Lectures on semi-Markov processes

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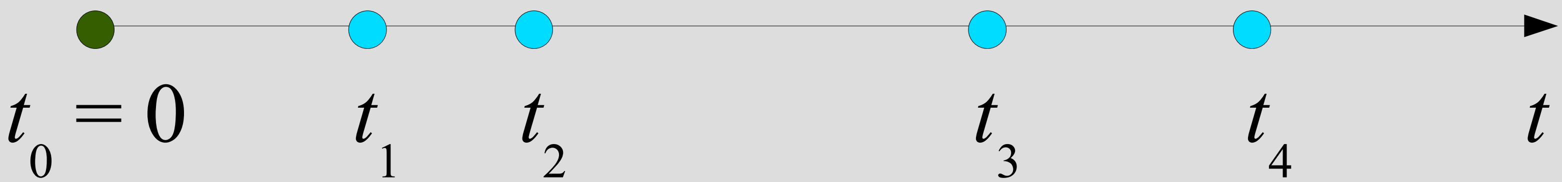
The Poisson process (PP) I

The Poisson process $N(t)$ is a continuous-time stochastic process with stationary and independent increments (an instance of *Lévy process*). The increments $N(t) - N(s)$, $(0 \leq s < t)$ have the following distribution:

$$P[N(t)-N(s)=k]=\exp(-\lambda(t-s))\frac{(\lambda(t-s))^k}{k!}$$

with $k = 0, 1, 2, ...$

The PP II: Meaning



$N(t)$ is the random number of events from 0 up to time t. Setting $N(0) = 0$, one has

$$P[N(t)=k]=\exp(-\lambda t)\frac{(\lambda t)^k}{k!}$$

$$H(t)=\mathrm{E}[N(t)]=\lambda t$$

$$\mathrm{Var}[N(t)]=\lambda t$$

The PP III: Activity or rate

$$H(t) = \mathrm{E}[N(t)] = \lambda t$$ Renewal function

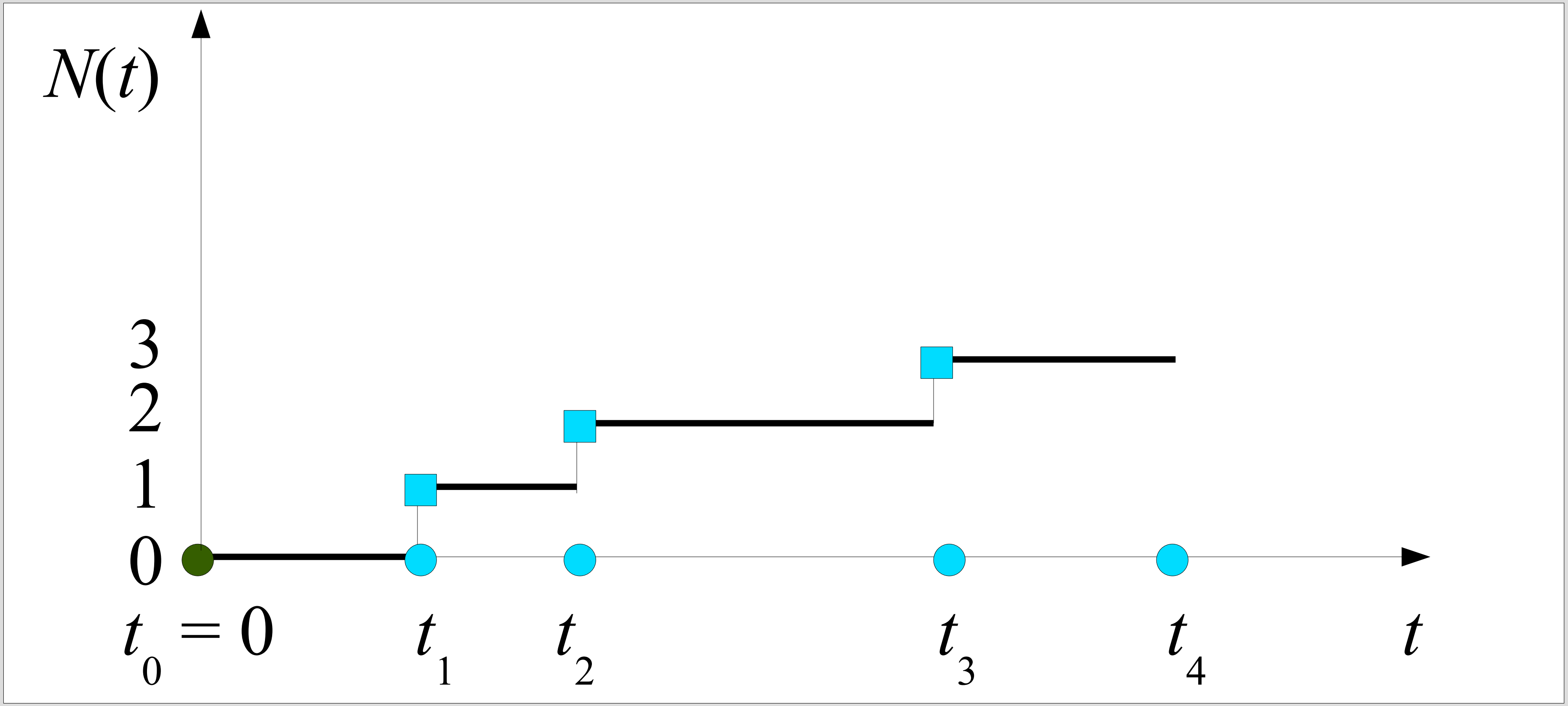
$$h(t) = \frac{d\,H(t)}{dt} = \lambda$$ Renewal density

The renewal density is the average number of events per unit time, it is also called *activity* or *rate.*

The activity of the Poisson process is constant!

The PP IV: Sample paths

The *realizations* (a.k.a. *sample paths*) of the Poisson process are step functions with all jumps equal to 1



Summary

- The Poisson process is a Lévy process: it has stationary and independent increments.
- The Poisson process is a time-homogeneous Markov process (not discussed, but good to know).
- The Poisson process counts random events arriving at a constant rate.
- The sample paths of the Poisson process are step functions with all jumps equal to 1.

Exercises on the PP

Exercise 1: The rate of a PP is $10\ s^{-1}$. Which is the expected number of events after 10 seconds?

Exercise 2: With the same rate as in Exercise 1, which is the probability of observing 1 event in 10 seconds?

Exercise 3: Study the Poisson distribution as a function of k. Hint: use $\lambda t = 100$