Lectures on semi-Markov processes

Enrico Scalas
The University of Eastern Piedmont
Department of Advanced Science and Technology

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Outline

- Motivation
- The random walk
- The Poisson process
- The exponential renewal process
- The compound Poisson process

Motivation I

$X(t)$ position of a particle at time t

$X(0)$ position of the particle at time $t_0 = 0$

$\Delta X(t) = X(t) - X(0)$

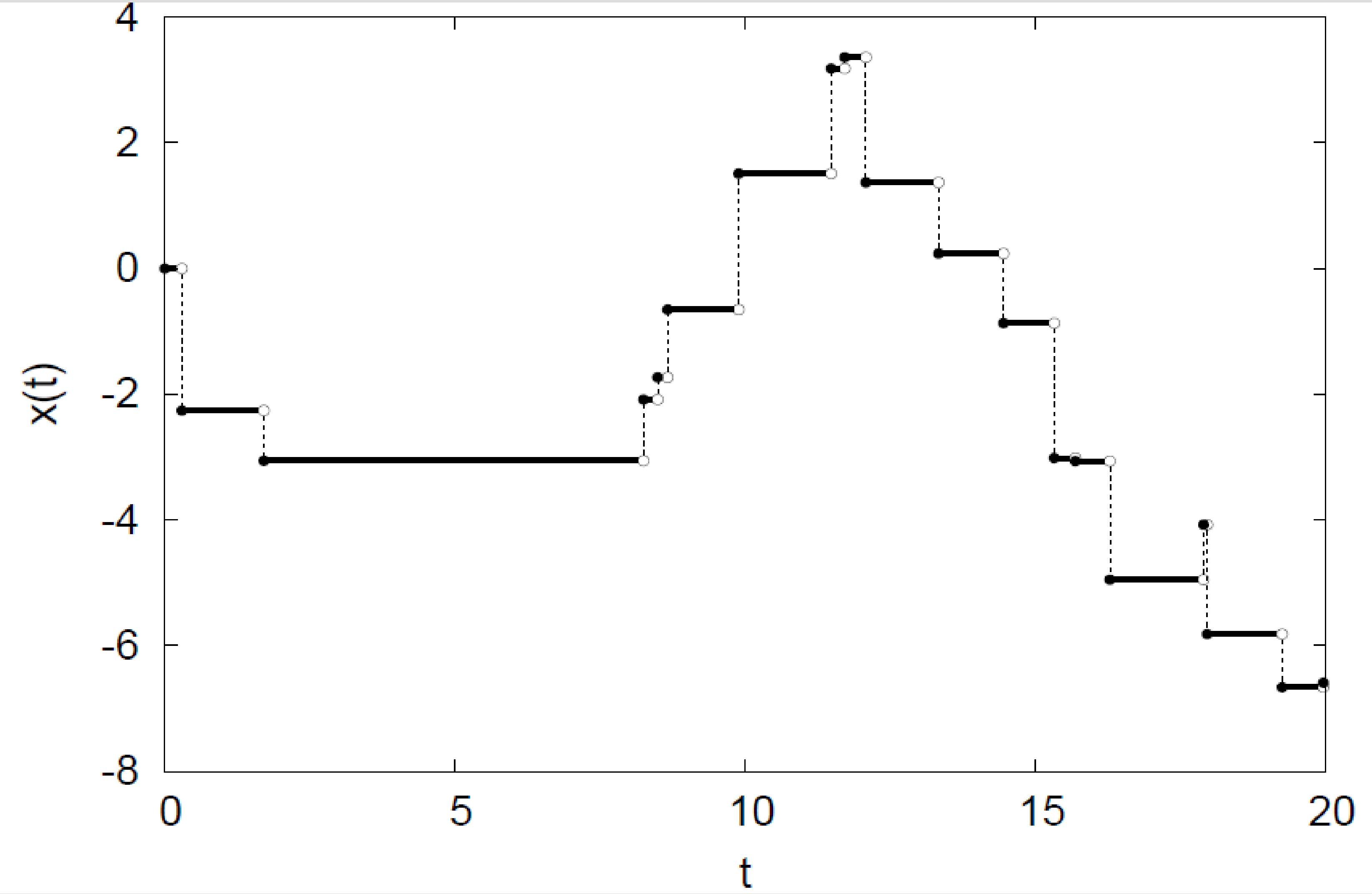
Usual initial condition $X(0) = 0$

$t_i, i = 0, 1, \ldots$ jump epochs

$\xi_i = X(t_i) - X(t_{i-1})$ jump sizes

$\tau_i = t_i - t_{i-1}$ residence times, durations, waiting times, etc.

Motivation II



Motivation III

$$X(t) = \sum_{i=1}^{N(t)} \xi_i$$

$N(t)$ is a *counting process*: the (random) number of jumps from 0 to t

We can also define $X_n = \sum_{i=1}^{n} \xi_i$ and write

$$X(t) = X_{N(t)}$$

This is a CTRW!

The random walk I

Assume ξ_i are independent and identically distributed random variables, then

$$X_n = \sum_{i=1}^{n} \xi_i$$

is a *random walk* (RW)!

The random walk II

If $X_n = \sum_{i=1}^{n} \xi_i$ then

$$X_n = X_{n-1} + \xi_n$$

a familiar definition in Econometrics!

The random walk III

Example: the symmetric dichotomous RW.

Recipe

1. You cast a fair coin n times getting a random sequence of heads (H) and tails (T);
2. If H then you write +1, if T you write -1, like that (n=5): HTTHT becomes +1-1-1+1-1;
3. Starting at 0 you sum (cumulative sum) and get 0,+1,0,-1,0,-1.

The random walk IV

We have $\xi_i = \pm 1$ and

$$P(\xi_i = +1) = P(\xi_i = -1) = 1/2$$

Can we compute $P(X_n = k | X_0 = 0)$?

Every sequence of length *n* has probability $1/2^n$. But how many sequences lead to k? Call this number $M(k,n)$.

The random walk V

One finds that

$$M(k,n) = \binom{n}{\frac{k+n}{2}} = \frac{n!}{h!(n-h!)}$$

where $h = (k+n)/2$.

The random walk VI

Eventually one gets

$$P(X_n = k | X_0 = 0) = M(k, n)\left(\frac{1}{2}\right)^n$$

that is

$$P(X_n = k | X_0 = 0) = \binom{n}{\frac{k+n}{2}}\left(\frac{1}{2}\right)^n$$

The random walk VII

What about ξ_i i.i.d. but with a generic continuous distribution with probability density function $w(\xi)$?
Let X and Y be two independent random variables with probability densities $w_X(x)$ and $w_Y(y)$. Let $Z = X+Y$, then one has

$$w_Z(z)=(w_X * w_Y)(z)=\int_{-\infty}^{+\infty} w_X(z-y)\,w_Y(y)\,dy$$

$$\int_{-\infty}^{+\infty} w_X(z-y)\,w_Y(y)\,dy=\int_{-\infty}^{+\infty} w_X(x)\,w_Y(z-x)\,dx$$

$(w_X * w_Y)(z)$ is called the *convolution product*.

The random walk VIII

But we have $X_n = \sum_{i=1}^{n} \xi_i$

Idea! Let us iterate the convolution product n times.
Call the result $w^{(n)}(x)$, then

$$p_n(x) = P(x \leqslant X_n \leqslant x + dx) = w^{(n)}(x)$$

Nice! But can we use this result? Indeed!

The random walk IX

Consider normally distributed jumps

$$w(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right)$$

We know that $\mathrm{E}(X_n) = n\,\mu$ and $\mathrm{Var}(X_n) = n\,\sigma^2$

Direct calculations show that

$$w^{(n)}(\xi) = \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{(\xi-n\mu)^2}{2n\sigma^2}\right)$$

The normal distribution does not change shape under convolution! It is a *stable* distribution!

Summary

- We wish to model pure jump diffusions.
- The ingredients are the random walk and the counting process.
- We have seen two examples of random walk.
- We have highlighted the role of the convolution product.
- We have seen that the normal distribution is stable with respect to the convolution product.