Lectures on semi-Markov processes

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The random walk X

Remember that $X_n = X_{n-1} + \xi_n$: this means that

$$p_n(x) = (p_{n-1} * w)(x) = \int_{-\infty}^{+\infty} p_{n-1}(u)\, w(x-u)\, du$$

given that X_{n-1} and ξ_n are independent.

The position at step n explicitly depends only on the position at step $n\text{-}1$ and not on the previous positions. The random walk is a *Markov process*. Indeed, it is a *Markov chain* (a discrete-time Markov process)!

What is a Markov process?

The conditional distribution of X_n, given $X_1, ..., X_{n-1}$ only depends on X_{n-1}:

$$P(X_n \leqslant x | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = P(X_n \leqslant x | X_{n-1} = x_{n-1})$$

In continuous time, for $0 \leq t_1 < ... < t_n \ (n > 1)$

$$P(X(t_n) \leqslant x | X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}) = P(X(t_n) \leqslant x | X(t_{n-1}) = x_{n-1})$$

The random walk XI

Let I_n denote the history of the walk up to step n. If we know I_n, we know X_n, the value of the walk at step n. Then, one has

$$E[X_n|I_m]=X_m+\sum_{i=m+1}^{n} E[\xi_i|I_m], m<n$$

Assume that $E[\xi_i]=0$, then

$$E[\xi_i|I_m]=E[\xi_i]=0, m+1\leq i\leq n$$

$$E[X_n|I_m]=X_m, m<n$$

If $E[\xi_i]=0$, the random walk is a *martingale*!

The random walk XII

- The discrete-time random walk is a Markov process.
- The discrete-time random walk is a martingale.
- We would like to extend these nice properties to the continuous-time case.

- Before that, some exercises.

Exercises on the random walk

Exercise 1: Consider the following asymmetric dichotomous random walk: $\xi_i = 1$ with probability p and $\xi_i = 0$ with probability $q=1-p$. Find $P(X_n = k | X_0 = 0)$

Exercise 2: Find the expected value and the variance of $P(X_n = k | X_0 = 0)$

Exercise 3: Write a Monte Carlo program that generates sample paths for the symmetric dichotomous random walk.