Lectures on semi-Markov processes

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Renewal processes

Given a sequence of positive independent and identically distributed random variables, with the usual meaning of *waiting times* (a.k.a. *sojourn times*, *residence times, ...*) $(\tau_i)_{i=1}^{\infty}$ define the *epochs*:

$$t_k = \sum_{i=1}^{k} \tau_i ;$$

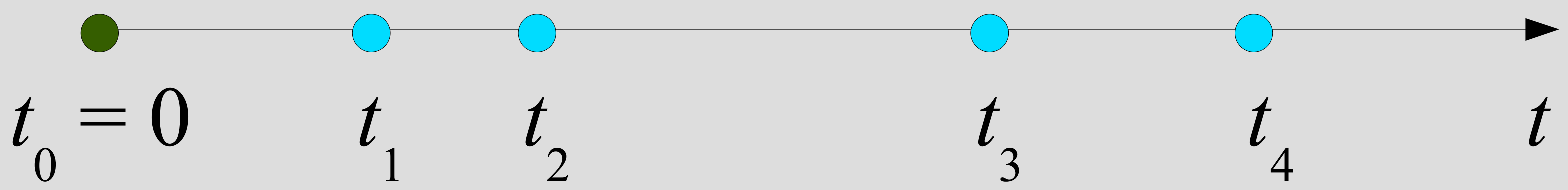
the sequence of the epochs is a *renewal process*! The process defined by

$$N(t) = \max(k : t_k \leq t)$$

is called *counting process*.

The exponential renewal process (ERP) I

Consider a Poisson process with rate λ:



Define $\tau_i = t_i - t_{i-1}$ with $\tau_1 = t_1$. One has

$$\Psi(t) = P(\tau_1 > t) = P(N(s) = 0, (0 \leq s < t))$$
$$P(N(t) = 0) = \exp(-\lambda t)$$

hence

$$\psi(t) = -\frac{d\Psi(t)}{dt} = \lambda \exp(-\lambda t)$$

The ERP II: ERP and PP

It is possible to prove that the $\tau_i = t_i - t_{i-1}$ are independent and identically distributed random variables. The proof is omitted. It is not easy!

Conversely, it is possible to prove that, given a sequence of positive independent and identically exponentially distributed random variables $(\tau_i)_{i=1}^{\infty}$ and defining $t_k = \sum_{i=1}^{k} \tau_i$, the corresponding counting process is Poisson!

The ERP III: Absence of memory

If τ is a random variable such that

$$P(\tau > t+s | \tau > t) = P(\tau > s)$$

then τ is an exponential random variable.
Indeed:

$$\Psi(t+s) = P(\tau > t+s) = P(\tau > t+s | \tau > t) P(\tau > t)$$

$$= P(\tau > s) P(\tau > t) = \Psi(t) \Psi(s)$$

meaning that $\Psi(t) = \exp(-\lambda t)$ for some λ.

The ERP IV: Simulation of PP

The previous results help in simulating the PP! Here is a possible pseudo-algorithm:

- repeat while $t_k = \sum_{i=1}^{k} \tau_i < t$;
- $i = i + 1$;
- set $i = 0$;
- generate $\tau_i \sim \exp(\lambda)$;
- end while;
- set $N(t) = \max(k : t_k \leq t)$

Summary

- The PP and the ERP are strictly related.
- The PP is the counting process of the ERP.
- The sample paths of the Poisson can be easily simulated thanks to this relationship.

Exercises on the ERP

Exercise 1: The rate of a PP is $10\ s^{-1}$. Which is the expected value of the corresponding exponential waiting times?

Exercise 2: And the variance?

Exercise 3: Show that the cumulative distribution function of an exponential random variable of parameter λ is $F(t)=1-\exp(-\lambda t)$.