Basic Concepts of Adaptive Finite Element Methods for Elliptic Boundary Value Problems

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Summer School 'Challenges in Applied Control and Optimal Control'

Basque Center for Applied Mathematics

Derio, July 4-8, 2011

The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

SOLVE $\Longrightarrow$ ESTIMATE $\Longrightarrow$ MARK $\Longrightarrow$ REFINE

SOLVE:	Numerical solution of the FE discretized problem
ESTIMATE:	Residual and hierarchical a posteriori error estimators Error estimators based on local averaging Goal oriented weighted dual approach Functional type a posteriori error bounds
MARK:	Strategies based on the max. error or the averaged error Bulk criterion for AFEMs
REFINE:	Bisection or 'red/green' refinement or combinations thereof

A Posteriori Error Estimation I

For a closed subspace $V \subset H^1(\Omega)$ we assume

$$a(\cdot,\cdot) : V \times V \to \mathbb{R}$$

to be a bounded, V-elliptic bilinear form, i.e.,

$$|a(v,w)| \leq C\|v\|_{k,\Omega}\|w\|_{k,\Omega}, \quad v,w \in V, \quad a(v,v) \geq \gamma\|v\|^2_{k,\Omega}, \quad v \in V,$$

for some constants $C > 0$ and $\gamma > 0$. We further assume $\ell \in V^*$ where V^* denotes the algebraic and topological dual of V and consider the variational equation: Find $u \in V$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in V.$$

It is well-known by the Lax-Milgram Lemma that under the above assumptions the variational problem admits a unique solution.

A Posteriori Error Estimation II

Example. The standard example is Poisson's equation: Let $\Gamma = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$ and $V := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$ and

$$a(v,w) := (\nabla v, \nabla w)_{0,\Omega}, \ v,w \in V, \quad \ell(v) := (f,v)_{0,\Omega} + (g,v)_{0,\Gamma_N}, \ v \in V,$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$. Then, the variational equation represents the weak form of Poisson's problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ n \cdot \nabla u &= g \quad \text{on } \Gamma_N, \end{aligned}$$

where n stands for the unit outward normal on Γ_N.

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A Posteriori Error Estimation III

Example cont'd. We remark that in case $\Gamma_N = \emptyset$ we have $V = H_0^1(\Omega)$ (pure homogeneous Dirichlet boundary conditions), whereas in case $\Gamma_D = \emptyset$ the appropriate function space is $V = \{v \in H^1(\Omega) \mid (v, \chi_\Omega)_{0,\Omega} = 0\}$, where χ_Ω stands for the characteristic function of Ω. In these cases, the V-ellipticity of the bilinear form $a(\cdot,\cdot)$ follows from the Poincaré-Friedrichs inequalities

$$\|v\|_{0,\Omega} \le C\Big(|v|_{1,\Omega} + |\int_\Gamma v\,ds|\Big),$$

$$\|v\|_{0,\Omega} \le C\Big(|v|_{1,\Omega} + |\int_\Omega v\,dx|\Big).$$

A Posteriori Error Estimation IV

An important issue in the theory of partial differential equations is the regularity of a solution. For instance, considering Poisson's problem, it is well-known that for a convex domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary Γ and $f \in L^2(\Omega)$ the weak solution $u \in V$ satisfies $u \in V \cap H^2(\Omega)$ and

$$\|u\|_{2,\Omega} \le C \|f\|_{0,\Omega}.$$

However, there is less regularity, if the domain Ω is no longer convex. A classical example is the so-called L-shaped domain: Consider the Poisson equation in

$$\Omega := (-1,+1) \times (0,1) \cup (-1,0) \times (-1,0],$$
$$\Gamma_D := \{0\} \times [-1,0] \cup [0,1] \times \{0\} \;,\; \Gamma_N := \Gamma \setminus \Gamma_D,$$

and assume $f \equiv 0$ and g such that $u(r,\varphi) = r^{2/3} \sin(\frac{2}{3}\varphi)$ is the exact solution of the problem. The solution belongs to $V \cap H^{5/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ (but not to $H^{5/3}(\Omega)$!) and has a singularity in the origin.

A Posteriori Error Estimation V

Finite element approximations are based on the Ritz-Galerkin approach: Given a finite dimensional subspace $V_h \subset V$ of test/trial functions, find $u_h \in V_h$ such that

$$a(u_h, v_h) \; = \; \ell(v_h), \quad v_h \in V_h.$$

Since $V_h \subset V$, the existence and uniqueness of a discrete solution $u_h \in V_h$ follows readily from the Lax-Milgram Lemma. Moreover, we deduce that the error $e_u := u - u_h$ satisfies the Galerkin orthogonality

$$a(u - u_h, v_h) \; = \; 0, \quad v_h \in V_h,$$

i.e., the approximate solution $u_h \in V_h$ is the projection of the solution $u \in V$ onto V_h with respect to the inner product $a(\cdot,\cdot)$ on V (elliptic projection). Using the Galerkin orthogonality, it is easy to derive the a priori error estimate

$$\|u - u_h\|_{1,\Omega} \; \leq \; M \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega},$$

where $M := C/\gamma$. This result tells us that the error is of the same order as the best approximation of the solution $u \in V$ by functions from the finite dimensional subspace V_h. It is known as Céa's Lemma.

A Posteriori Error Estimation VI

The Ritz-Galerkin method also gives rise to an a posteriori error estimate in terms of the residual $r : V \to \mathbb{R}$

$$r(v) \;:=\; \ell(v) \;-\; a(u_h, v), \quad v \in V.$$

In fact, it follows that for any $v \in V$

$$\gamma \|u - u_h\|_{1,\Omega}^2 \le a(u - u_h, u - u_h) = r(u - u_h) \le \|r\|_{-1,\Omega} \; \|u - u_h\|_{1,\Omega},$$

whence

$$\|u - u_h\|_{1,\Omega} \;\le\; \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}.$$

Reliability and Efficiency

Reliability and Efficiency of Error Estimators I

Definition. An error estimator η_h is called reliable, if it provides an upper bound for the error up to data oscillations osc_h^{rel}, i.e., if there exists a constant $C_{rel} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\|e_u\|_a \ \leq \ C_{rel}\, \eta_h \ + \ osc_h^{rel} \ .$$

On the other hand, an estimator η_h is said to be efficient, if up to data oscillations osc_h^{eff} it gives rise to a lower bound for the error, i.e., if there exists a constant $C_{eff} > 0$, independent of the mesh size h of the underlying triangulation, such that

$$\eta_h \ \leq \ C_{eff}\, \|e_u\|_a \ + \ osc_h^{eff} .$$

Finally, an estimator η_h is called asymptotically exact, if it is both reliable and efficient with $C_{rel} = C_{eff}^{-1}$.

Reliability and Efficiency of Error Estimators II

Remark. The notion 'reliability' is motivated by the use of the error estimator in error control. Given a tolerance tol, an idealized termination criterion would be

$$\|e_u\|_a \le \text{tol}.$$

Since the error $\|e_u\|_a$ is unknown, we replace it with the upper bound, i.e.,

$$C_{rel}\,\eta_h + \text{osc}_h^{rel} \le \text{tol}.$$

We note that the termination criterion both requires the knowledge of C_{rel} and the incorporation of the data oscillation term osc_h^{rel}. In the special case $C_{rel} = 1$ and $\text{osc}_h^{rel} \equiv 0$, it reduces to

$$\eta_h \le \text{tol}.$$

An alternative, but less used termination criterion is based on the lower bound, i.e., we require

$$\frac{1}{C_{eff}}\left(\eta_h - \text{osc}_h^{eff}\right) \le \text{tol}.$$

Typically, this criterion leads to less refinement and thus requires less computational time which motivates to call the estimator efficient.

Residual-Type A Posteriori Error Estimation

The Role of the Residual

The error estimate

$$\|u - u_h\|_{1,\Omega} \;\le\; \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\|v\|_{1,\Omega}}$$

shows that in order to assess the error $\|e_u\|_a$ we are supposed to evaluate the norm of the residual with respect to the dual space V^*, i.e.,

$$\|r\|_{V^*} \;:=\; \sup_{v \in V \setminus \{0\}} \frac{|r(v)|}{\|v\|_a}.$$

In particular, we have the equality

$$\|r\|_{V^*} \;=\; \|e_u\|_a,$$

whereas for the relative error of $r(v), v \in V$, as an approximation of $\|e_u\|_a$ we obtain

$$\frac{(\|e_u\|_a - r(v))}{\|e_u\|_a} \;=\; \frac{1}{2}\,\|v - \frac{e_u}{\|e_u\|_a}\|_a^2, \quad v \in V \text{ with } \|v\|_a = 1.$$

The goal is to obtain lower and upper bounds for $\|r\|_{V^*}$ at relatively low computational expense.

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Model problem: Let Ω be a bounded simply-connected polygonal domain in Euclidean space $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and consider the elliptic boundary value problem

$$Lu := -\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega ,$$

$$u = 0 \quad \text{on } \Gamma_D , \quad n \cdot a \nabla u = g \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $a = (a_{ij})_{i,j=1}^2$ is supposed to be a matrix-valued function with entries $a_{ij} \in L^\infty(\Omega)$, that is symmetric and uniformly positive definite. The vector n denotes the exterior unit normal vector on Γ_N. Setting

$$H^1_{0,\Gamma_D}(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \},$$

the weak formulation is as follows: Find $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$a(u,v) = \ell(v) \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega),$$

where

$$a(v,w) := \int_\Omega a \nabla v \cdot \nabla w \, dx, \quad \ell(v) := \int_\Omega f \, v \, dx + \int_{\Gamma_N} g \, v \, d\sigma \quad , \quad v \in H^1_{0,\Gamma_D}(\Omega).$$

FE Approximation: Given a geometrically conforming simplicial triangulation $\mathcal{T}_h$ of Ω, we denote by

$$S_{1,\Gamma_D}(\Omega;\mathcal{T}_h) \ := \ \{ \ v_h \in H^1_{0,\Gamma_D}(\Omega) \ | \ v_h\,|_T \in P_1(K) \ , \ T \in \mathcal{T}_h \ \}$$

the trial space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_h$. Note that $P_k(T) \ , \ k \geq 0$, denotes the linear space of polynomials of degree $\leq k$ on T. In the sequel we will refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D) \ , \ D \subseteq \bar{\Omega}$ as the sets of vertices and edges of $\mathcal{T}_h$ on D. We further denote by $|T|$ the area, by h_T the diameter of an element $T \in \mathcal{T}_h$, and by $h_E = |E|$ the length of an edge $E \in \mathcal{E}_h(\Omega \cup \Gamma_N)$. We refer to $f_T := |T|^{-1} \int_T f dx$ the integral mean of f with respect to an element $T \in \mathcal{T}_h$ and to $g_E := |E|^{-1} \int_E g ds$ the mean of g with respect to the edge $E \in \mathcal{E}_h(\Gamma_N)$.
The conforming P1 approximation reads as follows: Find $u_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h)$ such that

$$a(u_h, v_h) \ = \ \ell(v_h), \quad v_h \in S_{1,\Gamma_D}(\Omega;\mathcal{T}_h).$$

Representation of the Residual I

The residual r is given by

$$r(v) \ := \int_{\Omega} f \ v \ dx \ + \int_{\Gamma_N} g \ vds \ - \ a(u_h, v) \quad , \quad v \in V.$$

Applying Green's formula elementwise yields

$$a(u_h, v) = \sum_{T \in \mathcal{T}_h} \int_T a \ \nabla u_h \cdot \nabla v \ dx = \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [n \cdot a \ \nabla u_h] \ v \ ds + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_E n \cdot a \ \nabla u_h \ v \ ds,$$

where $[n \cdot a \ \nabla u_h]$ denotes the jump of the normal derivative of u_h across $E \in \mathcal{E}_h(\Omega)$ and where we have used that $\Delta u_h \equiv 0$ on $T \in \mathcal{T}_h$, since $u_h|_T \in P_1(T)$. We thus obtain

$$r(v) \ := \sum_{T \in \mathcal{T}_h} r_T(v) \ + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_N)} r_E(v).$$

Representation of the Residual II

Here, the local residuals $r_T(v), T \in \mathcal{T}_h$, are given by

$$r_T(v) := \int_T (f - Lu_h) v \, dx,$$

whereas for $r_E(v)$ we have

$$r_E(v) := -\int_E [n \cdot a \, \nabla u_h] v \, ds, \; E \in \mathcal{E}_h(\Omega),$$

$$r_E(v) := \int_E \Big(g - n \cdot a \, \nabla u_h \Big) v \, ds, \; E \in \mathcal{E}_h(\Gamma_N).$$

A Posteriori Error Estimator and Data Oscillations

The error estimator η_h consists of element residuals $\eta_T, T \in \mathcal{T}_h$, and edge residuals $\eta_E, E \in \mathcal{E}_H(\Omega \cup \Gamma_N)$, according to

$$\eta_h \ := \ \Big(\sum_{T \in \mathcal{T}_h} \eta_T^2 \ + \sum_{E \in \mathcal{E}_H(\Omega \cup \Gamma_N)} \eta_E^2 \Big)^{1/2},$$

where η_T and η_E are given by

$$\eta_T \ := \ h_T \ \|f_T - Lu_h\|_{0,T} \ , \ T \in \mathcal{T}_h,$$

$$\eta_E \ := \ \begin{cases} h_E^{1/2} \| [n \cdot a \ \nabla u_h] \|_{0,E} \ , \ E \in \mathcal{E}_h(\Omega), \\ h_E^{1/2} \| g_E - n \cdot a \ \nabla u_h \|_{0,E} \ , \ E \in \mathcal{E}_h(\Gamma_N) \end{cases} .$$

The a posteriori error analysis further invokes the data oscillations

$$\mathrm{osc}_h \ := \ \Big(\sum_{T \in \mathcal{T}_h} \mathrm{osc}_T^2(f) \ + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \mathrm{osc}_E^2(g) \Big)^{1/2},$$

where $\mathrm{osc}_T(f)$ and $\mathrm{osc}_E(g)$ are given by

$$\mathrm{osc}_T(f) := h_T \ \|f - f_T\|_{0,T}, \quad \mathrm{osc}_E(g) := h_E^{1/2} \ \|g - g_E\|_{0,E}.$$

Clément's Quasi-Interpolation Operator I

For $p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)$ we denote by φ_p the basis function in $S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$ with supporting point p, and we refer to D_p as the set

$$D_p \ := \bigcup \{ \ T \in \mathcal{T}_h \ | \ p \in \mathcal{N}_h(T) \ \}.$$

We refer to π_p as the L^2-projection onto $P_1(D_p)$, i.e.,

$$(\pi_p(v), w)_{0,D_p} \ = \ (v, w)_{0,D_p} \quad , \quad w \in P_1(D_p),$$

where $(\cdot, \cdot)_{0,D_p}$ stands for the L^2-inner product on $L^2(D_p) \times L^2(D_p)$. Then, Clément's interpolation operator P_C is defined as follows

$$P_C \ : \ L^2(\Omega) \longrightarrow S_{1,\Gamma_D}(\Omega, \mathcal{T}_h), \quad P_C v := \sum_{p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)} \pi_P(v) \ \varphi_P.$$

Clément's Quasi-Interpolation Operator II

Theorem. Let $v \in H^1_{0,\Gamma_D}(\Omega)$. Then, for Clément's interpolation operator there holds

$$\|P_C\, v\|_{0,T} \le C\, \|v\|_{0,D_T^{(1)}}, \quad \|P_C\, v\|_{0,E} \le C\, \|v\|_{0,D_E^{(1)}}, \quad \|\nabla P_C v\|_{0,T} \le C\, \|\nabla v\|_{0,D_T^{(1)}},$$

$$\|v - P_C\, v\|_{0,T} \le C\, h_T\, \|v\|_{1,D_T^{(1)}}, \quad \|v - P_C\, v\|_{0,E} \le C\, h_E^{1/2}\, \|v\|_{1,D_E^{(1)}}.$$

Further, we have

$$\Big(\sum_{T \in \mathcal{T}_h} \|v\|^2_{\mu,D_K^{(1)}}\Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1,$$

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} \|v\|^2_{\mu,D_E^{(1)}}\Big)^{1/2} \le C\, \|v\|_{\mu,\Omega}, \quad 0 \le \mu \le 1.$$

where $D_T^{(1)} := \bigcup \{ T' \in \mathcal{T}_h \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset \}$, $D_E^{(1)} := \bigcup \{ T' \in \mathcal{T}_h \mid \mathcal{N}_h(E) \cap \mathcal{N}_h(T') \neq \emptyset \}$.

Reliability of the A posteriori Error Estimator I

Theorem. There exist constants Γ_R and $\Gamma_{osc} > 0$ depending only on the shape regularity of $\mathcal{T}_h$ such that

$$\|e_u\|_a \leq \Gamma_R \, \eta_h \; + \; \Gamma_{osc} \, osc_h .$$

Proof. Setting $v = e_u$, we have

$$\|e_u\|_a \; = \; a(e_u, e_u) \; = \; r(e_u) \; = \; r(P_C e_u) \; + \; r(e_u - P_C e_u) .$$

For the first term on the right-hand side, by Galerkin orthogonality we obtain

$$r(P_C e_u) = \int_\Omega f \; P_C e_u \; dx \; + \; \int_{\Gamma_N} g \; P_C e_u \; ds \; - \; a(u_h, P_C e_u) \; = \; 0 .$$

Proof cont'd. On the other hand, for the second term on the right-hand side of Green's formula yields

$$r(e_u - P_C e_u) = \int_\Omega f\ (e_u - P_C e_u)\ dx + \int_{\Gamma_N} g\ (e_u - P_C e_u)\ ds + \sum_{T \in \mathcal{T}_h} \int_T \underbrace{\nabla \cdot\ a\ \nabla\ u_h}_{=\ -Lu_h}\ (e_u - P_C e_u)\ dx$$

$$- \sum_{T \in \mathcal{T}_h} \int_{\partial T} n_{\partial T} \cdot a\ \nabla u_h\ (e_u - P_C e_u)\ ds = \sum_{T \in \mathcal{T}_h} \int_T (f_T - Lu_h)\ (e_u - P_C e_u)\ dx$$

$$+ \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [n_E \cdot a\ \nabla u_h]\ (e_u - P_C e_u)\ ds + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_E (g_E - n_E \cdot a\ \nabla u_h)\ (e_u - P_C e_u)\ ds$$

$$+ \sum_{T \in \mathcal{T}_h} \int_T (f - f_T)\ (e_u - P_C e_u)\ dx + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_E (g - g_E)\ (e_u - P_C e_u)\ ds\ .$$

Proof cont'd. In view of the local approximation properties of Clément's quasi-interpolation operator, it follows that

$$
\begin{aligned}
r(e_u - P_C e_u) \le C \Big(& \sum_{T \in \mathcal{T}_h} h_T^2 \, \|f_T - L u_h\|_{0,T}^2 \Big)^{1/2} \, \|e_u\|_{1,\Omega} \\
+ & \sum_{E \in \mathcal{E}_h(\Omega)} h_E \, \|[n_E \cdot a \, \nabla u_h]_J\|_{0,E}^2 \Big)^{1/2} \, \|e_u\|_{1,\Omega} \\
+ & \sum_{E \in \mathcal{E}_h(\Gamma_N)} h_E \, \|g_E - n_E \cdot a \, \nabla u_h\|_{0,E}^2 \Big)^{1/2} \, \|e_u\|_{1,\Omega} + \Big(\sum_{T \in \mathcal{T}_h} h_T^2 \, \|f - f_T\|_{0,T}^2 \Big)^{1/2} \, \|e_u\|_{1,\Omega} \\
+ & \sum_{E \in \mathcal{E}_h(\Gamma_N)} h_E \, \|g - g_E\|_{0,E}^2 \Big)^{1/2} \, \|e_u\|_{1,\Omega} \Big),
\end{aligned}
$$

from which we may conclude.

Element and Edge Bubble Functions I

The element bubble function ψ_T is defined by means of the barycentric coordinates $\lambda_i^T, 1 \le i \le 3$, according to

$$\psi_T \; := \; 27 \, \lambda_1^T \, \lambda_2^T \, \lambda_3^T .$$

Note that $\text{supp}\; \psi_T = T_{int}$, i.e., $\psi_T \,|_{\partial T} = 0 \;,\; T \in \mathcal{T}_h$. On the other hand, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ and $T \in \mathcal{T}_h$ such that $E \subset \partial T$ and $p_i^E \in \mathcal{N}_h(E) \;,\; 1 \le i \le 2$, we introduce the edge-bubble functions ψ_E

$$\psi_E \; := \; 4 \, \lambda_1^T \, \lambda_2^T .$$

Note that $\psi_E \,|_{E'} = 0$ for $E' \in \mathcal{E}_h(T), E' \ne E$.

Element and Edge Bubble Functions II

The bubble functions ψ_T and ψ_E have the following important properties that can be easily verified taking advantage of the affine equivalence of the finite elements:

Lemma. There holds

$$\|p_h\|_{0,T}^2 \le C \int_T p_h^2 \, \psi_T \, dx, \quad p_h \in P_1(T),$$

$$\|p_h\|_{0,E}^2 \le C \int_E p_h^2 \, \psi_E \, d\sigma, \quad p_h \in P_1(E),$$

$$| \, p_h \, \psi_T \, |_{1,T} \le C \, h_T^{-1} \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_T\|_{0,T} \le C \, \|p_h\|_{0,T}, \quad p_h \in P_1(T),$$

$$\|p_h \, \psi_E\|_{0,E} \le C \, \|p_h\|_{0,E} \quad , \quad p_h \in P_1(E).$$

Element and Edge Bubble Functions III

For functions $p_h \in P_1(E)$, $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we further need an extension $p_h^E \in L^2(T)$ where $T \in \mathcal{T}_h$ such that $E \subset \partial T$. For this purpose we fix some $E' \subset \partial T$, $E' \neq E$, and for $x \in T$ denote by x_E that point on E such that $(x - x_E) \parallel E'$. For $p_h \in P_1(E)$ we then set

$$p_h^E \ := \ p_h(x_E).$$

Further, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we define $D_E^{(2)}$ as the union of elements $T \in \mathcal{T}_h$ containing E as a common edge

$$D_E^{(2)} \ := \bigcup \{ \ K \in \mathcal{T}_h \ \mid \ E \in \mathcal{E}_h(T) \ \}.$$

Element and Edge Bubble Functions IV

Lemma. There holds

$$| \, p_h^E \, \psi_E \, |_{1,D_E^{(2)}} \leq C \, h_E^{-1/2} \, \|p_h\|_{0,e}, \quad p_h \in P_1(E),$$

$$\|p_h^E \, \psi_E\|_{0,D_E^{(2)}} \leq C \, h_E^{1/2} \, \|p_h\|_{0,E}, \quad p_h \in P_1(E).$$

Further, for all $v \in V$ and $\mu = 0, 1$ there holds

$$\Big(\sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} h_E^{1-\mu} \, \|v\|^2_{\mu, D_E^{(2)}} \Big)^{1/2} \leq C \, \Big(\sum_{T \in \mathcal{T}_h} h_T^{1-\mu} \, \|v\|^2_{\mu,T} \Big)^{1/2}.$$

Efficiency of the A posteriori Error Estimator I

Theorem. There exist constants $\gamma_R, \gamma_E > 0$, depending only on the shape regularity of $\mathcal{T}_h$ such that

$$\gamma_R \ \eta_h \ - \ \gamma_E \ \mathrm{osc}_h \ \leq \ \|e\|_{1,\Omega}.$$

Proof. The proof will be given by a series of lemmas establishing the local efficiency of the estimator.

Lemma. Let $T \in \mathcal{T}_h$. Then there holds:

$$h_T \; \|f_T - Lu_h\|_{0,T} \le C \; \|e\|_{1,T} + h_T \; \|f - f_T\|_{0,T}.$$

Proof. We set $p_h := f_T$. Observing $\psi_T|_{\partial T} = 0$, by Green's formula

$$a|_T(u_h, p_h \; \psi_T) \;=\; - \int_T \nabla \cdot (a \; \nabla u_h) \; p_h \; \psi_T \; dx + \int_{\partial T} \underbrace{n_{\partial T} \cdot a \; \nabla u_h \; p_h \; \psi_T}_{= \; 0} \; ds.$$

Denoting by $\pi_h \psi_T$ the L^2-projection onto the linear space of elementwise constants and taking advantage of the properties of the bubble functions, it follows that

$$\|f_T - Lu_h\|_{0,T}^2 \le C \; \int_T (f_T - Lu_h) \; \pi_h \; \psi_T \; dx = \Big(\int_T f \; \pi_h \; \psi_T \; dx - a|_T(u_h, \pi_h \; \psi_T)$$

$$+ \int_T (\pi_h f - f) \; \pi_h \; \psi_T \; dx \Big) = \Big(a|_T(e_u, \pi_h \; \psi_T) + \int_T (f_T - f) \; \pi_h \; \psi_T \; dx \Big)$$

$$\le C \; h_T^{-1} \; \|e_u\|_{1,T} \; \|p_h\|_{0,T} + \|f_T - f\|_{0,T} \; \|p_h\|_{0,T},$$

Lemma. Let $E \in \mathcal{E}_h(\Omega)$. Then there holds:

$$h_E^{1/2} \| [n_E \cdot a \, \nabla u_h] \|_{0,E} \le C \, \|e\|_{1,D_E^{(2)}} + h_E \, \|f - f_T\|_{0,D_E^{(2)}} + h_E \, \|f_T - Lu_h\|_{0,D_E^{(2)}} .$$

Proof. We set $p_h^E := [n_E \cdot a \, \nabla u_h]$. In view of $\psi_E|_{E'} = 0, E' \neq E,$, Green's formula gives

$$\int_{\partial D_E^{(2)}} n_{\partial D_E^{(2)}} \cdot a \, \nabla u_h \; p_h^E \; \psi_E \; ds = a|_{D_E^{(2)}}(u_h, p_h^E \; \psi_E) \; + \int_{D_E^{(2)}} \underbrace{\nabla \cdot a \; \nabla u_h}_{= \; -Lu_h} \; p_h^E \; \psi_E \; dx,$$

where $p_h^E \psi_E$ is the L^2-projection onto the edgewise constants.

Proof cont'd. If we use the properties of the bubble functions, it follows that

$$\|[n_{\partial D_E^{(2)}} \cdot a\ \nabla u_h]\|_{0,E}^2 \le C \int_E [n_{\partial D_E^{(2)}} \cdot a\ \nabla u_h]\ p_h^E\ \psi_E\ ds = \int_{\partial D_E^{(2)}} [n_{\partial D_E^{(2)}} \cdot a\ \nabla u_h]\ p_h^E\ \psi_E\ ds$$

$$= \Big(a|_{D_E^{(2)}}(u_h, p_h^E\ \psi_E) - \int_{D_E^{(2)}} f\ p_h^E\ \psi_E dx + \int_{D_E^{(2)}} (f - f_T)\ p_h^E\ \psi_E dx + \int_{D_E^{(2)}} (f_T - Lu_h)\ p_h^E\ \psi_E dx \Big)$$

$$= -a|_{D_E^{(2)}}(e, p_h^E\ \psi_E) + \Big(\int_{D_E^{(2)}} (f - f_T)\ p_h^E\ \psi_E dx + \int_{D_E^{(2)}} (f_T - Lu_h)\ p_h^E\ \psi_E dx \Big)$$

$$\le C\ h_E^{-1/2}\ \|e_u\|_{1,D_E^{(2)}}\ \|p_h^E\|_{0,E} + h_E^{1/2}\ \|f - f_T\|_{0,D_E^{(2)}}\ \|p_h^E\|_{0,E} + h_E^{1/2}\ \|f_T - Lu_h\|_{0,D_E^{(2)}}\ \|p_h^E\|_{0,E}.$$

from which the assertion can be easily deduced.

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Lemma. Let $E \in \mathcal{E}_h(\Gamma_N)$. Then there holds:

$$h_E^{1/2} \|g_E - n_E \cdot \nabla u_h\|_{0,E} \le$$
$$C \; \|e\|_{1,D_E^{(2)}} + h_E^{1/2} \|g - g_E\|_{0,E} + h_E \; \|f - f_T\|_{0,D_E^{(2)}} + h_E \|f_T - L u_h\|_{0,D_E^{(2)}}.$$

Proof. We set $p_h^E := g_E - n_E \cdot a \, \nabla u_h$. Observing $\psi_E|_{E'} = 0, E' \neq E$, by Green's formula we obtain

$$\int_E n_E \cdot a \; \nabla u_h \; p_h^E \; \psi_E \; ds = \int_{\partial D_E^{(2)}} n_{\partial D_E^{(2)}} \cdot a \; \nabla u_h \; p_h^E \; \psi_E \; ds$$

$$= \; a|_{D_E^{(2)}}(u_h, p_h^E \; \psi_E) \; + \int_{D_E^{(2)}} \underbrace{\nabla \cdot a \; \nabla u_h}_{= \; -L u_h} \; p_h^E \; \psi_E \; dx.$$

Goal-Oriented Dual Weighted Approach

Goal-Oriented Dual Weighted Approach I

The goal oriented dual weighted approach allows to control the error $e_u := u - u_h$ with respect to a rather general error functional or output functional

$$J \ : \ V \subseteq H^1(\Omega) \to \mathbb{R}.$$

The goal oriented dual weighted approach strongly uses the solution $z \in V$ of the associated dual problem

$$a(v,z) \ = \ J(v) \quad , \quad v \in V,$$

and its finite element approximation $z_h \in V_h$, i.e.,

$$a(v_h, z_h) \ = \ J(v_h), \quad v_h \in V_h.$$

Using Galerkin orthogonality, we readily deduce that

$$J(e_u) \ = \ a(e_u, z) \ = \ a(e_u, z - v_h) \ = \ r(z - v_h), \quad v_h \in V_h,$$

where $r(\cdot)$ stands for the residual with respect to the computed finite element approximation u_h.

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Goal-Oriented Dual Weighted Approach II

Theorem. Let $u_h \in V_h := S_{1,\Gamma}(\Omega; \mathcal{T}_h(\Omega))$ be the conforming P1 approximation of the solution $u \in H_0^1(\Omega)$ of Poisson's equation with $f \in L^2(\Omega)$ and homogeneous Dirichlet boundary data. Then, the following error representation holds true

$$J(e_u) \;=\; \sum_{T \in \mathcal{T}_h(\Omega)} \Big((r_T, z - v_h)_{0,T} + (r_{\partial T}, z - v_h)_{0,\partial T} \Big), \quad v_h \in V_h,$$

where the element residuals r_T and the edges residuals $r_{\partial T}$ are given by

$$r_T := f, \quad T \in \mathcal{T}_h(\Omega), \quad r_{\partial T}|_E := \begin{cases} \frac{1}{2}\,\nu_E \cdot [\nabla u_h] \;, & E \in \mathcal{E}_h(\partial T \cap \Omega) \\ 0 \;, & E \in \mathcal{E}_h(\partial T \cap \Gamma) \end{cases}$$

Moreover, we have the error estimate

$$|J(e_u)| \;\le\; \eta_{DW} \;:=\; \sum_{T \in \mathcal{T}_h(\Omega)} \omega_T \, \rho_T,$$

where for $v_h \in V_h$ the element residuals ρ_T and the weights ω_T read

$$\rho_T := \Big(\|r_T\|_{0,T}^2 \;+\; h_T^{-1} \, \|r_{\partial T}\|_{0,\partial T}^2 \Big)^{1/2}, \quad \omega_T := \Big(\|z - v_h\|_{0,T}^2 \;+\; h_T \, \|z - v_h\|_{0,\partial T}^2 \Big)^{1/2}.$$

Goal-Oriented Dual Weighted Approach III

We remark that the previous result is not really a posteriori, since the solution $z \in V$ of the dual solution is not known. Therefore, information about the weights $\omega_T, T \in \mathcal{T}_h(\Omega)$ has to be provided either by means of an a priori analysis or by the numerical solution of the dual problem.

Theorem. Under the assumptions of the previous theorem let the error functional be given by

$$J(v) := \frac{(\nabla v, \nabla e_u)_{0,\Omega}}{\|\nabla e_u\|_{0,\Omega}}, \quad v \in V.$$

Then, there holds

$$\|\nabla e_u\|_{0,\Omega} \le C \Big(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^2 \, \rho_T^2 \Big)^{1/2}.$$

Proof. The dual solution $\mathrm{z} \in \mathrm{V}$ satisfies

$$a(v,z) \; = \; \frac{(\nabla v, \nabla e_u)_{0,\Omega}}{\|\nabla e_u\|_{0,\Omega}}, \quad v \in V,$$

from which we readily deduce the a priori bound

$$\|\nabla z\|_{0,\Omega} \; \leq \; 1.$$

In view of the basic error estimate it follows that

$$J(e_u) = \|\nabla e_u\|_{0,\Omega} \; \leq \Big(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^2 \; \rho_T^2 \Big)^{1/2} \Big(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^{-2} \; \omega_T^2 \Big)^{1/2}.$$

Choosing $v_h = P_C z$, where P_C is Clément's quasi-interpolation operator, we find

$$\inf_{v_h \in V_h} \Big(\sum_{T \in \mathcal{T}_h(\Omega)} (h_T^{-2} \; \|z - v_h\|_{0,T}^2 + h_T^{-1} \; \|z - v_h\|_{0,\partial T}^2 \Big)^{1/2} \; \leq C \; \|\nabla z\|_{0,\Omega}.$$

Using the last inequality in the previous one and observing the error representation gives the assertion.

Goal-Oriented Dual Weighted Approach IV

Theorem. Consider the conforming P1 approximation of Poisson's equation under homogeneous Dirichlet boundary conditions and assume that the solution $u \in V := H_0^1(\Omega)$ is 2-regular. Using the the error functional

$$J(v) \ := \ \frac{(v, e_u)_{0,\Omega}}{\|e_u\|_{0,\Omega}}, \quad v \in V,$$

gives rise to the a posteriori error estimate

$$\|e_u\|_{0,\Omega} \ \leq C \left(\sum_{T \in \mathcal{T}_h(\Omega)} h_T^4 \ \rho_T^2 \right)^{1/2}.$$

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Goal-Oriented Dual Weighted Approach V

Finally, we apply the goal-oriented dual weighted approach to the pointwise estimation of the error at some point $a \in \Omega$. Given some tolerance $\varepsilon > 0$, we consider the ball

$$K_\varepsilon(a) \ := \ \{x \in \Omega \mid |x - a| < \varepsilon\}$$

around the point a and define the regularized error functional

$$J(v) \ := \ |K_\varepsilon(a)|^{-1} \int\limits_{K_\varepsilon(a)} v \, dx.$$

The dual solution z of $a(v,z) \ = \ J(v)$ behaves like a regularized Green's function

$$z(x) \ \sim \ \log(r(x)) \ , \ r(x) \ := \ \sqrt{|x-a|^2 + \varepsilon^2}.$$

With the residual ρ_T we obtain

$$|(u - u_h)(a)| \ \sim \ \sum_{T \in \mathcal{T}_h(\Omega)} \frac{h_T^3}{r_T^2} \, \rho_T, \quad r_T := \max_{x \in T} r(x).$$