Lectures on semi-Markov processes

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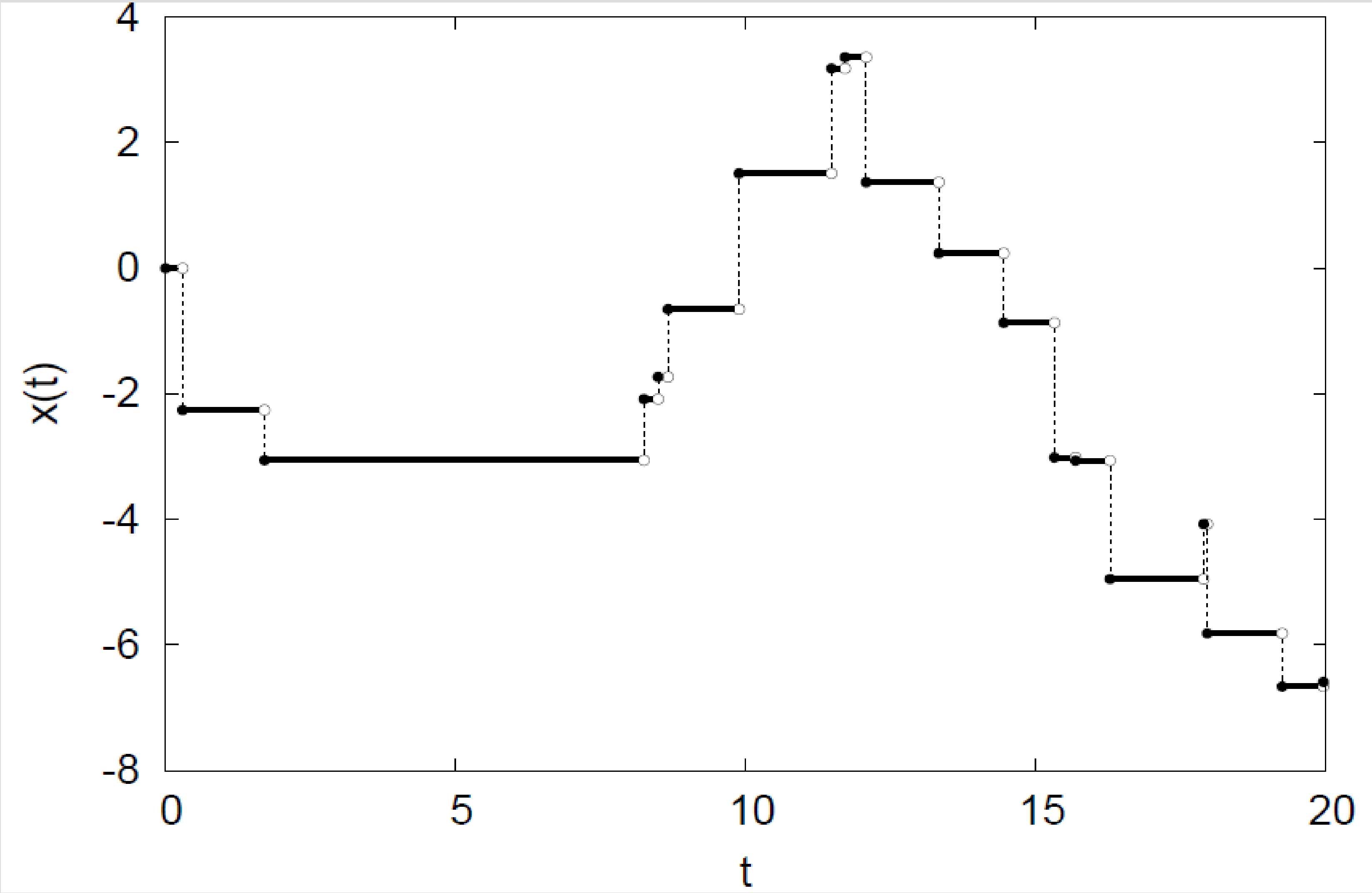
The normal compound Poisson process (NCPP) I



The NCPP II: Summary

$$X(t) = \sum_{i=1}^{N(t)} \xi_i$$

$$t_k = \sum_{i=1}^{k} \tau_i;$$

$$X(t) = X_{N(t)}$$

$N(t)$ is the Poisson *counting process.*

$N(t) = \max(k : t_k \leq t)$

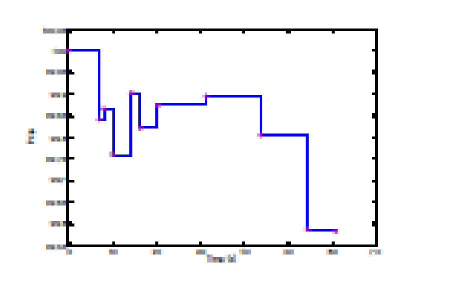
ξ_i are normally distributed i.i.d. r.v.;

τ_i are exponentially distributed i.i.d. r.v.;

ξ_i, τ_i are mutually independent.

The NCPP III: Simulation

Normal Compound Poisson Process (NCPP)



```
dt = exprnd(10,10,1); % generates 10 exponentially distributed deviates (durations)
time = cumsum(dt); % computes time instants of jumps
time = [0 time']; % time starts from 0
time = time’; % transposing from row to column vector
dx = normrnd(0,0.001,10,1); % generates 10 normally distributed deviates (log-returns)
x = cumsum(dx); % computes log-price
x = [0 x']; % initial log-price is zero (initial price acts as a numeraire)
x = x’; % transposing from row to column vector
price = 100*exp(x); % from log-price to price; initial price is 100
stairs(time,price) % stair plot
hold on % superposition of plots
plot(time,price,'o') % value of the price at instant of jumps
```

The NCPP IV: pdf I

We look for

$$p(x, t|0,0)$$

the probability density function (pdf) of being at position x at time t, given that the process started at 0 at time 0. What can we say?

The NCPP V: pdf II

At time t, $N(t)$ jumps have taken place. $N(t)$ can be $0, 1, 2, \ldots$, but it is finite! If $N(t) = 0$, then $X(t)$ must be 0, if $N(t) > 0$, $X(t)$ can be anywhere! Let say that $X(t) = x$, then, we can again use total probability to get:

$$p(x,t|0,0) = \delta(x)\Psi(t) + \sum_{i=1}^{\infty} P(N(t)=i)\, w^{(i)}(x)$$

What does this mean?

The NCPP VI: pdf III

$$p(x,t|0,0)=\delta(x)\Psi(t)+\sum_{i=1}^{\infty}P(N(t)=i)w^{(i)}(x)$$

Permanence in 0

Paths to x

$$\Psi(t)=P(N(t)=0)=\exp(-\lambda t)$$

$$P(N(t)=i)=\exp(-\lambda t)\frac{(\lambda t)^i}{i!}$$

$$w^{(i)}(x)=\frac{1}{\sqrt{2\pi i}\sigma}\exp\left(-\frac{(x-i\mu)^2}{2i\sigma^2}\right)$$

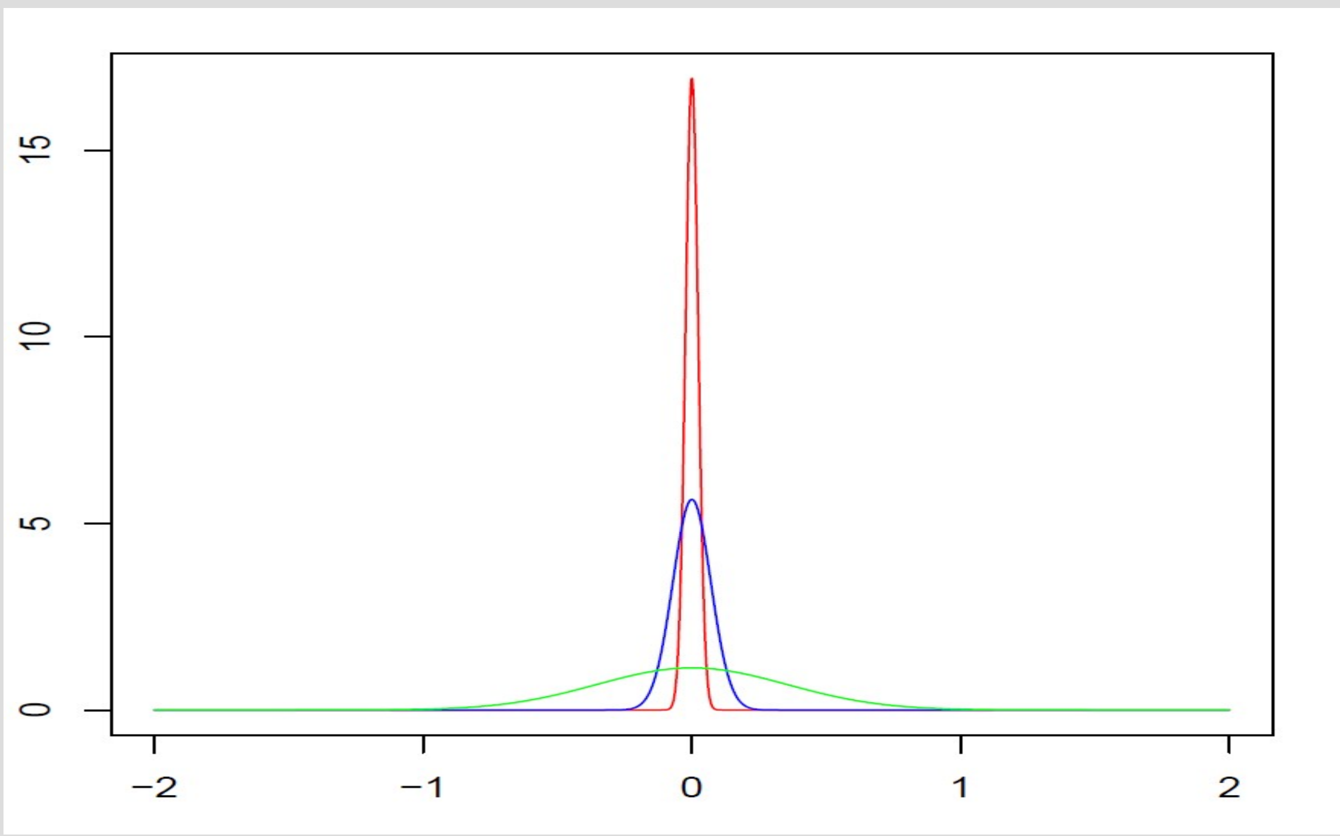
And $\delta(x)$?

The NCPP VII: Dirac's delta

$$\delta(x) = \lim_{a \to ٠} \delta_a(x)$$

$$\delta_a(x) = \frac{١}{a\sqrt{\pi}} \exp(-x^{٢}/a^{٢})$$

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = ١$$



Summary

- The introductory part of our lectures is over!
- The NCPP can be discussed with the same ideas used for the RW and the PP.
- It is a Lévy process (it has independent and stationary increments).
- Therefore, $p(x,t|\cdot,\cdot)$ fully characterizes the stochastic process.

- Why is this process a candidate for modeling high-frequency financial time series?