is similar to the 1-player case but more technical due to the use of non-anticipating strategies. Note that the upper values $\widetilde{T}$ and $\widetilde{V}$ satisfy a similar DPP. Let us introduce the two Hamiltonians for games *Isaacs' Lower Hamiltonian*

$$H(x,p) \equiv \min_{b\in B} \max_{a\in A}\{ -p\cdot f(x,a,b) \} - 1 \, .$$

Isaacs' Upper Hamiltonian

$$\widetilde{H}(x,p) \equiv \max_{a\in A} \min_{b\in B} \{ -p\cdot f(x,a,b) \} - 1 \, .$$

Theorem 9 (Evans-Souganidis [ES]) *1. If $\mathcal{R}\setminus\mathcal{T}$ is open and $T(\,\cdot\,)$ is continuous, then $T(\,\cdot\,)$ is a viscosity solution of*

$$H(x,\nabla T) = 0 \qquad in\ \mathcal{R}\setminus\mathcal{T}. \tag{HJI-L}$$

2. If $V(\,\cdot\,)$ is continuous, then it is a viscosity solution of

$$V + H(x,\nabla V) = 0 \qquad in\ \mathcal{T}^c.$$

The structural condition (SH) plays an important role for uniqueness.

Lemma 10 *Isaacs' Hamiltonians $H, \widetilde{H}$ satisfy the structural condition (SH).*

Then Comparison Theorem 6 applies and we get

Theorem 11 *If the lower value function $V(\,\cdot\,)\in BUC(\mathcal{T}^c)$, then V is the complete solution (maximal subsolution and minimal supersolution) of*

$$\begin{cases} u + H(x,Du) = 0 & in\ \mathcal{T}^c, \\ u = 0 & on\ \partial\mathcal{T}. \end{cases} \tag{BVP-I-L}$$

Thus V is the unique viscosity solution.

Note that for the upper value functions $\widetilde{T}$ and $\widetilde{W}$ the same results are valid with $H = \widetilde{H}$. We can give "capturability" conditions on the system ensuring $V, \widetilde{V} \in \mathrm{BUC}(\mathcal{T}^c)$.

However, those conditions are less studied for games because there are important pursuit-evasion games with discontinuous value, the games with "barriers" (*cfr.*[I]). It is important to note that in general the upper and the lower values are different. However, the Isaacs condition

$$H(x,p) = \widetilde{H}(x,p) \qquad \forall x,p, \tag{11}$$

guarantees that they coincide.

Corollary 12 *If $V, \widetilde{V} \in BUC(\mathcal{T}^c)$, then*

$$V \le \widetilde{V}, \qquad T \le \widetilde{T}\, .$$

*If Isaacs condition holds then $V = \widetilde{V}$ and $T = \widetilde{T}$, (*i.e.*the game has a value).*

Proof
Immediate from the comparison and uniqueness for (BVP-I-L). ∎

For numerical purposes, one can decide to write down an approximation scheme for either the upper or the lower value using the techniques of the next section. Before going to it, let us give some informations about the characterization of discontinuous value functions for games. This is an important issue because discontinuities appear even in classical pursuit-evasion games (*e.g.* in the homicidal chauffeur game that we will present and solve in Section 5). We will denote by $B(\Omega)$ the set of bounded real functions defined on Ω. Let us start with a definition which has been successfully applied to convex Hamiltonians.

Definition 2.5 (Discontinuous Viscosity Solutions) *Let $H(x, u, \cdot)$ be convex. We define,*

$$u^*(x) = \liminf_{y \to x} \ u(y), \qquad u_*(x) = \limsup_{y \to x} \ u(y)$$

We say that $u \in B(\Omega)$ is a viscosity solution if $\forall \, \phi \in C^1(\Omega)$, the following conditions are satisfied:
1. at every local maximum point x_0 for $u^* - \phi$,

$$H(x_0, u^*(x_0), \nabla\phi(x_0)) \leq 0.$$

2. at every local minimum point x_0 for $u_* - \phi$,

$$H(x_0, u_*(x_0), \nabla\phi(x_0)) \geq 0.$$

As we have seen, to obtain uniqueness one should prove that a comparison principle holds, *i.e.*for every subsolution w and supersolution W we have

$$w \leq W$$

Although this is sufficient to get uniqueness in the convex case the above definition will *not* guarantee uniqueness for nonconvex hamiltonians (*e.g.*min-max Hamiltonians).

Two new definitions have been proposed. Let us denote by S the set of subsolutions of our equation and by Z the set of supersolutions always satisfying the Dirichlet boundary condition on $\partial\Omega$.

Definition 2.6 (minmax solutions [Su 6]) *u is a* minmax *solution if there exists two sequences $w_n \in S$ and $W_n \in Z$ such that*

$$w_n = W_n = 0 \quad \text{on } \partial\Omega \tag{12}$$
$$w_n \text{ is continuous on } \partial\Omega \tag{13}$$
$$\text{and } \lim_n w_n(x) = u(x) = \lim_n W_n(x), \qquad x \in \overline{\Omega}. \tag{14}$$

Definition 2.7 (e-solutions, see *e.g.*[BCD]) *u is an e-solution (envelope solution) if there exists two non empty subsets*

$$S(u) \subset S \qquad Z(u) \subset Z$$

such that $\forall\, x \in \overline{\Omega}$

$$u(x) = \sup_{w \in S(u)} w(x) = \inf_{W \in Z(u)} W(x)$$

By the comparison Lemma there exists a unique e-solution.
In fact, if u and v are two e-solutions,

$$u(x) = \sup_{w \in S(u)} w(x) \leq \inf_{W \in Z(v)} W(x) = v(x)$$

and also

$$v(x) = \sup_{w \in S(v)} w(x) \leq \inf_{W \in Z(u)} W(x) = u(x)$$

It is interesting to note that in our problem the two definitions coincide.

Theorem 13 *Under our hypotheses, u is a minmax solution if and only if u is an e-solution.*

3 Numerical approximation

We will describe a method to construct approximation schemes for the Isaacs equation where we try to keep the essential informations of the game/control problem which is behind it. In this approach, the numerical approximation of the first order PDE is based on a time-discretization of the original control problem via discrete DP principle. Then, the functional equation for the time-discrete problem is "projected" on a grid to derive a finite dimensional fixed point problem. Naturally, one can also choose to construct directly an approximation scheme for the Isaacs equation based on classical methods for hyperbolic PDE, *e.g.*using a Finite Difference (FD) scheme. However this choice is not simpler from the point of view of implementation since it is well known that an up-wind correction is needed in the scheme to keep stability and obtain a converging scheme (*cfr.*[Str]). Moreover, proving convergence of the scheme by only PDE arguments is sometimes more complicated. The dynamic programming schemes which we present in this section have a built-in up-wind correction and convergence can be proved also using control arguments.

3.1 Time discretization

Let us start by the time discretization of the minimum time problem. The fact that there is only one player makes easier to describe the scheme and to introduce the basic ideas which are behind this approach. In the previous section, we have seen how one can obtain by the Kružkov change of variable (8) a characterization of the (rescaled) value function as the unique viscosity solution of (BVP-B). To build the approximation let us choose a time step $h = \Delta t > 0$ for the dynamical system and define the discrete times $t_m = mh$, $m \in \mathbb{N}$. We can obtain a discrete dynamical system associated to (D) just using any one-step scheme for the Cauchy problem. A well known example is the explicit Euler scheme which corresponds to the following discrete dynamical system

$$\begin{cases} x_{m+1} = x_m + hf(x_m, a_m) \\ x_0 = x \end{cases} \tag{D_h}$$

We will denote by $y_x(n;\{a_m\})$ the state at time nh of the discrete time trajectory verifying (D_h). Define the discrete analogue of the reachable set

$$\mathcal{R}_h \equiv \{x \in \mathbb{R}^N : \exists \text{ a sequence } \{a_m\} \text{ and } m \in \mathbb{N} \text{ such that } x_m \in \mathcal{T}\} \tag{15}$$

and

$$n_h(\{a_m\}, x) = \begin{cases} +\infty & x \notin \mathcal{R}_h \\ \min\{m \in \mathbb{N} : x_m \in \mathcal{T}\} & \forall x \in \mathcal{R}_h \end{cases} \tag{16}$$

$$N_h(x) = \min_{\{a_m\}} n_h(\{a_m\}, x), \tag{17}$$

Thus the discrete analogue of the minimum time function $T(\cdot)$ is $N_h(\cdot)h$.

Lemma 14 (Discrete Dynamic Programming Principle) *Let $h > 0$ be fixed. For all $x \in \mathcal{R}_h$, $0 \le n < N_h(x)$ (so that $x \notin \mathcal{T}$),*

$$hN_h(x) = \inf_{\{a_m\}} \{\, hn + N_h(y_x(n;\{a_m\}))\,\}\ . \tag{DDPP}$$

Sketch of the proof.
The "$\le$" inequality is easy since hN_h is clearly lower than any choice which makes n steps

on the dynamics and then is optimal starting from x_n. For the reverse inequality, by definition, for every $\epsilon > 0$ there exists a sequence $\{a^\epsilon_n\}$ such that

$$hN_h(x) > n_h(\{a^\epsilon_m\}, x) - \epsilon > hn + N_h(y_x(n; a^\epsilon_m))\ \ - \epsilon \tag{18}$$

Since ϵ is arbitrary, this concludes proof. □

As in the continuous problem, we apply the Kružkov change of variable

$$v_h(x) = 1 - e^{-h\,N_h(x)}. \tag{19}$$

Note that, by definition, $0 \le v_h \le 1$ and v_h has constant values on the set of initial points x which can be driven to $\mathcal{T}$ by the discrete dynamical system in the same number of steps (of constant width h).

Writing the discrete Discrete Dynamic Programming Principle for $n = 1$, and changing variable we get the following characterization of v_h

$$v_h(x) = S(v_h)(x) \qquad \text{on } \mathbb{R}^N \setminus \mathcal{T} \tag{HJB_h}$$

$$v_h(x) = 0 \qquad\qquad \text{on } \mathcal{T} \tag{BC_h}$$

where

$$S(v_h)(x) \equiv \min_{a\in A} \left[e^{-h} v_h(x + hf(x, a))\right] + 1 - e^{-h} \tag{20}$$

In fact, note that $x \in \mathcal{R}^c_h \equiv \mathbb{R}^N \setminus \mathcal{R}$ implies $x + hf(x, a) \in \mathcal{R}^c_h$ so we can easily extend v_h to $\mathcal{R}^c_h$ just defining

$$v_h(x) = 1 \qquad \text{on } \mathcal{R}^c_h$$

and get rid of $\mathcal{R}_h$ finally setting (HJB_h) on $\mathbb{R}^N \setminus \mathcal{T}$.

Theorem 15 *v_h is the unique bounded solution of $(HJB_h) - (BC_h)$.*

Sketch of the proof.
The proof directly follows from the fact that S defined in (20) is a contraction map in

$L^\infty(\mathbb{R}^N)$. In fact, one can easily prove (see [BF 1] for details) that for every $x \in \mathbb{R}^N$

$$|S(u)(x) - S(w)(x)| \le e^{-h}\|u - w\|_\infty, \qquad \text{for any } u, w \in L^\infty(\mathbb{R}^N). \tag{21}$$

□

In order to prove an error bound for the approximation we need to introduce some assumptions which are the discrete analogue of the local controllability assumptions of Section 1. Let us define the δ-neighbourhood of $\partial\mathcal{T}$

$$\mathcal{T}_\delta \equiv \partial\mathcal{T} + \delta B(0,1) \text{ , and } d(x) \equiv \text{ dist } (x, \partial\mathcal{T})$$

We are now able to prove the following upper bounds for our approximations

Lemma 16 *Under our assumptions on f and STLC , there exist some positive constants $\overline{h}$, δ such that*

$$v_h(x) \le C\,d(x) + h \ , \qquad \forall\, h < \overline{h} \ , \quad x \in \mathcal{T}_\delta$$

Theorem 17 *Let the assumptions of Lemma 16 be satisfied and let $\mathcal{T}$ be compact with nonempty interiour.*
Then, v_h converges to v locally uniformly in $\mathbb{R}^N$ for $h \to 0^+$

Sketch of the proof.
Since v_h is a discontinuous function we define the two semicontinuous envelopes

$$\underline{v} = \liminf_{\substack{h\to 0^+\\ y\to x}} v_h(y), \qquad \overline{v} = \limsup_{\substack{h\to 0^+\\ y\to x}} v_h(y)$$

Note that $\underline{v}$ (respectively $\overline{v}$) is lower (respectively upper) semicontinuous. The first step is to show that

1. $\overline{v}$ is a viscosity subsolution for (HJB)
2. $\underline{v}$ is a viscosity supersolution for (HJB)

Then, we want show that both the envelopes satisfy the boundary condition on $\mathcal{T}$. In fact, by Lemma 16,

$$v_h(x) \le C\,d(x) + h$$

which implies

$$|\overline{v}| \le C\,d(x) \tag{22}$$

$$|\underline{v}| \le C\,d(x) \tag{23}$$

so we have

$$\underline{v} = \overline{v} = 0 \quad \text{on } \partial\mathcal{T}$$

Since the two envelopes coincide on $\partial\mathcal{T}$ we can apply the comparison theorem for semi-continuous sub and supersolutions in [BP] and obtain

$$v = \underline{v} = \overline{v} \quad \text{on } \mathbb{R}^N .$$

□

We now want to prove an error estimate for our discrete time approximation. Let us assume Q is a compact subset of $\mathcal{R}$ where the following condition holds:

$$\begin{array}{l}\exists\, C_0 > 0 :\ \forall x \in Q \text{ there is a time optimal control with} \\ \text{total variation less than } C_0 \text{ bringing the system to } \mathcal{T}.\end{array} \tag{BV}$$

Theorem 18 ([BF 2]) *Let the assumptions of Lemma 16 be verified and let Q be a compact subset of $\mathcal{R}$ where (BV) holds. Then there exists two positive constants $\overline{h}$ and C such that*

$$|v(x) - v_h(x)| \le Ch \qquad \forall x \in Q, h \le \overline{h} \tag{E}$$

The above results show that the rate of convergence for the scheme based on the Euler scheme is 1, which is exactly what we expected.

Now let us go back to games. The same time discretization can be written for the dynamics and natural extensions of N_h and v_h are easily obtained. The crucial point is to prove that the discrete dynamic programming principle holds true and that the upper value of the discrete game is the unique solution in $L^\infty(\mathbb{R}^N)$ of the external Dirichlet problem

$$v_h(x) = S(v_h)(x) \qquad \text{on } \mathbb{R}^N \setminus \mathcal{T} \tag{HJI_h}$$

$$v_h(x) = 0 \qquad \text{on } \partial\mathcal{T} \tag{BC_h}$$

where the fixed point operator now is

$$S(v_h)(x) \equiv \max_{b\in B} \min_{a\in A} \left[e^{-h} v_h(x + hf(x,a,b)) \right] + 1 - e^{-h}$$

The next step is to show that the discretization (HJI_h)–(BC_h) is convergent to the upper value of the game. A detailed presentation goes beyond the purposes of this paper, the interested reader will find these results in [BS 3]. We just give the main convergence result for the continuous case.

Theorem 19 *Let v_h be the solution of (HJI_h)–(BC_h), Let $\mathcal{T}$ be compact with nonempty interiour, the assumptions on f be verified, v be continuous.Then, v_h converges to v locally uniformly in $\mathbb{R}^N$ for $h \to 0^+$.*

3.2 Space discretization (1 player)

In order to solve the problem numerically we need a (finite) grid so we have to restrict our problem to a compact subdomain. A typical choice is to replace the whole space with a hypercube containing $\mathcal{T}$. We consider a triangular mesh of Q made by triangles S_j, $j \in J$ denoting by k the size of the mesh (this means that $k = \Delta x \equiv \max_j\{diam(S_j)\}$. Let us just remark that one can always decide to build a structured grid for Q as is the case for FD schemes, although for dynamic programming schemes this is not compulsory. Moreover, the use of triangles instead of uniform rectangular cells can be a clever and better choice when the boundary of $\mathcal{T}$ is rather complicated. We will denote by x_i, the nodes of the mesh (the vertices of the triangles S_j), typically i is a multi-index, $i = (i_1, \ldots, i_N)$ where N is the dimension of the state space of the problem. We denote by L the global number of nodes.

To simplify the presentation let us take a rectangle Q in $\mathbb{R}^2$, $Q \supset \mathcal{T}$ and present the scheme for $N = 2$. Moreover, we map the matrix of the values at the nodes (where v_{i_1,i_2} is the value corresponding to the node x_{i_1,i_2}) on to a vector V of dimension L by the usual representation by rows where the element v_{i_1,i_2} goes into V_m for $m = (i_1 - 1)N_{columns} + i_2$. This ordering allows us to locate the nodes and their values by a single index so from now on $i \in I \equiv \{1, \ldots, L\}$. We will divide the nodes into three subsets, the algorithm will perfom different operations in the three subsets. Let us introduce the sets of indices

$$I_{\mathcal{T}} \equiv \{i \in I : x_i \in \mathcal{T}\} \tag{24}$$

$$I_{\text{out}} \equiv \{i \in I : x_i + hf(x_i, a) \notin Q \, \forall a \in A\} \tag{25}$$

$$I_{\text{in}} \equiv \{i \in I : x_i + hf(x_i, a) \in Q\} \tag{26}$$

We can describe the *fully discrete scheme* simply writing (HJB_h) at every node of the grid such that $i \in I_{in}$ adding the boundary conditions on $\mathcal{T}$ and on ∂Q (more precisely, on the part of the boundary where the vectorfield points outward for every control). The fully discrete scheme is

$$v(x_i) = \min_{a \in A}[\beta v(x_i + hf(x_i, a)] + 1 - \beta, \quad \text{for } i \in I_{\text{in}} \tag{27}$$

$$v(x_i) = 0 \quad \text{for } i \in I_{\mathcal{T}} \tag{28}$$

$$v(x_i) = 1 \quad \text{for } i \in I_{out} \tag{29}$$

where $\beta = e^{-h}$. Note that the condition on I_{out} assigns to those nodes a value greater than the maximum value inside $Q \setminus \mathcal{T}$. It is like saying that once the trajectory leaves Q it will never come back to $\mathcal{T}$ (which is obviously false). Nonetheless the condition is reasonable since we will never get the information that the real trajectory (living in the whole space) can get back to the target unless we compute the solution in a larger domain containing Q. The solution we compute in Q is correct only if I_{out} is empty, if this is not the case the solution is correct in a subdomain of Q and is greater than the real solution everywhere in Q. This means that the reachable set is approximated from the inside.

We look for a solution of the above problem in the space of piecewise linear functions

$$W^k \equiv \{w : Q \to [0, 1] : w \in C(Q), \nabla w = constant \text{ in } S_j\} \tag{30}$$

For any $i \in I_{in}$, there exists at least one control such that $z_i(a) \equiv x_i + hf(x_i, a) \in Q$. Associated to $z_i(a) \in Q$ there is a unique vector of coefficients, $\lambda^i_j(z_i(a))$, $i, j \in I$ such that

$$0 \leq \lambda^i_j(a) \leq 1, \quad \sum_{j=1}^{L} \lambda^i_j(z_i(a)) = 1 \text{ and } z_i(a) = \sum_{j=1}^{L} \lambda^i_j(z_i(a))x_j$$

The coefficients $\lambda^i_j(z_i(a))$ are the local (baricentric) coordinates of the point $z_i(a)$ with respect to the vertices of the triangle containing $z_i(a)$. The above conditions just say that z can be written in a unique way as a convex combination of the nodes. Since we are looking for a solution in W^k, it is important to note that for any $w \in W^k$, $w(z_i(a)) = \sum_j \lambda^i_j w_j$ where $w_j = w(x_j)$, $j = 1, \ldots, L$. For $z \notin Q$ we set $w(z) = 1$.

Let us define componentwise the operator $S : \mathbb{R}^L \to \mathbb{R}^L$ corresponding to the fully discrete scheme

$$[S(U)]_i \equiv \begin{cases} \min\limits_{a \in A}[\beta \Lambda^i(a) \cdot U] + 1 - \beta \ , & \forall\, i \in I_{in} \\ 0 & \forall\, i \in I_{\mathcal{T}} \\ 1 & \forall\, i \in I_{out} \end{cases} \tag{31}$$

where $\Lambda^i(a) \equiv (\lambda^i_1(z_i(a)), \ldots, \lambda^i_L(z_i(a)))$ The following theorem shows that S has a unique fixed point.

Theorem 20 *The operator S defined in (31) has the following properties:*

i) S is monotone, i.e.*$U \leq V$ implies $S(U) \leq S(V)$;*

ii) $S : [0,1]^L \to [0,1]^L$;

iii) S is a contraction mapping in the max norm $\|W\|_\infty = \max_i |W_i|$,

$$\|S(U) - S(V)\|_\infty \leq \beta \|U - V\|_\infty$$

Sketch of the proof.

i) To prove that S is monotone it suffices to show that for $U \leq V$,

$$S(U)_i \leq S(V)_i, \text{ for } i \in I_{in}.$$

In fact, for any $i \in I_{in}$, we have

$$S(U)_i - S(V)_i \leq \beta \Lambda^i(\widehat{a}) \cdot (U - V)$$

where $\widehat{a}$ is the control where the minimum for $S(V)$ is achieved. The proof just follows by the fact that all the λ^i_j are nonnegative.

ii) Then, for any $U \in [0,1]^L$

$$1 - \beta = S_i(0) \leq S_i(U) \leq S_i(\mathbf{1}) = 1, \qquad \forall\, i \in I_{in}$$

where $\mathbf{1} \equiv (1, 1, \ldots, 1)$. This concludes the proof.
iii). For any $i \in I_{in}$

$$S_i(U) - S_i(V) \leq \beta \Lambda^i(\widehat{a})(U - V)$$

and $\|\Lambda^i(\widehat{a})\|_\infty \leq 1$, $\forall\, a \in A$ which implies

$$\|S_i(U) - S(V)\|_\infty \leq \beta \|U - V\|_\infty.$$

□

Although one can compute the fixed point starting from any initial guess U^0 it is more efficient to start from an initial guess in the set of discrete supersolutions U^+,

$$U^+ \equiv \{U \in [0,1]^L : U \geq S(U)\}$$

This will guarantee monotone convergence. In fact, let us consider the fixed point sequence

$$U^{n+1} \equiv S(U^n) \tag{32}$$

Taking $U^0 \in U^+$, the monotonicity of S implies

$$U^0 \geq S(U^0) = U^1, \quad U^1 \geq U^2, \quad U^2 \geq U^3 \quad \ldots$$

and U^n converges to U^* monotonically decreasing by the fixed point argument. A typical choice for $U^0 \in U^+$ is

$$U_i^0 = \begin{cases} 0 & \forall\, i \in I_{\mathcal{T}} \\ 1 & \text{elsewhere} \end{cases}$$

It is also interesting to remark that in the algorithm the information flows from the target to the other nodes of the grid. In fact, on the nodes in $Q \setminus \mathcal{T}$ we have $U_i^0 = 1$ but these values immediately decrease in a neighbourhood of $\mathcal{T}$ since, by the local controllability assumption, the Euler scheme drives them to the target in just one step. At the next iteration other values, in a larger neighbourhood of $\mathcal{T}$, will decrease due to the same mechanism and so on.

3.3 Fully discrete scheme for games

Let us get back to zero-sum games. Using the change of variable $v(x) \equiv 1 - e^{-T(x)}$ we can set the Isaacs equation in $\mathbb{R}^N$ obtaining

$$\begin{cases} v(x) + \min_{b \in B} \max_{a \in A} [-f(x,a,b) \cdot \nabla v(x)] = 1 & \text{in } \mathbb{R}^N \setminus \mathcal{T} \\ v(x) = 0 & \text{for } x \in \partial \mathcal{T} \end{cases} \tag{HJI}$$

Assume we want to solve the equation in Q, an hypercube in R^N. As we have seen, in the case of a single player we need to impose boundary conditions on ∂Q or, at least, on I_{out}. However, the situation for games is much more complicated. In fact, setting the value of the solution outside Q equal to 1 (as in the single player case) will imply that the pursuer looses every time the evader drives the dynamics outside Q. On the contrary, setting the value to 0 outside Q will give a great advantage to the pursuer. One way to define more unbiased boundary conditions is the following. Assume that $Q = Q_1 \cap Q_2$, where Q_i, $i = 1, 2$ are subsets of $\mathbb{R}^{N/2}$ which can be interpreted as the set of constraints for the i-th player. For example, in $\mathbb{R}^2$ we can consider as Q_1 a vertical strip and as Q_2 an horizontal strip and compute in the rectangle Q which is the intersection of those strips. According to this construction, we penalize the pursuer if the dynamics exits Q_1 and the evader if the dynamics exits Q_2. When the dynamics exits Q_1 *and* Q_2 we have assign a value, *e.g.*giving an advantage to one of them (in the following scheme we are giving advantage to the evader).

The discretization in time and space leads to a fully discrete scheme

$$\begin{aligned} w(x_i) &= \max_b \min_a [\beta w(x_i + hf(x_i, a, b))] + 1 - \beta \quad \text{for } i \in I_{\text{in}} && (33) \\ w(x_i) &= 1 \qquad \text{for } i \in I_{out_2} && (34) \\ w(x_i) &= 0 \qquad \text{for } i \in I_{\mathcal{T}} \cup I_{out_1} && (35) \end{aligned}$$

where $\beta \equiv e^{-h}$ and

$$I_{in} = \{i : x_i + hf(x_i, a, b) \in Q \setminus \mathcal{T} \text{ for any } a \in A, b \in B\} \tag{36}$$

$$I_{\mathcal{T}} = \{i : x_i \in \mathcal{T} \cap Q\} \tag{37}$$

$$I_{out_1} = \{i : x_i \notin Q_2\} \tag{38}$$

$$I_{out_2} = \{i : x_i \notin Q_2 \setminus Q\} \tag{39}$$

Theorem 21 *The operator S defined in (33) has the following properties:*

i) S is monotone, i.e.*$U \leq V$ implies $S(U) \leq S(V)$;*

ii) $S : [0, 1]^L \to [0, 1]^L$;

iii) S is a contraction mapping in the max norm,

$$\|S(U) - S(V)\|_\infty \leq \beta \|U - V\|_\infty$$

The proof of the above theorem is a generalization of that of Theorem 20 and can be found in [BFS1].

The above result guarantees that there is a unique fixed point U^* for S. Naturally the numerical solution w will be obtained extending by linear interpolation the values of U^* and it will depend on the discretization steps $h = \Delta t$ and $k = \Delta x$. Let us state the first convergence result for continuous value functions

Theorem 22 *Let $\mathcal{T}$ be the closure of an open set with Lipschitz boundary, "*diam $Q \to +\infty$*" and v be continuous. Then, for $h \to 0^+$ and $\frac{k}{h} \to 0^+$, $w^{h,k}$ converges to v locally uniformly on the compact sets of $\mathbb{R}^N$.*

Note that the requirement "diam $Q \to +\infty$" is just a technical trick to avoid to deal with boundary conditions on Q (a similar statement can be written in the whole space just working on an infinite mesh). We conclude this section quoting a convergence result which holds also in presence of discontinuities (barriers) for the values function. Let w_n^ϵ be the sequence generated by the numerical scheme with target $\mathcal{T}_\epsilon = \{x : d(x, \mathcal{T}) \leq \epsilon\}$.

Theorem 23 *For all x there exists the limit*

$$\overline{w}(x) = \lim_{\substack{\varepsilon\to 0^+\\ n\to+\infty\\ n\geq n(\epsilon)}} w_n^{\epsilon}(x)$$

and it coincides with the lower value V of the game with target $\mathcal{T}$, i.e.$\overline{w} = V$*. Convergence is uniform on every compact set where V is continuous.*

Can we know *a-priori* what is the accuracy of the method in the approximation of the value function? This result is necessary to understand how far we are from the real solution when we compute our approximate solution. To simplify, let us assume that the Lipschitz constant for f $L_f \leq 1$ and that v is Lipschitz continuous. Then,

$$\|w^{h,k} - v\|_\infty \leq Ch^{1/2}\left(1 + \left(\frac{k}{h}\right)^2\right)$$

The proof of the above error estimate is rather technical and can be found in [S 4].

4 Approximation of optimal feedback and trajectories

One of the goals of every approximation for control problems and games is to compute discrete synthesis of feedback controls. It is interesting to note that the algorithm proposed and analyzed in the previous section computes an approximate optimal control at every point of the grid. Since the numerical solution w has been extended to Q by interpolation we can also compute an approximate optimal feedback at every point of $x \in Q$, *i.e.*for the control problem we can define the *feedback map* $F : Q \to A$. In order to construct this map, let us introduce the notation

$$I^k(x,a) \equiv e^{-h}w(x + hf(x,a)) + 1 - e^{-h}. \tag{40}$$

where we indicate by the index k the fact that the above function also depends on the space discretization. Note that $I^k(x,\cdot)$ has a minimum over A, but the minimum point may be not unique.
We want to construct a *selection*, e.g. take a strictly convex ϕ and define

$$A_x^k = \{\hat{a} \in A : I_x^k(x,\hat{a}) = \min_A I^k(x,a)\} \tag{41}$$

The selection is

$$a_x^* = \text{arg min }_{A_x^k}\ \phi(a) \tag{42}$$

In this way we are able to compute our approximate optimal trajectories, we define the piecewise constant control

$$a^k(s) = a^*_{x_{m,h}} \quad s \in [mh,(m+1)h[\tag{43}$$

where $x_{m,h}$ is the state of the Euler scheme, at the iteration m. Error estimates of the approximation of feedbacks and optimal trajectories are available for control problems in [BCD] [F 1, F 2].

For games, the algorithm computes an approximate optimal control couple (a^*, b^*) at every point of the grid. Again by w we can also compute an approximate optimal feedback at every point $x \in Q$.

$$(a^*(x), b^*(x)) \equiv \text{argminmax}\{e^{-h}w(x + hf(x, a, b))\} + 1 - e^{-h} \tag{44}$$

If that control is not unique then we can select a unique couple, *e.g.*minimizing two convex functionals. A typical choice is to introduce an inertial criterium to stabilize the trajectories, *i.e.*if at step $n+1$ the set of optimal couples contains (a^*_n, b^*_n) we keep it.

We end this section giving some informations on recent developments which we cannot include in this presentation. Some acceleration methods have been implemented to reduce the amount of floating point operation needed to compute the fixed point: Gauss-Seidel iterations, monotone acceleration methods [St] and approximation in policy space [TG 2, St]. Moreover, an effort has been made to reduce the size of the problem by means of the domain decomposition technique, see [QV] for a general presentation of the method and several applications to PDEs. This approach has produced parallel codes for the Isaacs equation [FLM, St, FSt].

5 Numerical experiments

Let us examine some classical games and look at their numerical solutions. We will focus our attention to the accuracy in the approximation of the value function as well as to the accuracy in the approximation of optimal feedbacks and trajectories. In the previous sections we always assumed that the sets of controls A and B were compact. In the algorithm and in the numerical tests we have used a discrete finite approximation for those sets which allows to compute the min-max by comparison. For example, we will consider the following discrete sets

$$A = \Big\{a_1 + j\frac{a_2 - a_1}{c-1}\Big\}, \qquad j = 0, \dots, c-1;$$
$$B = \Big\{b_1 + j\frac{b_2 - b_1}{c-1}\Big\}, \qquad j = 0, \dots, c-1;$$

where $[a_1, a_2]$ and $[b_1, b_2]$ represent the control sets respectively for the pursuer P and the evader E. Finally, note that all the value functions represented in the pictures have values in $[0, 1]$ because we have computed the fixed point after the Kružkov change of variable.

5.1 The Tag-Chase Game

Two boys P and E are running one after the other in the plane $\mathbb{R}^2$. P wants to catch E in minimal time whereas E wants to avoid the capture. Both of them are running with constant velocity and can change their direction instantaneously. This means that the dynamics of the system is

$$f_P(y, a, b) = v_P a \qquad f_E(y, a, b) = v_E b$$

where v_P and v_E are two scalars representing the maximum speed for P and E and the admissible controls are taken in the sets

$$A = B = B(0, 1).$$

Let us give a more explicit version of the dynamics which is useful for the discretization. Let us denote by (x_P, y_P) the position of P and by (x_E, y_E) the position of E, we can write the dynamics as

$$\begin{cases} \dot{x}_P = v_P \sin\theta_P \\ \dot{y}_P = v_P \cos\theta_P \\ \dot{x}_E = v_E \sin\theta_E \\ \dot{y}_E = v_E \cos\theta_E \end{cases} \tag{45}$$

where $\theta_P \in [a_1, a_2] \subseteq [-\pi, \pi]$ is the control for P and $\theta_E \in [b_1, b_2] \subseteq [-\pi, \pi]$ is the control for E, θ_P and θ_E are the angles between the y axis and the velocities for P and E (see Figure 5.1).

We say that E has been captured by P if their distance in the plane is lower than a given threshold $\epsilon > 0$. Introducing $z \equiv (x_P, y_P, x_E, y_E)$ we can say that the capture occurs whenever $z \in \widetilde{\mathcal{T}}$ where

$$\widetilde{\mathcal{T}} \equiv \left\{ z \in \mathbb{R}^4 : \sqrt{(x_P - x_E)^2 + (y_P - y_E)^2} < \epsilon \right\}. \tag{46}$$

The Isaacs equation is set in $\mathbb{R}^4$ since every player belongs to $\mathbb{R}^2$. However the result of the game just depends on the relative positions of P and E, since their dynamics are homogeneous. In order to reduce the amount of computations needed to compute the value function we describe the game in a new coordinate system introducing the variables

$$\tilde{x} = (x_E - x_P)\cos\theta - (y_E - y_P)\sin\theta \tag{47}$$
$$\tilde{y} = (x_E - x_P)\sin\theta - (y_E - y_P)\cos\theta \tag{48}$$
$$\tag{49}$$

The new system (called relative coordinates system) has the origin fixed on the position of P and moves with this player (see Figure 5.1). Note that the y axis is oriented from P to E. In the new coordinates the dynamics (45) becomes

$$\begin{cases} \dot{x} = v_E \sin\theta_E - v_P \sin\theta_P \\ \dot{y} = v_E \cos\theta_E - v_P \cos\theta_P \end{cases} \tag{50}$$

and the target (46) is

$$\mathcal{T} \equiv \left\{ (x, y) : \sqrt{x^2 + y^2} < \epsilon \right\}.$$

It is important to note that the above change of variables greatly simplifies the numerical solution of the problem for three different reasons. The first is that we now solve the Isaacs equation in $\mathbb{R}^2$ and we need a grid of just M^2 nodes instead of M^4 nodes (here M denotes the number of nodes in one dimension). The second reason is that we now have a compact target in $\mathbb{R}^2$ whereas the original target (46) is unbounded. This is a major advantage since we can choose a fixed rectangular domain Q in $\mathbb{R}^2$ such that it contains $\mathcal{T}$ and compute the solution in it. Finally, we get rid of the boundary conditions on ∂Q (see Section 3).

It is easily seen that the game has always a value and that the only interesting case is $v_P > v_E$ (if the opposite inequality holds true capture is impossible if E plays optimally). In this situation the best strategy for E is to run at maximal velocity in the direction opposite to P along the line passing through the initial positions of P and E. The optimal strategy for P is to run after E at maximal velocity. The corresponding minimal time of capture is

$$T(x_P, y_P, x_E, y_E) = \frac{\sqrt{(x_E - x_P)^2 + (y_E - y_P)^2}}{v_P - v_E}$$

or, in relative coordinates,

$$T(x, y) = \frac{\sqrt{x^2 + y^2}}{v_P - v_E}.$$

Let us comment some numerical experiments. We have chosen $Q = [-1, 1]^2$

$$v_P = 2, \quad v_E = 1, \quad A = B = [-\pi, \pi].$$

Figures 2, 3 correspond to the following discretization

# Nodes	Δt	ϵ	# Controls
23×23	0.05	0.20	P=41 E=41

The value function is represented in the relative coordinate system, so P is fixed at the origin and the value at every point is the minimal time of capture (after Kružkov transform). As one can see in Figure 2, the behaviour is correct since it correspond to a (rescaled) distance function. The optimal trajectories for the initial positions $P = (0.3, 0.3)$, $E = (0.6, -0.3)$ are represented in Figure 3.

5.2 The Tag-Chase game with constraints on the directions

This game has the dynamics (45). The only difference with respect to the Tag-Chase game is that now the pursuer P has a constraint on his displacement directions. He can choose his control in the set $\theta_P \in [a_1, a_2] \subseteq [-3/4\pi, 3/4\pi]$. The evader can still choose his control as $\theta_E \in [b_1, b_2] = [-\pi, \pi]$, i.e.

$$A = [\theta_1, \theta_2] \text{ and } B = [-\pi, \pi].$$

In the numerical experiment below we have chosen $v_P = 2$, $v_E = 1$, $A = [\frac{3}{4}\pi, \frac{3}{4}\pi]$ and $B = [-\pi, \pi]$. As one can see in Figure 4 the time of capture at points which are below the origin and which cannot be reached by P in a direct way have a value bigger than at the symmetric points (above the origin). This is clearly due to the fact that P has to zig-zag to those points because the directions pointing directly to them are not allowed (Figure 5).

5.3 Zermelo navigation problem

A boat B moves with constant velocity in a river and it can change its direction istantaneously. The water of the river flows with a velocity σ and the boat tries to reach an island in the middle of the river (the target) maneuvering against water current. We choose a system of coordinates such that the velocity of the current is $(\sigma, 0)$ (see Figure 6). In the new system, the dynamics of the boat is described by

$$\begin{cases} \dot{x} = \sigma + v_B \cos a, \\ \dot{y} = v_B \sin a, \end{cases}$$

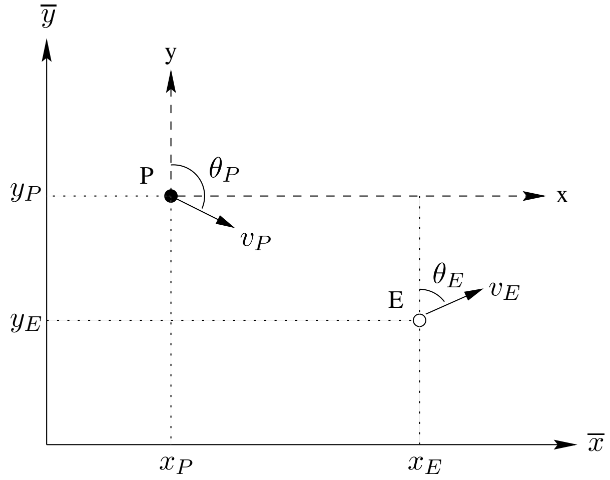

Figure 1: The Tag-Chase game

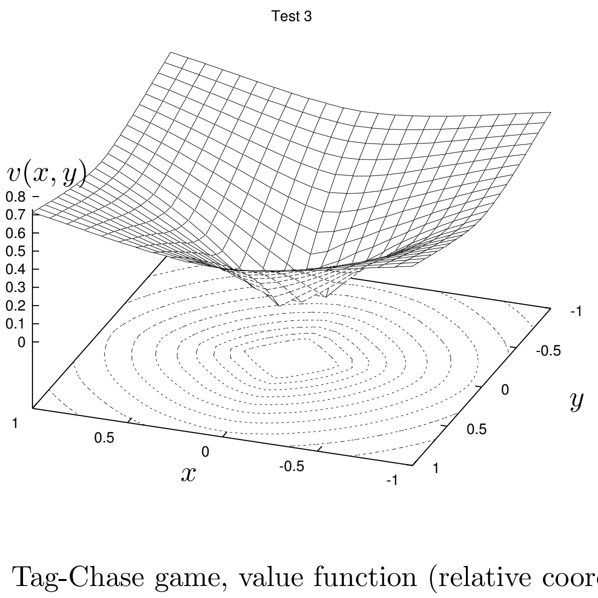

Figure 2: Tag-Chase game, value function (relative coordinates)

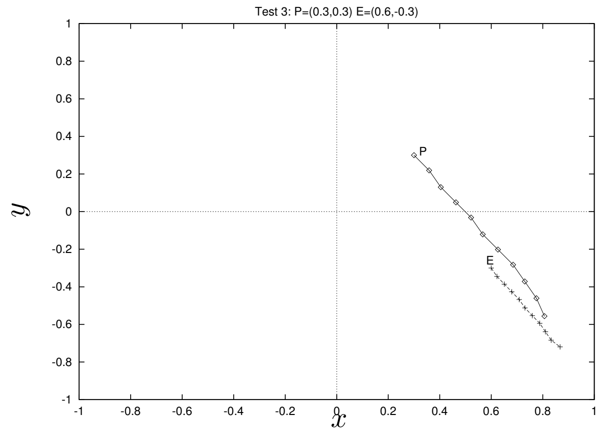

Figure 3: Tag-Chase game, optimal trajectoires

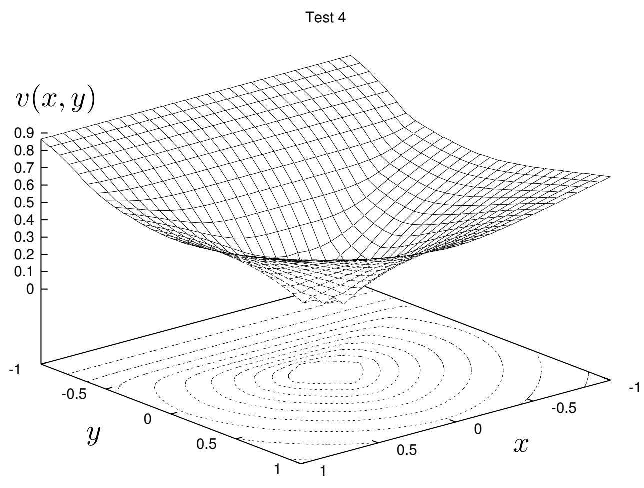

Figure 4: Constrained Tag-Chase game, value function

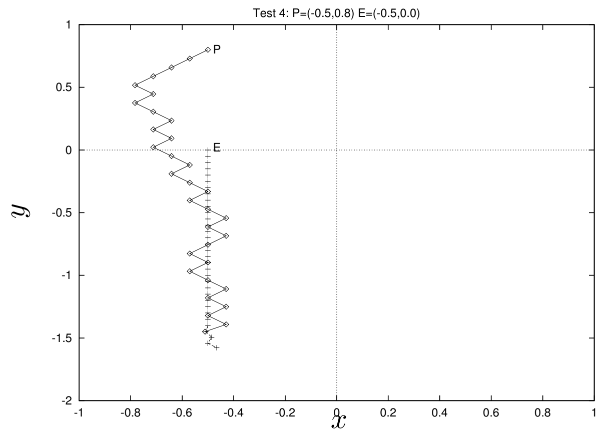

Figure 5: Constrained Tag-Chase game, optimal trajectories

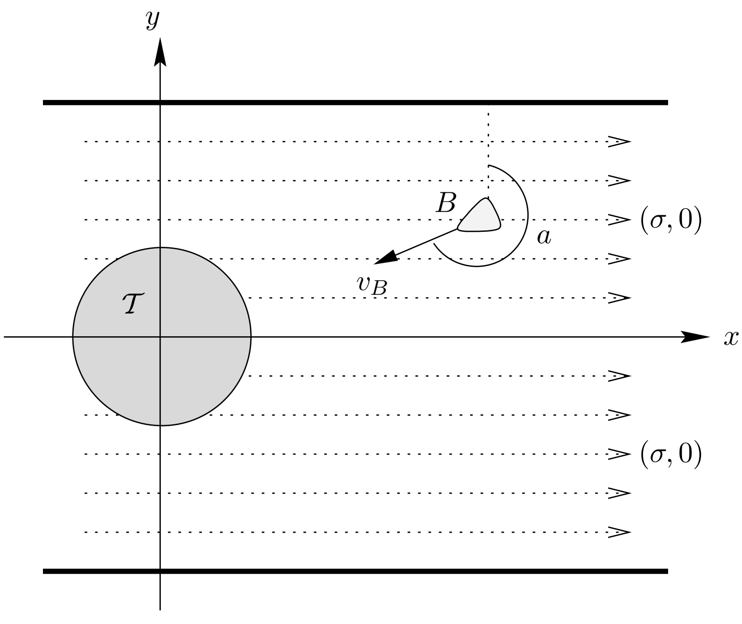

Figure 6: Zermelo navigation problem

where $a \in [-\pi, \pi]$ is the control over the boat direction. Let $z(t) = \big(x(t), y(t)\big)$.

It is easy to see that $v_B > \sigma$ is a sufficient condition for the boat to reach the island from any initial condition in the river. This is not the case if the opposite condition holds true (see below). Let us introduce the reachable set

$$\mathcal{R} \equiv \big\{x_0 : \exists t > 0, a(\cdot) \in \mathcal{A} \text{ such that } y(t; x_0, a(\cdot)) \in \mathcal{T}\big\} . \tag{51}$$

This is the set of initial positions from which the boat can reach the island. Let us choose $\mathcal{T} = \big\{(0,0)\big\}$, then a simple geometric argument shows that

$$\mathcal{R} = \begin{cases} \mathbb{R}^2 & \text{se } v_B > \sigma, \\ \big\{(x,y) : x < 0 \text{ or } x = y = 0\big\} & \text{if } v_B = \sigma, \\ \big\{(x,y) : x < 0, |y| \le -x\, v_B(\sigma^2 - v_B^2)^{-\frac{1}{2}}\big\} & \text{if } 0 \le v_B < \sigma. \end{cases} \tag{52}$$

The result is obvious for $v_B \ge \sigma$. Now assume $0 < v_B < \sigma$. The motion of the boat at every point is determined by the (vector) sum of its velocity v_B and the current velocity σ.

The maximum angle which is allowed to the boat direction is

$$\overline{\theta} \equiv \frac{\pi}{2} + \arctan\left(\frac{v_B}{\sqrt{\sigma^2 - v_B^2}}\right)$$

and the equation of the line with this slope passing from the origin is

$$y = -x\, \frac{v_B}{\sqrt{\sigma^2 - v_B^2}} ,$$

which explains (52). Let us examine the results of two tests. In both tests we have set $Q = [-1, 1]^2$ and the parameters of the discretization are

# Nodes	Δt	ϵ	# Controls
80×80	0.05	0.10	B=36

In the first test, $\sigma = 1$ and $v_B = 1.4$. The value function is presented in Figure 7. Since $v_B > \sigma$ the reachable set is the whole river as predicted by the above argument.

In the second test, $\sigma = 1$ and $v_B = 0.6$. As one can see in Figure 8 the reachable set is stricly contained in Q since in this case $v_B < \sigma$.

5.4 The lady in the lake

A lady E is swimming in a circular lake, she can change her direction istantaneously. A man P is waiting for her on the shore, he is not able to swim so he can not enter the lake but he wants to meet her when she leaves the lake. P runs on the shore with a maximum velocity v_P and he can change istantaneously his direction (switching from clockwise to counterclockwise) (see Figure 9).

Naturally the lady would like to leave the lake at some point but she also wants to avoid the man. We assume that on the ground the velocity of the lady in greater than that of the man so that meeting can only occur on the shore, this is the interesting case.

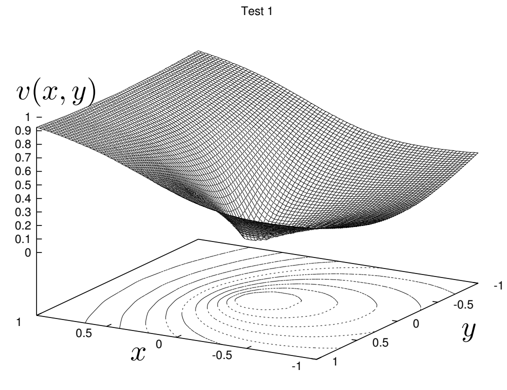

Figure 7: Zermelo problem, value function for $\sigma = 1$, $v_B = 1.4$

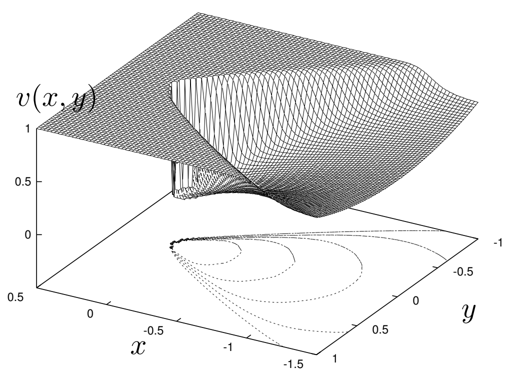

Figure 8: Zermelo problem, value function for $\sigma = 1$, $v_B = 0.6$

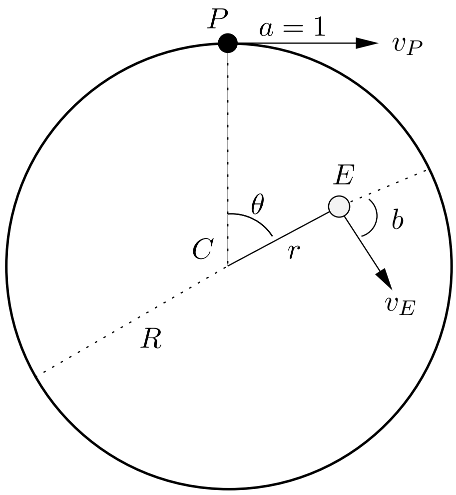

Figure 9: The lady in the lake problem

The lady wants to maximize the angular distance PE (measured from the center of the lake) at the time when she exits from the water; naturally P wants to minimize the same distance.

It is natural to describe the dynamics in polar coordinates. We will denote by θ the angle $P\widehat{C}E$ (where C is the center of the lake) and with r the distance of E from C. We denote by R the radius of the lake. The dynamics of the game is

$$\begin{cases} \dot{\theta} = \frac{v_E \sin b}{r} - \frac{v_P a}{R}, \\ \dot{r} = v_E \cos b. \end{cases}$$

where v_E and v_P are two scalars representing the maximum velocities of E and P and $a \in A \equiv [-1, 1]$ and $b \in B \equiv [-\pi, \pi]$. As we made in the tag-chase game we choose a system of coordinates centered on P. The control a for P is bounded, $|A| < 1$, and its sign corresponds to the direction followed by P to move along the shore (as usual, $+$ means counterclockwise). The control b for E is the angle between the direction chosen by E and the segment CE, so $b \in [-\pi, \pi]$.

P (respectively E) wants to minimize (maximize)

$$\big|\theta(T)\big|$$

where $T \equiv \min\big\{t : r(t) = R\big\}$ is the moment when E exits the lake and $\theta \in [-\pi, \pi]$. Our value function will be (in polar coordinates)

$$v(r, \theta) \equiv \min_a \max_b \big|\theta(T)\big|.$$

Let us now compute $v(r, \theta)$ by a geometric argument. Let us assume $v_P = +1$. In the relative coordinate system the E velocity is the (vector) sum v_r of two components: the velocity vector of E in the lake (v_E) and a vector v_P which is due to the rotation of P around the lake, this vector is opposite to the direction of P and its modulus is equal to $r(t)/R$ (see Figure 10).

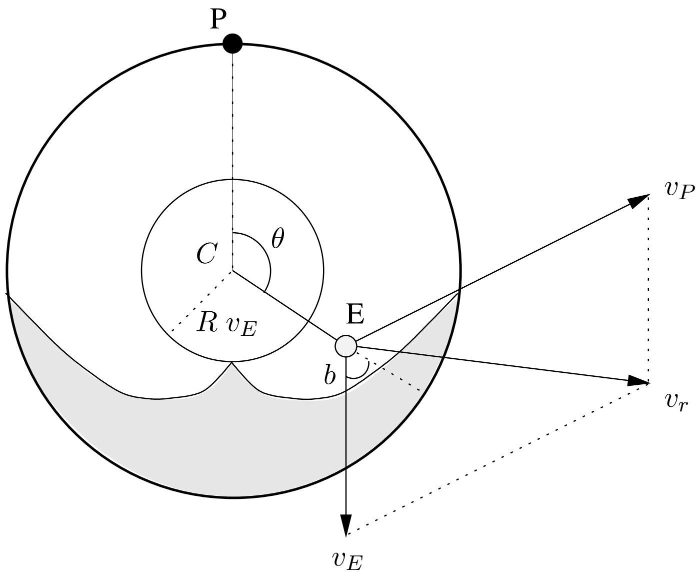

Figure 10: The lady in the lake, equilibrium strategies

The best strategy for E is to choose b in order to keep v_P orthogonal to v_r. For any other choice of b the angle between CE and v_r will decrease, *i.e.* E would move more rapidly toward P which is exactly what the lady wants to avoid. Then, the equilibrium strategy is $b = b^*$ where $sin(b^*) = Rv_E/r(t)$. Note that the above argument to reconstruct an optimal strategy for E fails if $r(t) < Rv_E$. In fact, in that case E has an angular velocity greater than P and can always reach a point which is opposite with respect to the position of P (which means $\theta(t) = \pi$). Once E reaches the position $(\theta = \pi, r = Rv_P)$, her optimal strategy consists in running around (remember, the lady runs faster than the man on the shore). Following [BO] we can obtain the value of the game

$$\left|\theta(t)\right| = \pi + \arccos v_E - \frac{1}{v_E}\sqrt{1 - v_E^2}$$

where we have assumed $v_P = 1$. The above argument is always true for any initial condition belonging to the circle or radius Rv_E. If the initial position of E is outside that circle she can always enter it swimming in the direction of the center C. It is interesting to note that from some initial positions outside the circle or radius Rv_E, E has a better strategy. These initial positions belong to the area which is shadowed in Figure 10. This area is bounded on one side by the lake's shore and by the two equilibrium trajectories which start at the point $(\theta = \pi, r = Rv_E)$. In this area the optimal trajectories can be obtained by a direct integration of the Isaacs equation (see [BO], p. 369 for details). Let us set the following values for the parameters: $R = 1$, $v_P = 1$ and $v_E = 0.3$. For the discretization we have used the following values

# Nodes	Δt	ϵ	# Controls
40×40	0.05	0.10	P=2 E=36

The value function is represented in Figure 11. One can observe that, according to above discussion, the time of capture is constant everywhere but in the shadowed area of Figure

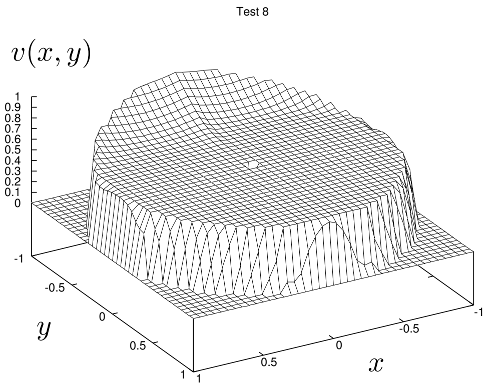

Figure 11: The lady in the lake problem, value function

10. For this problem we do not present optimal trajectories since they are not uniquely defined. In fact, in this problem there is no running cost (the lady can wait forever swimming in the center of the lake) so that the lady can always decide to stop her motion along an equilibrium path, go around everywhere in the lake and get back to the path without affecting the value of the game.

5.5 The Homicidal chauffeur

The presentation of this game follows [FSt]. Let us consider two players (P and E) and the following dynamics:

$$\begin{cases} \dot{x}_P = v_P \sin\theta, \\ \dot{y}_P = v_P \cos\theta, \\ \dot{x}_E = v_E \sin b, \\ \dot{y}_E = v_E \cos b, \\ \dot{\theta} = \frac{R}{v_P} a, \end{cases} \tag{53}$$

where $a \in A \equiv [-1, 1]$ and $b \in B \equiv [-\pi, \pi]$ are the two player's controls. The pursuer P is not free in his movements, he is constrained by a minimum curvature radius R. The target is defined as in the Tag-Chase game. Also in this example we have used the reduced coordinate system (50). We have considered the homicidal chauffeur game where $Q = [-1, 1]^2$, $v_P = 1$, $v_E = 0.5$, $R = 0.2$ and the following discretization parameters:

# Nodes	Δt	ϵ	# Controls
120×120	0.05	0.10	P=36 E=36

Figure 13 shows the value function of the game. Note that when E is in front of P the behaviour of the two players is analogous to the tag-chase game: in this case, indeed, the constraint on P's radius turn does not come into action (Figure 14). However, on the P sides the value function has two higher lobes. In fact, to reach the corresponding points of the domain, the pursuer must first turn around himself to be able to catch E following a straight line (see Figure 15). Finally, behind P there is a region where capture is impossible ($v = 1$) because the evader has the time to exit Q before the pursuer can catch him. Figure 17 shows a set of optimal trajectories near a barrier in the relative coordinates system . Figure 16 is taken from [Me] and shows the optimal trajectories which have been obtained by analytical methods. One can see that our results are quite accurate since the approximate trajectories (Figure 17) look very similar to the exact solutions (Figure 16). Moreover, in the numerical approximation the barrier curve is clearly visible: that barrier cannot be crossed if both the players behave optimally. It divides the initial positions from which the trajectories point directly to the origin from those corresponding to trajectories reaching the origin after a round trip.

6 Conclusions and open problems

As we have seen, the dynamic programming approach can be used to compute the solution of two-persons zero-sum differential games in low dimension. The accuracy in the reconstruction of the value functions, optimal feedbacks and trajectories is rather satisfactory although it strongly depends on the discretization steps and this has a dramatic impact on the number of floating point operations necessary to compute the solutions. There are several open problems from the theoretical as well as from the algorithmic point of view:

1. From the theoretical point of view if would be nice to establish sharp bounds for approximation of discontinuous value functions in terms of the discretization steps. The extension to high-order schemes also deserves attentions because the use of this schemes can produce a significative reduction of the grid points required for a given accuracy and, on turn, this will reduce the number of floating point operations. Moreover, the accuracy in the reconstruction of optimal feedbacks and trajectories is still an open problem.

2. Several points should be investigated to improve the algorithms. The first is to develop an efficient acceleration method for the fixed point scheme resulting from the discretization of the Isaacs equation. Fast Marching Methods have been proposed for convex Hamiltonians (basically for eikonal type equation) but the extension to non convex Hamiltonian is still open. Another improvement could be obtained using adaptive grids which would allow to concentrate the nodes where the singularity of the value function appear, (*i.e.* near the lines of discontinuity of Dv or v).

3. Can the DP approach be extended to other classes of differential games? For example, one could start from more general non-cooperative games. Once the characterization of Nash equilibria and Pareto optima has been derived we could try to develop numerical schemes for them.

The above program would require strong efforts and several years to be accomplished.

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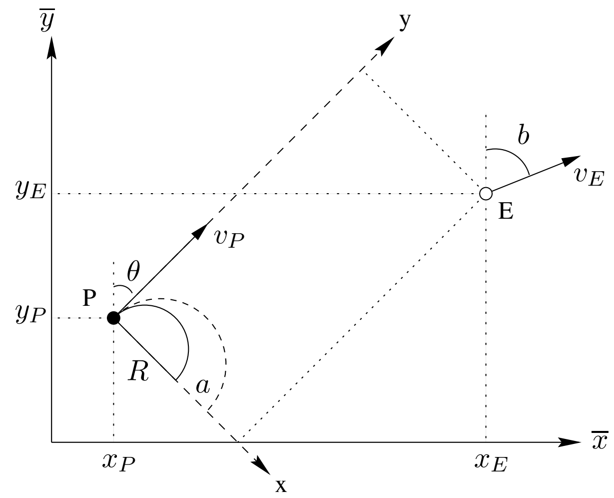

Figure 12: The Homicidal Chauffeur problem

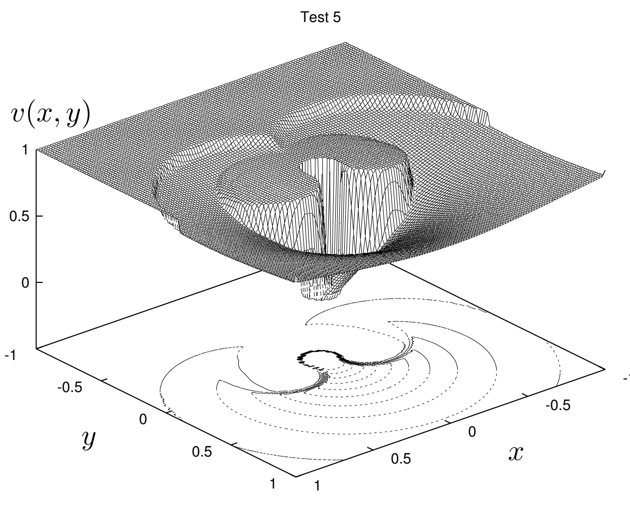

Figure 13: Homicidal Chauffeur, value function

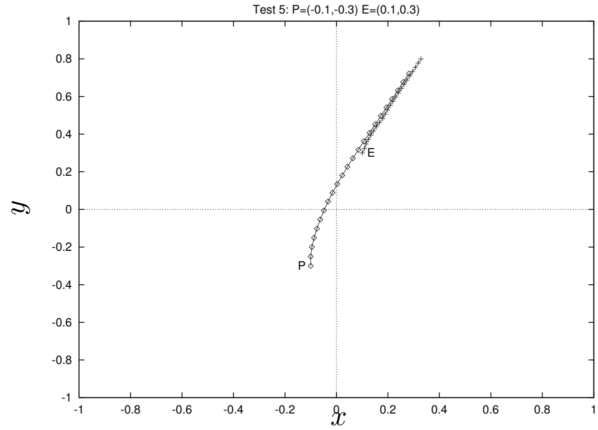

Figure 14: Homicidal Chauffeur, optimal trajectories

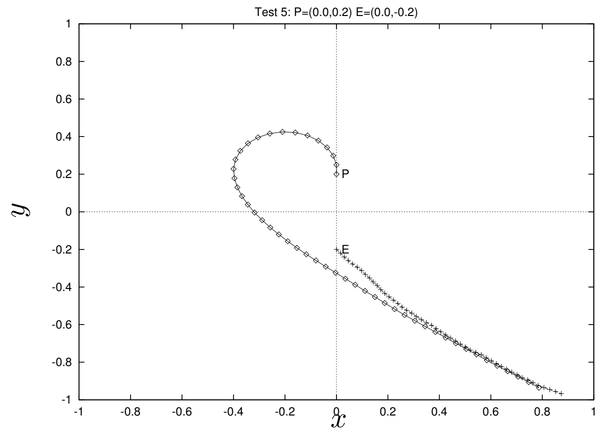

Figure 15: Homicidal Chauffeur, optimal trajectories

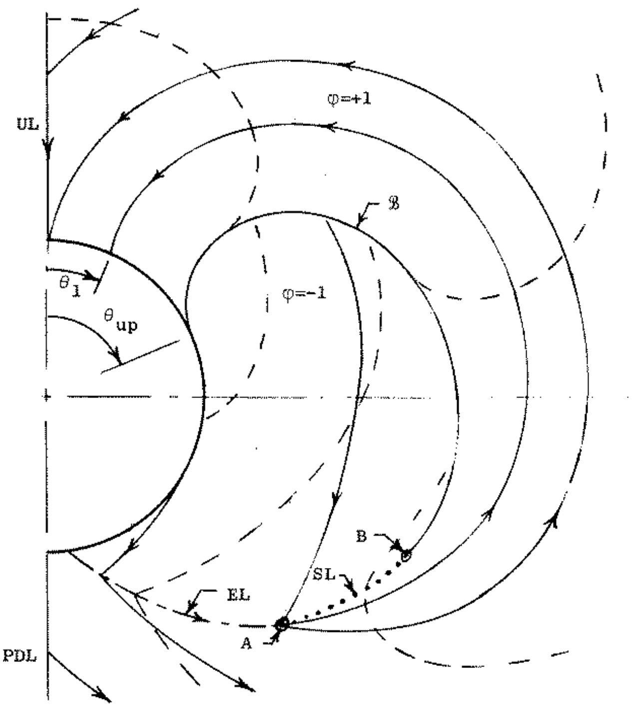

Figure 16: Homicidal Chauffeur, optimal trajectories (Merz Thesis)

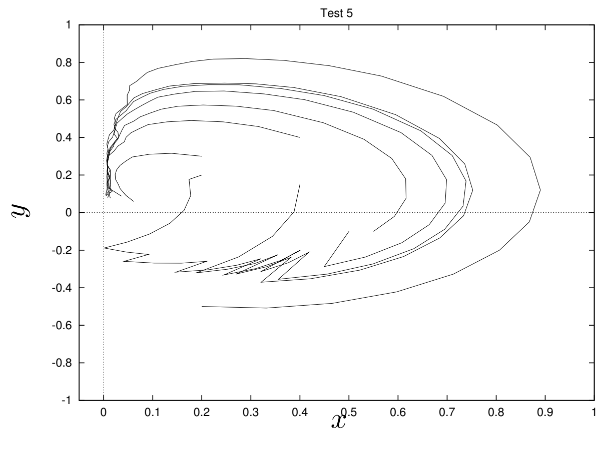

Figure 17: Homicidal Chauffeur, optimal trajectories (computed)

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