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Hyperbolic Conservation Laws

\[ \partial_t u + \nabla \cdot f(u) = 0 \]

\[ u = (u_1, \cdots, u_m)^\top, \quad x = (x_1, \cdots, x_d), \quad \nabla = (\partial_{x_1}, \cdots, \partial_{x_d}) \]

\[ f = (f_1, \cdots, f_d) : \mathbb{R}^m \to (\mathbb{R}^m)^d \quad \text{is a nonlinear mapping} \]

\[ f_i : \mathbb{R}^m \to \mathbb{R}^m \quad \text{for} \quad i = 1, \cdots, d \]

\[ \partial_t A(u, u_t, \nabla u) + \nabla \cdot B(u, u_t, \nabla u) = 0 \]

\[ A, B : \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m)^d \to \mathbb{R}^m \text{ are nonlinear mappings} \]

Connections and Applications:

- **Fluid Mechanics and Related**: Euler Equations and Related Equations
  Gas, shallow water, elastic body, reacting gas, plasma, ....

- **Special Relativity**: Relativistic Euler Equations and Related Equations

- **General Relativity**: Einstein Equations and Related Equations

- **Differential Geometry**: Isometric Embeddings, Nonsmooth Manifolds..
Diverse Approaches in Sciences:
- Experimental data
- Large and small scale computing by a search for effective numerical methods
- Modelling (Asymptotic and Qualitative)
- Rigorous proofs for prototype problems and an understanding of the solutions

Two Strategies as a first step:
- Study good, simpler nonlinear models with physical motivations;
- Study special, concrete nonlinear problems with physical motivations

Meanwhile, extend the results and ideas to:
- Study the Euler equations in gas dynamics and elasticity
- Study nonlinear systems that the Euler equations are the main subsystem or describe the dynamics of macroscopic variables such as MHD, Euler-Poisson Equations, Combustion, Relativistic Euler Equations, .........
- Study more general hyperbolic systems and related problems
- Develop further new mathematical ideas, techniques, approaches, as well as new mathematical theories
1. Important Multidimensional Models

2. Multidimensional Steady Problems

3. Multidimensional Self-Similar Problems
UTSD equation (E-9) in transonic aerodynamics:

\[
\begin{align*}
\partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) + \partial_y v &= 0, \\
\partial_x v - \partial_y u &= 0,
\end{align*}
\]

or the Zabolotskaya-Khokhlov equation (E-10):

\[
\partial_x \left( \partial_t u + u \partial_x u \right) + \partial_{yy} u = 0.
\]
Unsteady Transonic Small Disturbance Equation (E-9)

UTSD equation (E-9) in transonic aerodynamics:

\[
\begin{align*}
\partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) + \partial_y \nu &= 0, \\
\partial_x \nu - \partial_y u &= 0,
\end{align*}
\]

or the Zabolotskaya-Khokhlov equation (E-10):

\[
\partial_x (\partial_t u + uu_x) + \partial_{yy} u = 0.
\]

(E-9) describes the potential flow field near the reflection point in weak shock reflection, which determine the leading order approximation of geometric optical expansions. It can be also used to formulate asymptotic equations for the transition from regular to Mach reflection for weak shocks. It also describes high-frequency waves near singular rays (Hunter 1986).
Unsteady Transonic Small Disturbance Equation (E-9)

**UTSD equation (E-9)** in transonic aerodynamics:

\[
\begin{align*}
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) + \frac{\partial}{\partial y} v &= 0, \\
\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u &= 0,
\end{align*}
\]

or the **Zabolotskaya-Khokhlov equation (E-10)**:

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u \right) + \frac{\partial}{\partial y} \frac{\partial}{\partial y} u = 0.
\]

(E-9) describes the potential flow field near the reflection point in weak shock reflection, which determine the leading order approximation of geometric optical expansions. It can be also used to formulate asymptotic equations for the transition from regular to Mach reflection for weak shocks. It also describes high-frequency waves near singular rays (Hunter 1986).

(E-10) was first derived by Timman (1964) in the context of transonic flows and by Zabolotskaya-Khokhlov (1969) in nonlinear acoustics which describes the diffraction of nonlinear acoustic beams. Cramer-Seebass (1978) used (E-7) to study caustics in nearly planar sound waves. The same equation arises as a weakly nonlinear equation for cusped caustics.
Steady Transonic Small Disturbance Equation (E-11)

\[ \partial_x (u \partial_x u) + \partial_{yy} u = 0 \]

or

\[ u \partial_{xx} u + \partial_{yy} u + (\partial_x u)^2 = 0. \]

Elliptic: \( u > 0 \)

Hyperbolic: \( u < 0 \)

This is a nonlinear version of the celebrated linear equations of mixed type:

Tricomi Equation: \( \partial_{xx} u + x \partial_{yy} u = 0 \) (hyperbolic degeneracy at \( x = 0 \))

Keldysh Equation: \( x \partial_{xx} u + \partial_{yy} u = 0 \) (parabolic degeneracy at \( x = 0 \))

J. Hunter, C. Morawetz, B. Keyfitz, S. Canic, G. Lieberman, ···
Pressureless Euler Equations \((E-12)\)

Separating the pressure from the Euler equations in \(\mathbb{R}^2\), motivated by the flux-splitting scheme used by Argarwal-Halt (1994) in numerical computations for airfoil flows. It may be obtained from the infinite Mach number limit from the full Euler equations.

The pressureless Euler equations, modelling the motion of free particles which stick under collision, \((E-12)\):

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_y (\rho v) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_y (\rho uv) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho uv) + \partial_y (\rho v^2) &= 0, \\
\partial_t (\rho E) + \partial_x (\rho u E) + \partial_y (\rho v E) &= 0.
\end{align*}
\]

All the characteristic families are linear degenerate; but solutions may become measure solutions due to concentration under collision.

Zeldovich 1970
Pressure-Gradient System (E-13)

\[
\begin{align*}
\partial_t \rho &= 0, \\
\partial_t (\rho u) + \partial_x p &= 0, \\
\partial_t (\rho v) + \partial_y p &= 0, \\
\partial_t (\rho E) + \partial_x (up) + \partial_y (vp) &= 0.
\end{align*}
\]

For small velocity and large gas constant \( \gamma \), \( \rho E = \frac{1}{2} \rho (u^2 + v^2) + \frac{1}{\gamma - 1} p \) is dominated by \( \frac{1}{\gamma - 1} p \), then, by setting \( p = (\gamma - 1)P, t = \frac{1}{\gamma - 1} \tau \), we have

\[
\begin{align*}
\partial_\tau u + \partial_x P &= 0, \\
\partial_\tau v + \partial_y P &= 0, \\
\partial_\tau P + P \partial_x u + P \partial_y v &= 0.
\end{align*}
\]

Eliminating \((u, v)\), we obtain the nonlinear wave equations (E-13):

\[
\partial_\tau \left( \frac{1}{P} \partial_\tau P \right) - \Delta P = 0.
\]

Yuxi Zheng, Kyungwoo Song, · · ·
Nonlinear Wave System (E-14)

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_y (\rho v) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p) + \partial_y (\rho uv) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho uv) + \partial_y (\rho uv + p) &= 0,
\end{align*}
\]

For small velocity and irrotational flow, ignore the nonlinear velocity terms and denote \((m, n) = (\rho u, \rho v)\) as momenta:

\[
\begin{align*}
\partial_t \rho + \partial_x m + \partial_y n &= 0, \\
\partial_t m + \partial_x p &= 0, \\
\partial_t n + \partial_y p &= 0.
\end{align*}
\]

Eliminating \((m, n)\), we have

\[
\partial_{tt} \rho - \Delta p(\rho) = 0
\]

Suny Canic, Barbara Keyfitz, · · ·
Euler Equations for the Chaplygin Gas (E-15)

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \quad \mathbf{v} \in \mathbb{R}^d, \quad \mathbf{x} \in \mathbb{R}^d \\
\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) &= 0.
\end{align*}
\]

For the **Chaplygin gas**, also called the **von Karmen gas**, we have

\[ p(\rho) = -\frac{1}{\rho}, \]

**Advantage**: The pressure waves are contact discontinuities, and their location is often known apriori, instead of being a free boundary.

**Drawback**: Since the pressure is uniformly bounded, some Riemann problems yield a concentration of mass along a co-dimension one subset.

Serre 2009: Multidimensional shock waves and Riemann problems
Gauss-Codazzi Equations in the Fluid Dynamics Formalism for Isometric Embedding (E-16)

The Codazzi Equations (i.e. the Balance of Momentum Equations):

\[
\begin{align*}
\partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= -\Gamma_{22}^{(2)}(\rho v^2 + p) - 2\Gamma_{12}^{(2)}\rho uv - \Gamma_{11}^{(2)}(\rho u^2 + p), \\
\partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= -\Gamma_{22}^{(1)}(\rho v^2 + p) - 2\Gamma_{12}^{(1)}\rho uv - \Gamma_{11}^{(1)}(\rho u^2 + p),
\end{align*}
\]

The Gauss Equation (i.e. the Bernoulli Relation):

\[p = -\sqrt{q^2 + K}, \quad q^2 = u^2 + v^2, \quad K \text{ —Gaussian curvature}\]

Constitutive relation—the Chaplygin type gas: \( p(\rho) = -\frac{1}{\rho} \)

\( \Gamma^{(k)}_{ij} \) — Christoffel symbols, depending on the metric \( g_{ij} \) up to their 1st derivatives, \( i, j, k = 1, 2 \).

Define the sound speed: \( c^2 = p'(\rho) \). Then \( c^2 = 1/\rho^2 = q^2 + K \).
Gauss-Codazzi Equations in the Fluid Dynamics Formalism for Isometric Embedding (E-16)

The Codazzi Equations (i.e. the Balance of Momentum Equations):

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\partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= -\Gamma^{(2)}_{22}(\rho v^2 + p) - 2\Gamma^{(2)}_{12} \rho uv - \Gamma^{(2)}_{11}(\rho u^2 + p), \\
\partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= -\Gamma^{(1)}_{22}(\rho v^2 + p) - 2\Gamma^{(1)}_{12} \rho uv - \Gamma^{(1)}_{11}(\rho u^2 + p),
\end{align*}
\]

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Define the sound speed: \( c^2 = p'(\rho) \). Then \( c^2 = 1/\rho^2 = q^2 + K \).

\( c^2 > q^2 \) and the “flow” is subsonic when \( K > 0 \),

\( c^2 < q^2 \) and the “flow” is supersonic when \( K < 0 \),

\( c^2 = q^2 \) and the “flow” is sonic when \( K = 0 \).

The Gaussian Curvature $K$ on a Torus: Doughnut Surface or Toroidal Shell
Lax System: Complex Inviscid Burgers Equation (E-17)

\( f(u) \) – Complex valued function of a single complex variable \( u = u + vi \)

\( u = u(t, z) \) – Complex valued function with \( z = x + yi \) and \( t \in \mathbb{R} \)

Lax System: \( \partial_t \bar{u} + \partial_z f(u) = 0, \quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \)

For \( u = u + iv \) and \( \frac{1}{2}f(u) = a(u, v) + b(u, v)i \):

\[
\begin{align*}
\partial_t u + \partial_x a(u, v) + \partial_y b(u, v) &= 0, \\
\partial_t v - \partial_x b(u, v) + \partial_y a(u, v) &= 0.
\end{align*}
\]

When \( f(u) = u^2 = u^2 + v^2 + 2uv i \), the complex Burger equation:

\[
\begin{align*}
\partial_t u + \frac{1}{2}\partial_x (u^2 + v^2) + \partial_y (uv) &= 0, \\
\partial_t v - \partial_x (uv) + \frac{1}{2}\partial_y (u^2 + v^2) &= 0.
\end{align*}
\]

This is a symmetric hyperbolic system with an entropy \( \eta(u, v) = u^2 + v^2 \), so that local well-posedness of classical solutions can be inferred directly.

For the 1-D case, this is an archetype of hyperbolic conservation laws with umbilic degeneracy: Schaeffer-Shearer: 1976, Chen-Kan: 1995, 2001, ⋯
1 Important Multidimensional Models

2 Multidimensional Steady Problems

3 Multidimensional Self-Similar Problems
Two-D Steady Euler Equations (E-18): \((x, y) \in \mathbb{R}^2\)

\[
\begin{aligned}
\partial_x(\rho u) + \partial_y(\rho v) &= 0, \\
\partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= 0, \\
\partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= 0, \\
\partial_x(\rho u(E + \frac{p}{\rho})) + \partial_y(\rho v(E + \frac{p}{\rho})) &= 0,
\end{aligned}
\]

Constitutive Relations:

\[
(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S)), \quad E = e + \frac{1}{2}(u^2 + v^2)
\]

\(\rho\)—fluid density \(\quad (u, v)\)—fluid velocity \(\quad p\)—scalar pressure

\(S\)—entropy \(\quad e\)—internal energy \(\quad \theta\)—temperature

For a polytropic gas,

\[
p = R\rho\theta, \quad e = \frac{p}{(\gamma - 1)\rho}, \quad \gamma = 1 + \frac{R}{c_v}
\]

\[
p = p(\rho, S) = \kappa\rho^\gamma e^{S/c_v}, \quad e = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v}
\]

\(R > 0, c_v > 0, \kappa > 0\) are constants, \(\gamma > 1\) is the adiabatic exponent
Eigenvalues for the $1^{st}$ and $4^{th}$ families:

$$\lambda_j = \frac{uv + (-1)^j c \sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 4.$$ 

$\implies$

Supersonic when $u^2 + v^2 > c^2$: Shock waves, Rarefaction waves
Subsonic when $u^2 + v^2 < c^2$: Elliptic equations

Eigenvalues for the $2^{nd}$ and $3^{rd}$ families:

$$\lambda_i = \frac{v}{u}, \quad i = 2, 3$$ 

$\implies$ Two transport equations

The $2^{nd}$ families: Vortex sheets
The $3^{rd}$ families: Entropy waves

*New Phenomena:* Compressible vortex sheets
\[ \nabla \cdot (\rho (\nabla \Phi) \nabla \Phi) = 0 \]

For a \(\gamma\)-law gas, \(p = p(\rho) = \rho^\gamma/\gamma, \gamma > 1\), is the normalized pressure. Then the normalized Bernoulli’s law:

\[ q^2 - q_{cr}^2 \]
Steady Potential Flow Equation (E-5)

\[ \nabla \cdot (\rho (\nabla \Phi) \nabla \Phi) = 0 \]

For a \( \gamma \)-law gas, \( p = p(\rho) = \rho^\gamma / \gamma, \gamma > 1 \), is the normalized pressure. Then the normalized Bernoulli’s law:

\[ \rho = \hat{\rho}(q^2) := (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma - 1}}, \quad q = \sqrt{u^2 + v^2} \text{ – flow speed.} \]

Define \( c = \sqrt{1 - \frac{\gamma - 1}{2} q^2} \) (sonic speed), \( q_{cr} \equiv \sqrt{\frac{2}{\gamma + 1}} \) (critical speed). We rewrite Bernoulli’s law in the form

\[ q^2 - q_{cr}^2 = \frac{2}{\gamma + 1} (q^2 - c^2). \]
Steady Potential Flow Equation (E-5)

\[ \nabla \cdot (\rho (\nabla \Phi) \nabla \Phi) = 0 \]

For a \( \gamma \)-law gas, \( p = p(\rho) = \rho^\gamma / \gamma, \gamma > 1 \), is the normalized pressure. Then the normalized Bernoulli’s law:

\[ \rho = \hat{\rho}(q^2) := (1 - \frac{\gamma - 1}{2} q^2)^{\frac{1}{\gamma - 1}}, \quad q = \sqrt{u^2 + v^2} - \text{flow speed}. \]

Define \( c = \sqrt{1 - \frac{\gamma - 1}{2} q^2} \) (sonic speed), \( q_{cr} := \sqrt{\frac{2}{\gamma + 1}} \) (critical speed). We rewrite Bernoulli’s law in the form

\[ q^2 - q_{cr}^2 = \frac{2}{\gamma + 1} (q^2 - c^2). \]

Then the flow is subsonic (elliptic) when \( q < q_{cr} \),

sonic (degenerate state) when \( q = q_{cr} \),

supersonic (hyperbolic) when \( q > q_{cr} \).
Airfoil Problems I: \( \nabla \cdot (\rho(\nabla \Phi)\nabla \Phi) = 0, \quad x \in \mathbb{R}^2, \quad v_\infty = 0 \)

**Obstacle Boundary** \( \partial \Omega_1 \): Solid curve in (a); Solid closed curve in (b).

**Far-field Boundary** \( \partial \Omega_2 \): Dashed line segments in both (a) and (b).

**Domain** \( \Omega \): bounded by \( \partial \Omega_1 \) and \( \partial \Omega_2 \).

**Boundary conditions on the obstacle** \( \partial \Omega \):

\[
\begin{aligned}
\nabla \Phi \cdot n &= 0 \quad \text{on} \quad \partial \Omega_1, \\
\text{Consistent far-field boundary conditions on} \quad \partial \Omega_2,
\end{aligned}
\]

where \( n \) is the unit normal pointing into the flow region on \( \partial \Omega \).

In case (b), the circulation about the boundary \( \partial \Omega_2 \) is zero.
Problem 1: Existence of global entropy solutions

Yes: When the far-field velocity \((u_\infty, 0)\) with \(u_\infty \leq q^*\) for some \(q^* < q_{cr}\):

There exists a global subsonic (not necessarily strictly subsonic) flow

Bers-Shiffman 1958, · · ·, Chen-Dafermos-Slemrod-Wang 2007

Open: Existence of global transonic entropy solutions when \(u_\infty \in (q^*, q_{cr})\)?

References:


Chen-Slemrod-Wang: ARMA 189 (2008), 159–188

A vanishing viscosity method
Wedge Problems

**Supersonic-Supersonic:** Smooth supersonic: Gu, Schaeffer, Li, S. Chen
Discontinuous supersonic: Y. Zhang, Chen-Zhang-Zhu, Chen-Li,

**Supersonic-Subsonic:** Fang, Chen-Fang, Chen-J. Chen-Feldman...
**Wedge/Cone Problems II: Cone Problems**

**Supersonic-Supersonic:** Smooth supersonic: Chen-Xin-Yin, ⋯  
Discontinuous supersonic: Lien-T.-P. Liu, Chen-Zhang-Zhu

**Supersonic-Subsonic:** Chen-Fang, Yin et al, ⋯

**Require:** The incoming flow is sufficiently supersonic: In progress...

**Problem 2: Existence and stability of shock-fronts as long as the incoming flow is supersonic**

![Diagram of shock fronts and cones](image)
Transonic Nozzle Problems

Problem 3. Existence of subsonic-supersonic-subsonic solutions past a transonic nozzle

The problem involves two types of transonic flow with two free boundaries:

- Free boundary from the subsonic to supersonic flow through a continuous transition;
- Free boundary from the supersonic to subsonic flow through a transonic shock.

Existence and stability of supersonic-subsonic solutions:
Chen-Yuan; Xin-Yin, ....


Chen-Zhang-Zhu (SIAM 2007), Chen-Kukreja (2011): $|M^\pm| > 1$: $L^1$-Stability

M. Bae (Preprint 2011): $|M^\pm| < 1$: Structural Stability

Question: Stability of two or multiple steady compressible vortex sheets?

Backgrounds: Motion and structure of galactic jets in astrophysics....
Two-Dimensional Steady Euler Equations: \((x, y) \in \mathbb{R}^2\):

\[
\begin{align*}
\partial_x(\rho u) + \partial_y(\rho v) &= 0, \\
\partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= 0, \\
\partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= 0, \\
\partial_x(\rho u(E + \frac{p}{\rho})) + \partial_y(\rho v(E + \frac{p}{\rho})) &= 0,
\end{align*}
\]

Constitutive Relations:

\((e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S)), \quad E = e + \frac{1}{2}(u^2 + v^2)\)

- \(\rho\)—fluid density
- \((u, v)\)—fluid velocity
- \(p\)—scalar pressure
- \(S\)—entropy
- \(e\)—internal energy
- \(\theta\)—temperature

For a polytropic gas,

\[
p = R\rho\theta, \quad e = \frac{p}{(\gamma - 1)\rho}, \quad \gamma = 1 + \frac{R}{c_v}
\]

\[
p = p(\rho, S) = \kappa\rho^{\gamma}e^{S/c_v}, \quad e = \frac{\kappa}{(\gamma - 1)}\rho^{\gamma - 1}e^{S/c_v}
\]

\(R > 0, c_v > 0, \kappa > 0\) are constants, \(\gamma > 1\) is the adiabatic exponent.
The 2\textsuperscript{nd} vortex sheet and 3\textsuperscript{rd} entropy wave 
\( y = \chi_i(x) \):
\[
\frac{dy}{dx} = \lambda_i = \nu / u, \quad i = 2, 3
\]

The 2\textsuperscript{nd}-family vortex sheet curves in the phase space:

\[ C_2(U_0) : \quad U = (u_0 e^{\sigma_2}, v_0 e^{\sigma_2}, p_0, \rho_0)^\top, \quad S = S_0, \]

with strength \( \sigma_2 \) and slope \( \nu_0 / u_0 \), determined via the eigenvector \( r_2 \) by
\[
\frac{dU}{d\sigma_2} = r_2(U) = (u, \nu, 0, 0)^\top, \quad U|_{\sigma_2=0} = U_0.
\]

The 3\textsuperscript{rd}-family entropy wave curves in the phase space:

\[ C_3(U_0) : \quad U = (u_0, v_0, p_0, \rho_0 e^{\sigma_3})^\top, \quad S = S_0 - c_v \gamma \sigma_3, \]

with strength \( \sigma_3 \) and slope \( \nu_0 / u_0 \), determined via the eigenvector \( r_3 \) by
\[
\frac{dU}{d\sigma_3} = r_3(U) = (0, 0, 0, \rho)^\top, \quad U|_{\sigma_3=0} = U_0.
Shock Waves and Rarefaction Waves

**Eigenvalues for the 1\textsuperscript{st} and 4\textsuperscript{th} families:**

\[ \lambda_j = \frac{uv + (-1)^j c \sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 4. \]

**The \( j \)-\textsuperscript{th}-family rarefaction wave curves in the phase space:**

\[ R_j(U_0) : \quad dp = c^2 d\rho, \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v)dv = dp \quad \text{for } \rho < \rho_0, \ u > c. \]

**The \( j \)-\textsuperscript{th}-family shock wave curves in the phase space:**

\[ S_j(U_0) : \quad [p] = \frac{c_0^2}{b} [\rho], \ [u] = -s_j [v], \ \rho_0(s_j u_0 - v_0)[v] = [p] \]

\[ s_j = \frac{u_0 v_0 + (-1)^j \sqrt{u_0^2 + v_0^2 - \overline{c}^2}}{u_0^2 - \overline{c}^2} \quad \text{for } \rho > \rho_0, \ u > c \]

where \( \overline{c}^2 = \frac{c_0^2}{b} \frac{\rho}{\rho_0} \), \( b = \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} \frac{\rho}{\rho_0} \), and \( \rho_0 < \rho \) is equivalent to the entropy condition on the shock wave. \( S_j(U_0) \) contacts with \( R_j(U_0) \) at \( U_0 \) up to second-order.
Two-Phase Free Boundary Problems

Free Boundary: \[ \Gamma := \{ y = \chi(x) : x > 0 \}, \quad \chi'(x) = \frac{v(x,\chi(x))}{u(x,\chi(x))} \]

Free Bdry Conditions: \[ [\rho] = 0, [\frac{\nu}{u}] = 0; [\rho] = 0 \text{ or } [u] = [v] = 0 \]

*Unlike the shock case, the free boundary \( \Gamma \) is now a characteristic surface with the characteristic boundary conditions.
The vortex sheet/entropy wave problem can be formulated into the following IBVP:

**Cauchy Condition:**

\[ U|_{x=0} = \begin{cases} U_1, & 0 < y < y_0^*, \\ U_2, & y > y_0^* \end{cases} \]  

(1)

**Boundary Condition:**

\[(u, v) \cdot n = 0 \quad \text{on the wedge boundary } \{y = g(x)\}. \]  

(2)
Lateral Riemann Solutions

Figure: Wave curves in the $(u, v)$-plane for the lateral Riemann problem
There exist $\epsilon_0 > 0$ and $C > 0$ such that, if $TV(g'(\cdot)) < \epsilon$ for $\epsilon \leq \epsilon_0$, then there exists a pair of functions

$$U \in BV_{loc}(\Omega) \cap L^\infty(\Omega), \quad \chi \in Lip(\mathbb{R}_+; \mathbb{R}_+)$$

with $\chi(0) = y_0^*$ such that

- $U$ is a global entropy solution of the IBVP in $\Omega$ satisfying

  $$TV\{U(x, \cdot) : [g(x), \infty)\} \leq C \ TV(g'(\cdot)) \quad \text{for every} \ x \in [0, \infty);$$

- The curve $\{y = \chi(x)\}$ is a strong vortex sheet/entropy wave with $\chi(x) > g(x)$ for any $x > 0$,

  $$\sup_{g(x) < y < \chi(x)} |U(x, y) - U_1| \leq C\epsilon, \quad \sup_{y > \chi(x)} |U(x, y) - U_2| \leq C\epsilon,$$

  $$\lim_{x \to \infty} \sup \{ |\frac{v(x, y)}{u(x, y)} - g'_{\infty}| : y > g(x)\} + \lim_{x \to \infty} |\chi'(x) - g'_{\infty}| = 0;$$

- There exists a constant $p_{\infty} > 0$ such that

  $$\lim_{x \to \infty} \sup \{ |p(x, y) - p_{\infty}| : y > g(x)\} = 0.$$
Estimates on Weak Wave Interactions: Classical

\[
\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, \beta_4),
\]
\[
\{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)
\]

Then

\[
\gamma_i = \alpha_i + \beta_i + O(1) \Delta(\alpha, \beta),
\]

\[
\Delta(\alpha, \beta) = (|\alpha_4| + |\alpha_3| + |\alpha_2|)|\bar{\theta}_1| + |\alpha_4|(|\bar{\theta}_2| + |\bar{\theta}_3|) + \sum_{j=1,4} \Delta_j(\alpha, \bar{\theta})
\]

with

\[
\Delta_j(\alpha, \bar{\theta}) = \begin{cases} 
0, & \alpha_j \geq 0, \bar{\theta}_j \geq 0; \\
|\alpha_j||\bar{\theta}_j|, & \text{otherwise.}
\end{cases}
\]
Estimates on the Weak Wave Reflections on the Boundary

\[
\{U_m, U_a\} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, 0), \; \{U_k, U_m\} = (0, 0, 0, \alpha_4) \text{ with } (u_k, v_k) \cdot n_k = 0.
\]

Then \( \exists 1 \; U_{k+1} \) s.t.
\[
\{U_{k+1}, U_a\} = (0, 0, 0, \delta_4), \; U_{k+1} \cdot (n_{k+1}, 0, 0) = 0, \; \text{and}
\]
\[
\delta_4 = \alpha_4 + K_{b1} \bar{\theta}_1 + K_{b2} \bar{\theta}_2 + K_{b3} \bar{\theta}_3 + K_{b0} \omega_k,
\]

where \( K_{bj}, j = 0, 1, 2, 3, \) are \( C^2 \)-functions of \( \bar{\theta}_3, \bar{\theta}_2, \bar{\theta}_1, \alpha_4, \omega_{k+1}, U_a \) with
\[
K_{b1}\big|_{\{\omega_k=\alpha_4=\bar{\theta}_1=\bar{\theta}_2=\bar{\theta}_3=0, U_a=U_1\}} = 1,
\]
\[
K_{bi}\big|_{\{\omega_k=\alpha_4=\bar{\theta}_1=\alpha_2=\bar{\theta}_3=0, U_a=U_1\}} = 0, \; i = 2, 3,
\]

and \( K_{b0} \) is bounded. In particular, \( K_{b0} < 0 \) at the origin.
Weak Waves Approach the Strong Vortex Sheet/Entropy Wave from Below: Essential Feature

\[
\{U_b, U_m\} = (0, \alpha_2, \alpha_3, \alpha_4), \quad \{U_m, U_a\} = (\beta_1, \sigma_2, \sigma_3, 0), \\
\{U_b, U_a\} = (\delta_1, \sigma'_2, \sigma'_3, \delta_4)
\]

Then

\[
\begin{align*}
\delta_1 &= \beta_1 + K_{11} \alpha_4 + O(1) \Delta', \\
\delta_4 &= K_{14} \alpha_4 + O(1) \Delta', \\
\sigma'_2 &= \sigma_2 + \alpha_2 + K_{12} \alpha_4 + O(1) \Delta', \\
\sigma'_3 &= \sigma_3 + \alpha_3 + K_{13} \alpha_4 + O(1) \Delta',
\end{align*}
\]

with

\[
|K_{11}| \{\alpha_4 = \alpha_3 = \alpha_2 = 0, \sigma'_2 = \sigma_{20}, \sigma'_3 = \sigma_{30}\} < 1,
\]

and \(\sum_{j=2}^{4} |K_{1j}|\) is bounded, where \(\Delta' = |\beta_1| (|\alpha_2| + |\alpha_3|)\).
Weak Waves Approach the Strong Vortex Sheet/Entropy Wave from Above

\[
\begin{align*}
\{U_b, U_m\} &= (0, \sigma_2, \sigma_3, \alpha_4), & \{U_m, U_a\} &= (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, 0) \\
\{U_b, U_a\} &= (\delta_1, \sigma'_2, \sigma'_3, \delta_4).
\end{align*}
\]

Then

\[
\begin{align*}
\delta_1 &= K_{21} \bar{\theta}_1 + O(1) \Delta'' , & \sigma'_2 &= \sigma_2 + \bar{\theta}_2 + K_{22} \bar{\theta}_1 + O(1) \Delta'' , \\
\sigma'_3 &= \sigma_3 + \bar{\theta}_3 + K_{23} \bar{\theta}_1 + O(1) \Delta'' , & \delta_4 &= \alpha_4 + K_{24} \bar{\theta}_1 + O(1) \Delta'',
\end{align*}
\]

with \(\sum_{j=1}^{4} |K_{2j}|\) is bounded, where \(\Delta'' = |\alpha_4| (|\bar{\theta}_2| + |\bar{\theta}_3|)\).
Glimm-type Functional $F(J)$ on the Mesh Curves $J$

$$F(J) = C^*(|\sigma_2^J - \sigma_{20}| + |\sigma_3^J - \sigma_{30}|) + L^1(J) + L^2(J) + KQ(J),$$

with

$$L^1(J) = K_1^*L_0(J) + K_{11}^*L_1^1(J) + K_{12}^*L_2^1(J) + K_{13}^*L_3^1(J) + L_4^1(J),$$

$$L^2(J) = K_{21}^*L_1^2(J) + K_{22}^*L_2^2(J) + K_{23}^*L_3^2(J) + K_{24}^*L_4^2(J),$$

$$Q(J) = \sum \{||\alpha_i||||\beta_j|| : \text{both } \alpha_i \text{ and } \beta_j \text{ cross } J \text{ and approach}\},$$

$$L_0(J) = \sum \{|\omega(C_k)| : C_k \text{ the corner points in } J^+ \text{ and the boundary}\},$$

$$L_i^j(J) = \sum \{||\alpha_j|| : \alpha_j \text{ crosses } J \text{ in region } (i)\}, \quad i = 1, 2, j = 1, 2, 3, 4,$$

$(\sigma_2^J, \sigma_3^J)$—Strength of the strong vortex sheet/entropy wave crossing $J$,

where $K$ and $C^*$ will be chosen and

$$K_1^* > |K_{b0}|, \quad K_{1j}^* > |K_{bj}|, \quad j = 1, 2, 3,$$

$$K_{24}^* < \frac{1 - |K_{11}|K_{11}^*}{|K_{14}|}, \quad K_{21}^* > |K_{21}|K_{11}^* + |K_{24}|K_{24}^*,$$

while $K_{12}^*, K_{13}^*, K_{22}^*$, and $K_{23}^*$ are arbitrarily large positive constants. These conditions can be achieved from our discussions of the properties of $K_{bj}$,
Near the Strong Vortex Sheet/Entropy Waves (I)

\[(L^1 + L^2)(J) - (L^1 + L^2)L(I)\]
\[\leq (|K_{14}|K_{14}^* + |K_{11}|K_{11}^* - 1)|\alpha_4| - (K_{12}^* + O(1)|\tilde{\theta}_1|)|\alpha_2| - (K_{13}^* + O(1)|\tilde{\theta}_1|)|\alpha_3|\]

with \(|K_{14}|K_{14}^* + |K_{11}|K_{11}^* - 1 < 0\) by our choice of the constants.

Furthermore, since \(Q(I)\) can always be bounded by \(L(I)\) and

\[|\sigma_2^J - \sigma_2^I| + |\sigma_3^J - \sigma_3^I|\]
\[\leq (|K_{12}| + |K_{13}|)|\alpha_4| + (1 + O(1)|\tilde{\theta}_1|)|\alpha_2| + (1 + O(1)|\tilde{\theta}_1|)|\alpha_3|\]

with \(|K_{12}|\) and \(|K_{13}|\) bounded, we can choose \(C^*\) suitably small such that

\[F(J) - F(I) \leq C^*(|\sigma_2^J - \sigma_2^I| + |\sigma_3^J - \sigma_3^I|)\]
\[+ (L^1(J) + L^2(J) + KQ(J)) - (L^1(I) + L^2(I) + KQ(I)) \leq 0.\]
Near the Strong Vortex Sheet/Entropy Waves (II)

\[(L^1 + L^2)(J) - (L^1 + L^2)(I) \leq (|K_{21}|K_{11}^* + |K_{24}|K_{24}^* - K_{21}^*)|\bar{\theta}_1| - (K_{22}^* + O(1)|\alpha_4|)|\bar{\theta}_2| - (K_{23}^* + O(1)|\alpha_4|)|\bar{\theta}_3|\]

with 
\[|K_{21}|K_{11}^* + |K_{24}|K_{24}^* - K_{21}^* < 0\]

by our choice of the constants.

Similar to the analysis for Case (I), we again have \(F(J) \leq F(I)\).
Remarks and Connections

- Glimm schemes, wave-front tracking schemes...
- Contact discontinuities for 1-D strictly hyperbolic systems:
  Corli and Sablé-Tougeron (1997), Sablé-Tougeron (1993), .......
- Connections between the stability of steady compressible vortex sheets/entropy waves and long-time asymptotic stability of unsteady compressible vortex sheets/entropy waves in supersonic flow
- Three-dimensional, steady vortex sheets/entropy waves??
1. Important Multidimensional Models

2. Multidimensional Steady Problems

3. Multidimensional Self-Similar Problems
Shock Wave Patterns around a Wedge (airfoils, inclined ramps, ...) Complexity of Reflection-Diffraction Configurations Was First Identified and Reported by Ernst Mach 1879

Experimental Analysis: 1940s $\rightarrow$ von Neumann, Bleakney, Bazhenova Glass, Takyama, Henderson, ...
A New Mach Reflection-Diffraction Pattern:
A. M. Tesdall and J. K. Hunter: TSD, 2002
A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008
B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114
Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor**  
  *Shock Wave Reflection Phenomena*  
  
  **Experimental results before 1991**  
  **Various proposals for transition criteria**

- **Milton Van Dyke**  
  *An Album of Fluid Motion*  
  
  **Various photographs of shock wave reflection phenomena**

- **Journals by Springer:**
  
  - *Shock Waves*
  - *Combustion, Explosion, and Shock Waves*

- **Richard Courant & Kurt Otto Friedrichs**  
  *Supersonic Flow and Shock Waves*  
Scientific Issues

- Structure of the Shock Reflection-Diffraction Patterns
- Transition Criteria among the Patterns
- Dependence of the Patterns on the Parameters
  - wedge angle $\theta_w$, adiabatic exponent $\gamma \geq 1$
  - incident-shock-wave Mach number $M_s$

... ...

Interdisciplinary Approaches:

- Experimental Data and Photographs
- Large or Small Scale Computing
  Colella, Berger, Deschambault, Glass, Glaz, Woodward,....
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  Yu. Dem’yanov, Panasenko, ....
- Asymptotic Analysis
  Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian...
  Morawetz: CPAM 1994
- Rigorous Mathematical Analysis?? (Global Solutions)
  Existence, Stability, Regularity, Bifurcation, .......
2-D Riemann Problem for Hyperbolic Conservation Laws

\[ \partial_t U + \nabla_x \cdot F(U) = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \]

Books and Survey Articles:

Kurganov-Tadmor 2002, · · ·

Theoretical Roles:  Asymptotic States and Attractors
Local Structure and Building Blocks…
TUNG CHANG, GUI-QIANG CHEN AND SHULI YANG

**Fig. 5.5a**
Density contour curves

**Fig. 5.5b**
Self-Mach number contour curves

**Fig. 5.5c**
Pressure contour curves
Fig. 5.6a
Density contour curves

Fig. 5.6b
Self-Mach number contour curves

Fig. 5.6c. Pressure contour curves
Riemann Solutions vs General Entropy Solutions

Asymptotic States and Attractors

Observation (C–Frid 1998):

- Let $R(\frac{\mathbf{x}}{t})$ be the unique piecewise Lipschitz continuous Riemann solution with Riemann data: $R|_{t=0} = R_0(\frac{\mathbf{x}}{|\mathbf{x}|})$
- Let $U(t, \mathbf{x})$ be a bounded entropy solution with initial data:
  $$U|_{t=0} = R_0(\frac{\mathbf{x}}{|\mathbf{x}|}) + P_0(\mathbf{x}), \quad R_0 \in L^\infty(S^{d-1}), \quad P_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$$
- The corresponding self-similar sequence $U^T(t, \mathbf{x}) := U(Tt, T\mathbf{x})$ is compact in $L^1_{loc}(\mathbb{R}^{d+1})$

$$\Rightarrow \quad \text{ess lim}_{t \to \infty} \int_\Omega |U(t, t\xi) - R(\xi)| \, d\xi = 0 \quad \text{for any } \Omega \subset \mathbb{R}^d$$

Building Blocks and Local Structure

Local structure of entropy solutions

Building blocks for numerical methods
Full Euler Equations (E-1): \((t, x) \in \mathbb{R}^3_+ := (0, \infty) \times \mathbb{R}^2\)

\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= 0 \\
\partial_t (\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e) + \nabla \cdot (\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e + p) \mathbf{v} &= 0
\end{aligned}
\]

Constitutive Relations: \(p = p(\rho, e)\)

- \(\rho\)–density, \(\mathbf{v} = (v_1, v_2)\)–fluid velocity, \(p\)–pressure
- \(e\)–internal energy, \(\theta\)–temperature, \(S\)–entropy

For a polytropic gas: \(p = (\gamma - 1) \rho e, \quad e = c_v \theta, \quad \gamma = 1 + \frac{R}{c_v}\)

\[
p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \quad e = e(\rho, S) \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v},
\]

- \(R > 0\) may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- \(c_v > 0\) is the specific heat at constant volume
- \(\gamma > 1\) is the adiabatic exponent, \(\kappa > 0\) is any constant under scaling
Euler Equations for Potential Flow: (E-3): $v = \nabla \Phi$

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \nabla \Phi) &= 0, \quad \text{(conservation of mass)} \\
\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + \frac{\rho^\gamma - 1}{\gamma - 1} &= \frac{\rho_0^\gamma - 1}{\gamma - 1}, \quad \text{(Bernoulli's law)};
\end{align*}
\]

or, equivalently,

\[
\partial_t \rho (\partial_t \Phi, \nabla \Phi, \rho_0) + \nabla \cdot (\rho (\partial_t \Phi, \nabla \Phi, \rho_0) \nabla \Phi) = 0,
\]

with

\[
\rho(\partial_t \Phi, \nabla \Phi, \rho_0) = \left( \rho_0^\gamma - 1 - (\gamma - 1)(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2) \right)^\frac{1}{\gamma - 1}.
\]

We will see the Euler equations for potential flow is actually EXACT in an important region of the solution to the shock reflection problem.
Discontinuities of Solutions

\[ \partial_t u + \nabla \cdot f(u) = 0, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d \]

An oriented surface \( \Gamma(t) \) with unit normal \( n = (n_t, \cdots, n_d) \in \mathbb{R}^d \) in the \((t, x)\)-space is a discontinuity of a piecewise smooth entropy solution \( U \) with

\[ u(t, x) = \begin{cases} 
  u^+(t, x), & (t, x) \cdot n > 0, \\
  u^-(t, x), & (t, x) \cdot n < 0,
\end{cases} \]

if the Rankine-Hugoniot Condition is satisfied

\[ (u^+ - u^-, f(u^+) - f(u^-)) \cdot n = 0 \quad \text{along } \Gamma(t). \]

The surface \((\Gamma(t), u)\) is called a Shock Wave if the Entropy Condition (i.e., the Second Law of Thermodynamics) is satisfied:

\[ (\eta(u^+) - \eta(u^-), q(u^+) - q(u^-)) \cdot n > 0 \quad \text{along } \Gamma(t), \]

for some \((\eta(u), q(u))\): \( \nabla^2 \eta(u) \geq 0, q_j(u) = \eta(u)f_j(u), \ j = 1, \cdots, d \)

Example: For the full Euler equations: \( (\eta(u), q(u)) = (-\rho S, -\rho v S) \).
Two Types of Discontinuities

Noncharacteristic Discontinuities: Shock Waves:

Characteristic Discontinuities: Vortex Sheets/Entropy Waves

(i) \( (p_+, \rho_+) = (p_-, \rho_-), \mathbf{v}_+ \neq \mathbf{v}_- \)

(ii) \( (p_+, \mathbf{v}_+) = (p_-, \mathbf{v}_-), \rho_+ \neq \rho_- \)
Initial-Boundary Value Problem: $0 < \rho_0 < \rho_1$, $v_1 > 0$

Initial condition at $t = 0$:

$$(v, p, \rho) = \begin{cases} 
(0, 0, \rho_0, \rho_0), & |x_2| > x_1 \tan \theta_w, x_1 > 0, \\
(v_1, 0, p_1, \rho_1), & x_1 < 0;
\end{cases}$$

Slip boundary condition on the wedge bdry: $v \cdot \nu = 0$.

Invariant under the Self-Similar Scaling: $(t, x) \longrightarrow (\alpha t, \alpha x)$, $\alpha \neq 0$
Seek Self-Similar Solutions

\[(v, p, \rho)(t, x) = (v, p, \rho)(\xi, \eta), \quad (\xi, \eta) = \left(\frac{x_1}{t}, \frac{x_2}{t}\right)\]

\[
\begin{aligned}
(\rho U)\xi + (\rho V)\eta + 2\rho &= 0, \\
(\rho U^2 + p)\xi + (\rho UV)\eta + 3\rho U &= 0, \\
(\rho UV)\xi + (\rho V^2 + p)\eta + 3\rho V &= 0, \\
(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))(\xi) + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))(\eta) + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) &= 0,
\end{aligned}
\]

where \(q = \sqrt{U^2 + V^2}\) and \((U, V) = (v_1 - \xi, v_2 - \eta)\) is the pseudo-velocity.

**Eigenvalues:** \(\lambda_0 = \frac{V}{U}\) (repeated), \(\lambda_{\pm} = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}\),

where \(c = \sqrt{\gamma p/\rho}\) is the **sonic speed**

**When the flow is pseudo-subsonic:** \(q < c\), the system is **hyperbolic-elliptic composite-mixed**
Boundary Value Problem in the Unbounded Domain

Slip boundary condition on the wedge boundary:

\[(U, V) \cdot \nu = 0 \quad \text{on} \quad \partial D\]

Asymptotic boundary condition as \( \xi^2 + \eta^2 \to \infty \):

\[(U + \xi, V + \eta, p, \rho) \to \begin{cases} (0, 0, p_0, \rho_0), & \xi > \xi_0, \eta > \xi \tan \theta_w, \\ (v_1, 0, p_1, \rho_1), & \xi < \xi_0, \eta > 0. \end{cases}\]
When $\theta_w = \pi/2$, the reflection becomes the normal reflection, for which the incident shock normally reflects and the reflected shock is also a plane.
Local Theory for Regular Reflection (cf. Chang-C. 1986)

\[ \exists \theta_d = \theta_d(M, \gamma) \in (0, \frac{\pi}{2}) \text{ such that, when } \theta_W \in (\theta_d, \frac{\pi}{2}), \text{ there exist two states } (2) = (U^a, V^a, p^a, \rho^a) \text{ and } (U^b, V^b, p^b, \rho^b) \text{ such that} \]

\[ |(U^a, V^a)| > |(U^b, V^b)| \text{ and } |(U^b, V^b)| < c^b. \]

Stability Criterion (C-Feldman 2005) as \( \theta_W \to \frac{\pi}{2} \):

Choose \( (2) = (U^a_2, V^a_2, p^a_2, \rho^a_2) \).
von Neumann Criteria & Conjectures (1943)

Local Theory for Regular Reflection (cf. Chang-C. 1986)

\[ \exists \theta_d = \theta_d(M_s, \gamma) \in (0, \frac{\pi}{2}) \text{ such that, when } \theta_W \in (\theta_d, \frac{\pi}{2}), \text{ there exist} \]

two states \((2) = (U^a_2, V^a_2, p^a_2, \rho^a_2)\) and \((U^b_2, V^b_2, p^b_2, \rho^b_2)\) such that

\[ |(U^a_2, V^a_2)| > |(U^b_2, V^b_2)| \text{ and } |(U^b_2, V^b_2)| < c^b_2. \]

Stability Criterion (C-Feldman 2005) as \(\theta_W \to \frac{\pi}{2}\):

Choose \((2) = (U^a_2, V^a_2, p^a_2, \rho^a_2).\)

Detachment Criterion: There is no Regular Reflection Configuration when the wedge angle \(\theta_W \in (0, \theta_d).\)

Sonic Conjecture: There exists a Regular Reflection Configuration when \(\theta_W \in (\theta_s, \frac{\pi}{2}), \) for \(\theta_s > \theta_d\) such that \(|(U^a_2, V^a_2)| > c^a_2\) at \(A.\)
Detachment Criterion vs Sonic Criterion $\theta_c > \theta_s$: $\gamma = 1.4$

Global Theory?

\[(U, V) \cdot \nu = 0\]

subsonic?
Euler Eqs. under Decomposition: \((U, V) = \nabla \varphi + W, \ \nabla \cdot W = 0\)

\[
\begin{aligned}
\nabla \cdot (\rho \nabla \varphi) + 2\rho + \nabla \cdot (\rho \nabla W) &= 0, \\
\nabla \left( \frac{1}{2} |\nabla \varphi|^2 + \varphi \right) + \frac{1}{\rho} \nabla \rho &= (\nabla \varphi + W) \cdot \nabla W + (\nabla^2 \varphi + I)W, \\
(\nabla \varphi + W) \cdot \nabla \omega + (1 + \Delta \varphi) \omega &= 0 \iff \nabla \cdot ((\nabla \varphi + W) \omega) + \omega = 0, \\
(\nabla \varphi + W) \cdot \nabla S &= 0.
\end{aligned}
\]

where \(S = c_v \ln(p \rho^{-\gamma})\) - Entropy; \(\omega = \text{curl} \ W = \text{curl}(U, V)\) - Vorticity

When \(\omega = 0\), \(S = \text{const}.\), and \(W = 0\) on a curve transverse to the fluid direction, then, in the region of the fluid trajectories past the curve,

\(W = 0, \ S = \text{const.} \Rightarrow W = 0, \ p = \text{const.} \rho^\gamma\)

Then we obtain the Potential Flow Equation (by scaling):

\[
\begin{aligned}
\nabla \cdot (\rho \nabla \varphi) + 2\rho &= 0, \\
\frac{1}{2}(|\nabla \varphi|^2 + \varphi) + \frac{\rho^{\gamma-1}}{\gamma - 1} &= \text{const.} > 0.
\end{aligned}
\]

J. Hadamard: Lecons sur la Propagation des Ondes, Hermann: Paris 1903
Potential Flow Dominates the Regular Reflection, provided that $\varphi \in C^{1,1}$ across the Sonic Circle

Potential Flow Equation

\[
\begin{align*}
\nabla \cdot (\rho \nabla \varphi) + 2\rho &= 0, \\
\frac{1}{2}|\nabla \varphi|^2 + \varphi + \frac{\rho^{\gamma-1}}{\gamma-1} &= \frac{\rho^*_0}{\gamma-1}
\end{align*}
\]
Potential Flow Equation

$$\nabla \cdot (\rho (\nabla \varphi, \varphi, \rho_0) \nabla \varphi) + 2 \rho (\nabla \varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = \text{const.} \implies \Delta \varphi + 2 = 0$

- Subsonic (Elliptic):

$$|\nabla \varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma + 1} (\rho_0^{\gamma - 1} - (\gamma - 1) \varphi)}$$

- Supersonic (Hyperbolic):

$$|\nabla \varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma + 1} (\rho_0^{\gamma - 1} - (\gamma - 1) \varphi)}$$
Global Theory?

\[(U, V) \cdot \nu = 0\]

\(\Gamma_{\text{sonic}}\)

subsonic?
Setup of the Problem for $\psi := \varphi - \varphi_2$ in $\Omega$

- $\text{div} \left( \rho(\nabla \psi, \psi, \xi, \eta, \rho_0)(\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) \right) + l.o.t. = 0 \quad (*)$
- $\nabla \psi \cdot \nu \big|_{\text{wedge}} = 0$
- $\psi \big|_{\Gamma_{\text{sonic}}} = 0 \implies \psi \nu \big|_{\Gamma_{\text{sonic}}} = 0$
- Rankine-Hugoniot Conditions on Shock $S$:
  
  \[ \begin{align*}
  [\psi]_S &= 0 \\
  [\rho(\nabla \psi, \psi, \xi, \eta, \rho_0)(\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) \cdot \nu]_S &= 0 \quad (**)
  \end{align*} \]
Setup of the Problem for $\psi := \varphi - \varphi_2$ in $\Omega$

- $\text{div} \left( \rho (\nabla \psi, \psi, \xi, \eta, \rho_0)(\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) + l.o.t. \right) = 0 \quad (*)$
- $\nabla \psi \cdot \nu |_{\text{wedge}} = 0$
- $\psi |_{\Gamma_{\text{sonic}}} = 0 \implies \psi \nu |_{\Gamma_{\text{sonic}}} = 0$
- Rankine-Hugoniot Conditions on Shock $S$:
  \[
  \begin{align*}
  [\psi]_S &= 0 \\
  [\rho (\nabla \psi, \psi, \xi, \eta, \rho_0)(\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) \cdot \nu]_S &= 0 \quad (**) 
  \end{align*}
  \]

Free Boundary Problem

- $\exists S = \{ \xi = f(\eta) \}$ such that $f \in C^{1,\alpha}$, $f'(0) = 0$ and
  \[
  \Omega_+ = \{ \xi > f(\eta) \} \cap D = \{ \psi < \varphi_1 - \varphi_2 \} \cap D 
  \]
- $\psi$ satisfy the free boundary condition (**) along $S$
Setup of the Problem for $\psi := \varphi - \varphi_2$ in $\Omega$

- $\text{div} \left( \rho (\nabla \psi, \psi, \xi, \eta, \rho_0) (\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) + \text{l.o.t.} \right) = 0 \quad (*)$
- $\nabla \psi \cdot \nu|_{\text{wedge}} = 0$
- $\psi|_{\Gamma_{\text{sonic}}} = 0 \implies \psi_\nu|_{\Gamma_{\text{sonic}}} = 0$
- Rankine-Hugoniot Conditions on Shock $S$:
  
  
  $[\psi]_S = 0$
  
  $[\rho (\nabla \psi, \psi, \xi, \eta, \rho_0) (\nabla \psi + \mathbf{v}_2 - (\xi, \eta)) \cdot \nu]_S = 0 \quad (**)$

Free Boundary Problem

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  $\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D$

- $\psi$ satisfy the free boundary condition (***) along $S$

- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+) \left\{ \begin{array}{l}
  \text{solves (*) in } \Omega_+, \\
  \text{is subsonic in } \Omega_+ \\
\end{array} \right.$

  with $(\psi, \psi_\nu)|_{\Gamma_{\text{sonic}}} = 0$, \quad $\nabla \psi \cdot \nu|_{\text{wedge}} = 0$
∃ \theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2}) such that, when \theta_W \in (\theta_c, \frac{\pi}{2}), there exist \( (\varphi, f) \) satisfying

- \( \varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega}) \) and \( f \in C^\infty(P_1P_2) \cap C^2(\{P_1\}) \);
- \( \varphi \in C^{1,1} \) across the sonic circle \( P_1P_4 \);
- \( \varphi \to \varphi_{NR} \) in \( W^{1,1}_{loc} \) as \( \theta_W \to \frac{\pi}{2} \).

⇒ \( \Phi(t, x) = t\varphi(\frac{x}{t}) + \frac{|x|^2}{2t} \), \( \rho(t, x) = (\rho_0^{\gamma-1} - (\gamma - 1)(\Phi_t + \frac{1}{2}|
abla\Phi|^2))^\frac{1}{\gamma-1} \) form a solution of the IBVP.
Approach for the Large-Angle Case

- **Cutoff Techniques by Shiffmanization**
  \[ \Rightarrow \text{Elliptic Free-Boundary Problem with Elliptic Degeneracy on } \Gamma_{\text{sonic}} \]

- **Domain Decomposition**
  - Near \( \Gamma_{\text{sonic}} \)
  - Away from \( \Gamma_{\text{sonic}} \)

- **Iteration Scheme**
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- **Corner Singularity Estimates**
  
  In particular, when the Elliptic Degenerate Curve \( \Gamma_{sonic} \) Meets the Free Boundary, i.e., the Transonic Shock
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- **Corner Singularity Estimates**
  - In particular, when the Elliptic Degenerate Curve \( \Gamma_{\text{sonic}} \) Meets the Free Boundary, i.e., the Transonic Shock

- **Removal of the Cutoff**
  - Require the Elliptic-Parabolic Estimates

?? Extend the Large-Angle to the Sonic-Angle \( \theta_s \)??
Elliptic Degeneracy

- **Linear**
  \[
  2x \psi_{xx} + \frac{1}{c^2} \psi_{yy} - \psi_x \sim 0
  \]
  \[
  \psi \sim Ax^{3/2} + h.o.t. \quad \text{when } x \sim 0
  \]

- **Nonlinear**
  \[
  \left\{
  \begin{aligned}
  (2x - (\gamma + 1)\psi_x) \psi_{xx} + \frac{1}{c^2} \psi_{yy} - \psi_x & \sim o(x^2) \\
  \psi|_{x=0} & = 0
  \end{aligned}
  \right.
  \]
  Ellipticity: \[
  \psi_x \leq \frac{2x}{\gamma+1}
  \]
  Apriori Estimate: \[
  |\psi_x| \leq \frac{4x}{3(\gamma+1)}
  \]
  \[
  \psi \sim \frac{x^2}{2(\gamma + 1)} + h.o.t. \quad \text{when } x \approx 0
  \]

\[ \varphi \in C^{1,1} \text{ but NOT in } C^2 \text{ across } P_1P_4; \]
\[ \varphi \in C^{1,1}(\{P_1\}) \cap C^{2,\alpha}(\bar{\Omega} \setminus (\{P_1\} \cup \{P_3\})) \cap C^{1,\alpha}(\{P_3\}) \cap C^\infty(\Omega); \]
\[ f \in C^2(\{P_1\}) \cap C^\infty(P_1P_2). \]

\[ \implies \text{ C-Feldman 2011: The global existence and the optimal regularity hold up to the sonic wedge-angle } \theta_s \text{ for any } \gamma \geq 1 \text{ for } u_1 < c_1; u_1 \geq c_1. \text{ (the von Neumann’s sonic conjecture)} \]
Existence for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$

**Issues:** As the wedge angle becomes smaller, prove the shock does not hit

(i) Wedge boundary,
(ii) Symmetry line $\Sigma$,
(iii) Sonic circle $\partial B_{c_1}(O_1)$ of state (1), where $O_1 = (u_1, 0)$,
(iv) Vertex point $P_3$. 

\[ c_1 > u_1 \]
Existence for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$

**Issues:** As the wedge angle becomes smaller, prove the shock does not hit

(i) Wedge boundary,
(ii) Symmetry line $\Sigma$,
(iii) Sonic circle $\partial B_{c_1}(O_1)$ of state (1), where $O_1 = (u_1, 0)$,
(iv) Vertex point $P_3$. This is unclear in the case $c_1 < u_1$. 
Existence for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$

Is attached case possible for regular reflection?

For irregular Mach reflection attached case appears to be possible, see Fig. 238 (page 144) of M. Van Dyke, *An Album of Fluid Motion*, The Parabolic Press: Stanford, 1982.
Theorem (C-Feldman). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 \leq c_1$, then a regular reflection solution $\varphi$ as our Theorem (2005) exists for all wedge angles $\theta_w \in (\theta_{sonic}, \pi/2)$.

The solution satisfies all properties stated in our Theorem (2005). In particular, $\varphi$ is $C^{1,1}$ near and across the sonic arc $P_1P_4$, and the shock is a $C^2$ curve, and $\varphi_2 \leq \varphi \leq \varphi_1$ in $\Omega$. 
Existence for $\theta_w \in (\theta_{\text{sonic}}, \frac{\pi}{2})$

Theorem (C-Feldman). If $\rho_1 > \rho_0 > 0$, $\gamma > 1$ satisfy $u_1 > c_1$, then a regular reflection solution $\varphi$ as in our Theorem (2005) exists for all wedge angles $\theta_w \in (\theta_c, \frac{\pi}{2})$, where

- either $\theta_c = \theta_{\text{sonic}},$

- or $\theta_c > \theta_{\text{sonic}}$ and for $\theta_w = \theta_c$ there exists an attached weak solution of regular reflection-diffraction problem.
The solution \( \varphi \) is called an **admissible solution** if

1. \( \varphi \in C^1(P_0P_1P_2P_3P_4) \), and \( P_0P_1P_2 \) is \( C^1 \) curve,
2. Equation is (strictly) elliptic in \( \Omega \setminus \overline{P_1P_4} \).
3. \( \varphi_2 \leq \varphi \leq \varphi_1 \) in \( \Omega \).
4. \( \varphi_1 - \varphi \) in \( \Omega \) monotonically non-increases in directions \( S_0 \) and \( S_1 \).
Large Angles $\Rightarrow$ Sonic Angle $\theta_{\text{sonic}}$

**Approach:** Apriori Estimates and Compactness

(a) Establish the strict inequalities in (iii) and the strict monotonicities in (iv) (thus $\varphi_1 - \varphi$ strictly decreases for a cone of directions, thus the shocks are Lipschitz graphs with uniform Lip estimates)

(b) Establish uniform bounds on $\text{diam}(\Omega), \|\varphi\|_{C^{0,1}(\Omega)}$, the monotonicities of $\varphi - \varphi_2$ near the sonic arc;

(c) Establish a uniform positive lower bound for the distance from the shock to the wedge, the sonic circle of state (1), and the uniform separation of the shock and the symmetry line;

(d) Make uniform regularity estimates for the solution and its shock in weighted/scaled Hölder norms (including near the sonic arc, which imply $C^1$ across the sonic arc);

(e) Prove that the uniform limit of admissible solutions is an admissible solution, and the uniform limit of the sequence of shocks is a shock.

**Continuity Method/Degree Theory $\Rightarrow$ Existence of Admissible Solutions for Large Wedge Angle:** $\Rightarrow$ von Neumann’s Sonic Conjecture
? Right space for vorticity $\omega$?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?
Further Developments on Shock Reflection-Diffraction

S.-X. Chen: Local Stability of Mach Configurations · · ·

D. Serre: Multi-D Shock Interaction for a Chaplygin Gas

S. Canic, B. Keyfitz, K. Jegdic, E. H. Kim:
Semi-Global Solutions for Shock Reflection Problems · · ·

V. Elling: Examples to the Sonic and Detachment Criteria

J. Glimm, X. Ji, J. Li, X. Li, P. Zhang, T. Zhang, and Y. Zheng:
Transonic Shock Formation in a Rarefaction Riemann Problem

Y. Zheng+al: Pressure-Gradient Equations, · · ·

?? Various Models for the Shock Reflection-Diffraction Problems??
Books and Survey Articles:

Kurganov-Tadmor 2002, · · ·

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks...
Multidimensional Problems vs New Mathematics

- Mixed and Composite Eqns. of Hyperbolic-Elliptic Type
  - Degenerate Elliptic Techniques
  - Degenerate Hyperbolic Techniques
  - Transport Equations with Rough Coefficients

  Naturally Arising in Many Fundamental Problems in Fluid Mechanics, Differential Geometry, Optimization, Elasticity, Relativity, ... 

- Free Boundary Techniques
- Regularity Estimates when a Free Boundary Meets a Degenerate Curve
- Boundary Harnack Inequalities
- Further Understanding of Compressible Vortex Sheets and Vorticity Waves
- Further Analysis of Divergence-Measure Vector Fields, ...
- New Measure-Theoretical Analysis, Geometric Measures, ...
- More Efficient Numerical Methods, ...
- .......

Gui-Qiang Chen (Oxford)