The inverse Calderón problem with Lipschitz conductivities

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Outline

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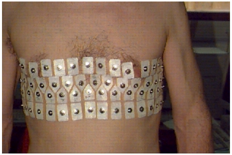
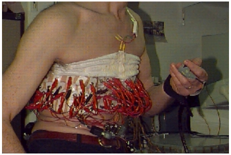
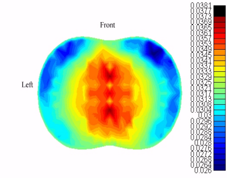
General goal of the Calderón problem

- The inverse Calderón problem consists of recovering the electric properties of a medium, namely the conductivity, by boundary measurements of many configurations of voltages and currents on its surface.
- The Calderón problem is the mathematical model for a medical imaging technique called electrical impedance tomography (EIT).
- EIT refers to a non-invasive medical imaging technique in which an image of the conductivity of part of the body is inferred from surface electrode measurements.





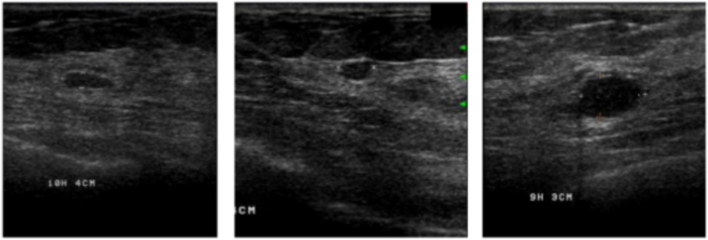


- EIT allows to monitor representative changes in the conductivities of tissues. It presents low resolution.

Some applications of EIT

- EIT is specially promising when monitoring lung functions since lung conductivity fluctuates intensely during the breath cycle.
- EIT has applications in breast cancer detection as a complementary technique to mammography and MRI since malignant breast tissues present higher conductivities (0.2 S) than healthy tissues (0.03 S).
- The success of mammography or MRI rests on their high resolution, however, they also present a low specificity, which is result of a relatively high rate of false positive.

benign | **Malign** | **benign**



Fibrotic Lesion | ***Carcinoma Grade II*** | ***Viscous cyst***

The mathematical model

- Let $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, be a bounded domain with boundary $\partial\Omega$. The case $n = 2$ is quite well understood (contributions due to Brown-Uhlmann, Nachmann, Astala-Päivärinta(-Lassas)).
- We suppose that the conductivity γ satisfies $c \leq \gamma \leq c^{-1}$.
- Given an electric potential on the boundary f, there is a unique solution u to the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \\ u\big|_{\partial\Omega} &= f. \end{aligned}$$

- u is the electric potential in the interior of Ω.
- Given that we can measure the induced current perpendicular to the boundary, we know the Dirichlet-to-Neumann map Λ_γ formally defined by

$$\Lambda_\gamma f = \gamma \nabla u \cdot n\big|_{\partial\Omega},$$

where n denotes the exterior unit normal to the boundary.

The Calderón problem

- The inverse Calderón problem consists of reconstructing γ from Λ_γ.
- Uniqueness: Does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply $\gamma_1 = \gamma_2$?
- Stability: Does there exist ω such that

$$\|\gamma_1 - \gamma_2\| \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)?$$

- Note that

$$\Lambda : \gamma \longmapsto \Lambda_\gamma$$

  is a non-linear problem map.

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Non-uniqueness for anisotropic conductivities

- Recall that uniqueness holds if

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \Rightarrow \quad \gamma_1 = \gamma_2.$$

The Calderón problem in anisotropic media has a simple obstruction to uniqueness (apparently due to Tartar):

- Given any anisotropic conductivity γ in Ω and any diffeomorphism $F : \overline{\Omega} \longrightarrow \overline{\Omega}$ satisfying $F|_{\partial\Omega} = \mathrm{Id}$, one has

$$\Lambda_\gamma = \Lambda_{F_*\gamma}.$$

Here $F_*\gamma$ is the pushforward conductivity

$$F_*\gamma(x) = \left. \frac{DF\,\gamma\,DF^t}{\det DF} \right|_{F^{-1}(x)}.$$

Uniqueness for Lipschitz conductivities

- Sylvester-Uhlmann proved uniqueness for isotropic smooth conductivities in 1988.
- In general, conductive media may present rough electrical properties, so it is relevant to know the minimal regularity assumptions on the conductivity to ensure uniqueness.
- Brown showed in 1996 that $C^{1,1/2+\varepsilon}$ was enough to ensure the uniqueness.
- Uhlmann conjectured (ICM 1998) that this should be true if the conductivities are assumed to be Lipschitz.
- That is to say, if the conductivities are assumed to satisfy

$$|\gamma(x) - \gamma(y)| \leq c|x - y|, \quad x, y \in \overline{\Omega}.$$

- This was proven by Haberman with $n = 3$ or 4 in 2014, and by Haberman-Tataru in 2011 with $n \geq 3$ for conductivities sufficiently close to one (with $\|\nabla \log \gamma\|_\infty$ sufficiently small).
- Our contribution has been to remove the smallness condition for all dimension $n \geq 3$.

Uniqueness theorem

Theorem (C-Rogers, 2014)

Let $n \geq 3$ and consider $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. Let $\gamma_1, \gamma_2 \in \mathrm{Lip}(\overline{\Omega})$ with $\gamma_1, \gamma_2 \geq c_0 > 0$. Then

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \Rightarrow \quad \gamma_1 = \gamma_2.$$

Our method basis on works of Sylvester-Uhlmann, Brown and Haberman-Tataru. It is different to Haberman's and it seems to be more suitable to obtain a reconstruction algorithm.

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Logarithmic stability under a-priori assumptions

- Alessandrini proved in 1988, under certain a-priori assumptions, logarithmic stability for this problem in $n \geq 3$: If $M^{-1} \leq \gamma_j$ and $\|\gamma_j\|_{H^s} \leq M$ for $s > n/2 + 2$, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty} \lesssim_M \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

  with

$$\omega(t) \leq |\log t|^{-\delta}, \qquad 0 < t < 1/e$$

  for $0 < \delta < 1$.
- Alessandrini also showed that these a-priori assumptions are necessary to prove the previous stability estimate.
- Mandache proved in 2001 that the optimal stability under these a-priori assumptions is logarithmic.
- The low resolution of EIT is connected with the (optimal) logarithmic stability of the inverse problem.

Resolution limit for EIT (Learnt from Alessandrini)

- Assume the conductivity γ to be piecewise constant:

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \mathbf{1}_{D_j}(x)$$

  where $D_1, \dots, D_N$ are known subdomains of Ω and $\gamma_1, \dots, \gamma_N$ are unknown constants.

- Alessandrini and Vessella proved in 2005 that

$$\|\gamma - \tilde{\gamma}\|_{L^\infty} \leq C_N \omega(\|\Lambda_\gamma - \Lambda_{\tilde{\gamma}}\|_*)$$

  with

$$\omega(t) \leq |t|, \qquad 0 < t < 1.$$

- Later Rondi showed in 2006 that

$$C_N \geq A e^{BN^{1/(2n-1)}}$$

  where A and B are absolute constants.

Resolution limit for EIT (Learnt from Alessandrini)

- Assume ε to be the error on the measured DN map and say we can tolerate an error up to $C_0\varepsilon$ on the reconstructed conductivity. The *error amplification tolerance* C_0 provides an upper bound on the number of subdomains $D_1, \ldots, D_N$:

$$N \leq \left(\frac{1}{B}\log\frac{C_0}{A}\right)^{2n-1}.$$

- Assuming that $|D_j| \sim r^n$ for some r, we have that $r \sim N^{-1/n}$. The number r can be interpreted as a *resolution parameter* and *resolution limit* is

$$r \geq \left(\frac{1}{B}\log\frac{C_0}{A}\right)^{-(2n-1)/n}.$$

- For fix C_0, no detail smaller than the resolution limit can be detected.

Stability theorem

Theorem (C-García-Reyes, 2012)

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$ with $n \geq 3$. Let M, δ and ε be real constants such that $M > 1$, $0 < \delta < 1$ and $0 < \varepsilon < 1$. Then,

$$\|\gamma_1 - \gamma_2\|_{C^{0,\delta}(\overline{\Omega})} \lesssim \left(\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{-1}\right)^{-\varepsilon^2(1-\delta)/(3n^2)}$$

for all $\gamma_1, \gamma_2 \in C^{1,\varepsilon}(\overline{\Omega})$ such that $\gamma_j > 1/M$ and $\|\gamma_j\|_{C^{1,\varepsilon}(\overline{\Omega})} \leq M$.

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Internal information from boundary data

- The Calderón problem is difficult because we are trying to detect internal information from boundary measurements.
- If u_j solves $\nabla(\gamma_j \nabla u_j) = 0$ in Ω, then

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = 0.$$

- Proving uniqueness from this requires to show density for certain class of solutions.
- If $\gamma_j \in L^\infty(\Omega)$, the class of solution has to satisfy

$$\nabla u_1 \cdot \nabla u_2 \in L^1(\Omega),$$

  which is somehow small class.
- Note that the smaller is the class the harder is to prove density.

More regular conductivities

- If $\gamma \in W^{2,\infty}(\Omega)$

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \Leftrightarrow \quad -\Delta v + \gamma^{-1/2} \Delta \gamma^{1/2} v = 0$$

  with $v = \gamma^{1/2} u$.
- If v_j solves $\Delta v_j + \gamma_j^{-1/2} \Delta \gamma_j^{1/2} v_j = 0$ in Ω, then

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \int_\Omega (\gamma_1^{-1/2} \Delta \gamma_1^{1/2} - \gamma_2^{-1/2} \Delta \gamma_2^{1/2}) v_1 v_2 \, dx = 0.$$

- Proving uniqueness under this regularity requires to show density for class of solutions satisfying that

$$v_1 v_2 \in L^1(\Omega).$$

- Note how assuming more regularity for γ_j allows to pass derivatives from the solutions to the conductivities.

Lipschtiz conductivities

Recall that $\gamma \in Lip(\overline{\Omega})$ means γ to be bounded and satisfy

$$|\gamma(x) - \gamma(y)| \leq c|x - y|, \quad x, y \in \overline{\Omega}.$$

- Its difference quotients ($\approx$ its first derivatives) are bounded.
- Therefore,

$$\gamma, \nabla\gamma \in L^\infty(\Omega) \quad \Leftrightarrow \quad \gamma \in W^{1,\infty}(\Omega).$$

- If $\gamma \in W^{1,\infty}(\Omega)$

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \Leftrightarrow \quad -\Delta v + qv = 0$$

with $v = \gamma^{1/2} u$ and $q = \gamma^{-1/2} \Delta \gamma^{1/2}$ in the sense of distributions:

$$\langle q\phi, \psi \rangle = \frac{1}{4} \int |\nabla \log \gamma|^2 \phi\psi \, dx - \frac{1}{2} \int \nabla \log \gamma \cdot \nabla(\phi\psi) \, dx.$$

Lipschitz conductivities

- If v_j solves $(-\Delta + q_j)v_j = 0$ in Ω and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then
$$\frac{1}{4}\int (|\nabla \log \gamma_1|^2 - |\nabla \log \gamma_2|^2) v_1 v_2 \, dx - \frac{1}{2}\int \nabla \log \frac{\gamma_1}{\gamma_2} \cdot \nabla (v_1 v_2) \, dx = 0.$$
- Proving uniqueness under this regularity requires to show density for class of solutions satisfying that
$$v_1 v_2, \nabla(v_1 v_2) \in L^1(\Omega).$$

Outline

What deserves to be kept in mind?

- The Calderón problem is a non-easy mathematical problem which models a medical imaging technique with promising applications as early detection of breast cancer.
- The difficulty of the Calderón problem comes up because we are trying to detect internal information from boundary measurements.
- The Calderón problem becomes much more delicate when the conductivity is not so smooth because the coefficient to be detect sits on higher order terms in the conductivity equation.