The inverse Calderón problem with Lipschitz conductivities

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Outline

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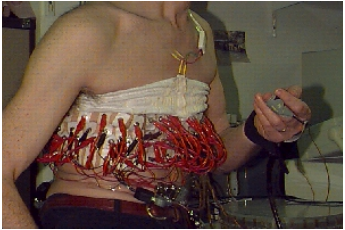
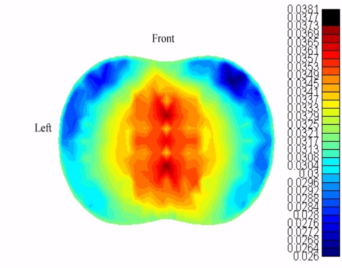
General goal of the Calderón problem

- The inverse Calderón problem consists of recovering the electric properties of a medium, namely the conductivity, by boundary measurements of many configurations of voltages and currents on its surface.





The mathematical model

- Let $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, be a bounded domain with boundary $\partial\Omega$. The case $n = 2$ is quite well understood (contributions due to Brown–Uhlmann, Nachmann, Astala–Päivärinta(–Lassas)).
- We suppose that the conductivity γ satisfies $c \leq \gamma \leq c^{-1}$.
- Given an electric potential on the boundary f, there is a unique solution u to the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \\ u\big|_{\partial\Omega} &= f. \end{aligned}$$

- u is the electric potential in the interior of Ω.
- Given that we can measure the induced current perpendicular to the boundary, we know the Dirichlet-to-Neumann map Λ_γ formally defined by

$$\Lambda_\gamma f = \gamma \nabla u \cdot n\big|_{\partial\Omega},$$

where n denotes the exterior unit normal to the boundary.

The Calderón problem

- The inverse Calderón problem consists of reconstructing γ from Λ_γ.
- We must first check Uniqueness:

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \Rightarrow \quad \gamma_1 = \gamma_2.$$

- Sylvester–Uhlmann proved uniqueness for smooth conductivities in 1988.
- In general, conductive media may present rough electrical properties, so it is relevant to know the minimal regularity assumptions on the conductivity to ensure uniqueness.
- Brown showed in 1996 that $C^{1,1/2+\varepsilon}$ was enough to ensure the uniqueness.

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Uniqueness for Lipschitz conductivities

- Uhlmann conjectured (ICM 1998) that this should be true if the conductivities are assumed to be Lipschitz.
- That is to say, if the conductivities are assumed to satisfy

$$|\gamma(x) - \gamma(y)| \leq c|x - y|, \quad x, y \in \overline{\Omega}.$$

- This was proven by Haberman with $n = 3$ or 4 in 2014, and by Haberman–Tataru in 2011 with $n \geq 3$ for conductivities sufficiently close to one (with $\|\nabla \log \gamma\|_\infty$ sufficiently small).
- Our contribution (also in 2014 but a couple of moths after Haberman's) has been to remove the smallness condition for all dimension $n \geq 3$.

Uniqueness theorem

Theorem (C–Rogers. Forum of Mathematics, Pi)

Let $n \geq 3$ and consider $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. Let $\gamma_1, \gamma_2 \in \mathrm{Lip}(\overline{\Omega})$ with $\gamma_1, \gamma_2 \geq c_0 > 0$. Then

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \Rightarrow \quad \gamma_1 = \gamma_2.$$

Our method basis on works of Sylvester–Uhlmann, Brown and Haberman–Tataru. It is different to Haberman's and it seems to be more suitable to obtain a reconstruction algorithm.

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The conductivity equation and the Schrödinger equation

- The conductivity equation is a second partial differential equation in divergence form:
$$\nabla \cdot (\gamma \nabla u) = 0.$$
- The unknown conductivity *sits* on the highest (or leading) order of derivatives of the solution u. From the equation in this form is very hard to obtain uniqueness.
- More regular conductivities may sit on lower order of derivatives of u:
$$\nabla \cdot (\gamma \nabla u) = 0 \quad \Leftrightarrow \quad \Delta u + \gamma^{-1} \nabla \gamma \cdot \nabla u = 0.$$
- And even in the zeroth order: write $u = \gamma^{-1/2} v$
$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= \nabla \cdot (\gamma^{1/2} \nabla v + \gamma^{1/2} \gamma^{1/2} \nabla \gamma^{-1/2} v) \\ &= \nabla \cdot (\gamma^{1/2} \nabla v - \nabla \gamma^{1/2} v) \qquad [\gamma^{1/2} \nabla \gamma^{-1/2} = -\gamma^{-1/2} \nabla \gamma^{1/2}] \\ &= \gamma^{1/2} \Delta v - \Delta \gamma^{1/2} v. \end{aligned}$$
- The Schrödinger equation:
$$\nabla \cdot (\gamma \nabla u) = 0 \quad \Leftrightarrow \quad \Delta v + q v = 0$$
with $v = \gamma^{1/2} u$ and $q = \gamma^{-1/2} \Delta \gamma^{1/2}$.

From the boundary to the interior

Proposition (An Alessandrini-type identity)

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \int_\Omega (q_1 - q_2) v_1 v_2 \, dx = 0,$$

for every v_1 and v_2 solving $-\Delta v_1 + q_1 v_1 = 0$ and $-\Delta v_2 + q_2 v_2 = 0$ in Ω.

- To prove density for class of solutions satisfying that $v_1 v_2 \in L^1(\Omega)$.
- By generating enough oscillatory solutions, this will yield

$$\int (q_1 - q_2) e^{-ik\cdot x} \, dx = 0, \quad \forall \; k \in \mathbb{R}^n \quad \Rightarrow \quad q_1 = q_2.$$

- This implies that

$$-\nabla \cdot \left(\gamma_1^{1/2} \gamma_2^{1/2} \nabla (\log \gamma_1^{1/2} - \log \gamma_2^{1/2}) \right) = 0,$$
$$(\log \gamma_1^{1/2} - \log \gamma_2^{1/2})|_{\partial\Omega} = 0.$$

- Thus, $\log \gamma_1^{1/2} = \log \gamma_2^{1/2} \Rightarrow \gamma_1 = \gamma_2$. [Sylvester–Uhlmann]

Why the reduction to Schrödinger equation?

- Without the reduction to Schrödinger equation (that means γ *sits* on the highest order of derivative of u) the integral identity would be:

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \int_{\Omega} (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 \, dx = 0.$$

- Proving uniqueness for $\gamma_j \in L^\infty(\Omega)$ requires to show density for solutions satisfying

$$\nabla u_1 \cdot \nabla u_2 \in L^1(\Omega).$$

- Note that the smaller is the class the harder is to prove density.
- When assuming more regularity for γ_j, we are allowed to pass derivatives from the solutions to the conductivities.

Complex geometrical optics solutions

The idea to prove uniqueness was to plug oscillatory solutions into the Alessandrini identity and prove density of the product:

$$0 = \int_\Omega (q_1 - q_2) v_1 v_2 \, dx \longrightarrow \int (q_1 - q_2) e^{-ik\cdot x} \, dx = 0 \quad \forall k \in \mathbb{R}^n.$$

The solutions are called complex geometrical optics (CGO) and look as

$$v_1 = e^{\zeta_1 \cdot x}(1 + w_1) \quad v_2 = e^{\zeta_2 \cdot x}(1 + w_2),$$

where

$$\zeta_1 = \tau\eta + i\Big(-\frac{1}{2}k + \Big(\tau^2 - \frac{|k|^2}{4}\Big)^{1/2}\theta\Big)$$
$$\zeta_2 = -\tau\eta + i\Big(-\frac{1}{2}k - \Big(\tau^2 - \frac{|k|^2}{4}\Big)^{1/2}\theta\Big),$$

with $\tau \geq 1$, $|\eta| = |\theta| = 1$, $\eta \cdot \theta = \eta \cdot k = \theta \cdot k = 0$ and w_1 and w_2 decay in some sense as $\tau \to \infty$.

- Note that $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$ so that $e^{\zeta_1 \cdot x}$ and $e^{\zeta_2 \cdot x}$ are harmonic.
- We also have $\zeta_1 + \zeta_2 = -ik$, so that

$$v_1 v_2 = e^{-ik\cdot x}(1 + w_1)(1 + w_2) = e^{-ik\cdot x} + e^{-ik\cdot x} w_1 (1 + w_2).$$

Existence of CGO solutions

- Note that for $v = e^{\zeta\cdot x}(1+w)$
$$(-\Delta+q)v = 0 \quad \Leftrightarrow \quad e^{-\zeta\cdot x}(-\Delta+q)e^{\zeta\cdot x}(1+w) = 0.$$
- Using that $\zeta\cdot\zeta = 0$, we see that
$$e^{-\zeta\cdot x}(-\Delta+q)e^{\zeta\cdot x} = (-\Delta - 2\zeta\cdot\nabla + q).$$
- It will suffice to find w (the **remainder term**) such that
$$(-\Delta - 2\zeta\cdot\nabla + q)w = -q.$$
- The symbol of the operator $-\Delta - 2\zeta\cdot\nabla$ is given by
$$p_\zeta(\xi) = |\xi|^2 - 2i\zeta\cdot\xi.$$
- Whenever $|\xi| \geq 4|\zeta| \sim \tau$
$$|p_\zeta(\xi)| \sim |\xi|^2 \sim (\tau^2 + |\xi|^2).$$
- As $-\Delta - 2\zeta\cdot\nabla + q$ is a zeroth order perturbation of $-\Delta - 2\zeta\cdot\nabla$, one could be optimistic and expect
$$\|(\tau^2+|\xi|^2)\widehat{w}\|_{L^2} \lesssim \|\widehat{q}\|_{L^2} \quad \Leftrightarrow \quad \|w\|_{H^k} \lesssim \tau^{k-1}\|q\|_{L^2},\ k = 0,1,2.$$

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Lipschtiz conductivities

Recall that $\gamma \in Lip(\overline{\Omega})$ means γ to be bounded and satisfy

$$|\gamma(x) - \gamma(y)| \leq c|x - y|, \quad x, y \in \overline{\Omega}.$$

- Its difference quotients ($\approx$ its first derivatives) are bounded.
- Therefore,

$$\gamma, \nabla\gamma \in L^\infty \quad \Leftrightarrow \quad \gamma \in W^{1,\infty}.$$

- If $\gamma \in W^{1,\infty}$

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \Leftrightarrow \quad -\Delta v + qv = 0$$

with $v = \gamma^{1/2} u$ and $q = \gamma^{-1/2} \Delta \gamma^{1/2}$ in the sense of distributions:

$$\langle q\phi, \psi \rangle = \frac{1}{4} \int |\nabla \log \gamma|^2 \phi\psi \, dx - \frac{1}{2} \int \nabla \log \gamma \cdot \nabla(\phi\psi) \, dx.$$

Note that $\nabla \log \gamma \in L^\infty$.

Lipschitz conductivities

- If v_j solves $(-\Delta + q_j)v_j = 0$ in Ω and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then

$$\frac{1}{4}\int (|\nabla \log \gamma_1|^2 - |\nabla \log \gamma_2|^2) v_1 v_2 \, dx - \frac{1}{2}\int \nabla \log \frac{\gamma_1}{\gamma_2} \cdot \nabla (v_1 v_2) \, dx = 0.$$

- Proving uniqueness under this regularity requires to show density for class of solutions satisfying that

$$v_1 v_2, \nabla(v_1 v_2) \in L^1(\Omega).$$

- The situation here can be saved because $\nabla(v_1 v_2) = v_2 \nabla v_1 + v_1 \nabla v_2$.
- Recall that $v = e^{\zeta \cdot x}(1 + w)$. Before we only used the decay of w in the L^2 norm. Now we will need to control also the H^1 norm.

The CGO solutions and the decay of the remainder term

In order to construct $v = e^{\zeta\cdot x}(1 + w)$ solving

$$(-\Delta + q)v = 0$$

It was enough to find a remainder term w satisfying

$$(-\Delta - 2\zeta\cdot\nabla + q)w = -q.$$

One can do so but the decay become worse:

$$\begin{aligned}\|w\|_{H^t} &\lesssim \tau^t \sup_{|y|\leq 1} \|\nabla\log\gamma(x) - \nabla\log\gamma(x - \tau^{-1}y)\|_{L^2_x} \qquad [0 \leq t \leq 1]\\ &\lesssim o(\tau^{t-s})\|\nabla\log\gamma\|_{H^s} \qquad [0 \leq s \leq 1]\end{aligned}$$

Warning: This decay estimate is only useful for the uniqueness problem when $s > 1/2$. This requires $\gamma \in H^{s+1}$, which is $1/2$ derivatives more than Lipschitz. Brown proved uniqueness with $\gamma \in C^{1,s}(\overline{\Omega})$.

How to improve the estimates:

- The remainder w depends on $\zeta = \zeta(\tau, \eta)$ but we do not need an estimate that holds for every τ and η, as the one above. We only need for some τ's and some η's.
- A way to detect if there are τ's and η's for which the above estimate can be improved is averaging in τ and η.

Spaces adapted to $-\Delta - 2\zeta\cdot\nabla$

We are going to prove the existence of w solution to

$$(-\Delta - 2\zeta\cdot\nabla + q)w = -q$$

in a family of spaces suitable to average in the parameters τ and η.

- It is a general fact that the surjectivity of $T = (-\Delta - 2\zeta\cdot\nabla + q)$ is a consequence of the injectivity of T^*.
- The injectivity follows from the *a priori* estimate for T^*:

$$\|\psi\|_{X_\zeta^{1/2}} \lesssim \|(-\Delta + 2\zeta\cdot\nabla + q)\psi\|_{X_\zeta^{-1/2}} \quad \forall\, \psi \in C_0^\infty(\Omega),$$

  where the norms adapted to the problem are given by

$$\|f\|^2_{X_\zeta^s} = \int \left(\left||\xi|^2 + 2i\zeta\cdot\xi\right|^2 + |\zeta|^2\right)^s |\widehat{f}(\xi)|^2\, d\xi.$$

When $\|\nabla \log\gamma\|_{L^\infty}$ is sufficiently small, the a priori estimate follows easily

$$\begin{aligned}\|\psi\|_{X_\zeta^{1/2}} &\lesssim \|(-\Delta + 2\zeta\cdot\nabla)\psi\|_{X_\zeta^{-1/2}}\\ &\lesssim \|(-\Delta + 2\zeta\cdot\nabla + q)\psi\|_{X_\zeta^{-1/2}} + \|q\psi\|_{X_\zeta^{-1/2}}\\ &\lesssim \|(-\Delta + 2\zeta\cdot\nabla + q)\psi\|_{X_\zeta^{-1/2}} + \|\nabla\log\gamma\|_{L^\infty}\|\psi\|_{X_\zeta^{1/2}}.\end{aligned}$$

Averaging in the parameters τ and η

From the previous a priori estimate one deduces that there exists w solution to

$$(-\Delta - 2\zeta\cdot\nabla + q)w = -q$$

that satisfies

$$\|w\|_{X_\zeta^{1/2}} \lesssim \|q\|_{X_\zeta^{-1/2}}$$

with $\zeta = \zeta(\tau,\eta)$. Averaging now

$$\begin{aligned}\frac{1}{\lambda}\int_S\int_\lambda^{2\lambda}\|w\|_{X_\zeta^{1/2}}\,d\tau d\eta &\lesssim \frac{1}{\lambda}\int_S\int_\lambda^{2\lambda}\|q\|_{X_\zeta^{-1/2}}\,d\tau d\eta\\ &\lesssim \sup_{|y|\leq 1}\|\nabla\log\gamma(x) - \nabla\log\gamma(x-\lambda^{-1/4}y)\|_{L_x^2}\\ &= o(1).\end{aligned}$$

- Haberman–Tataru introduced these spaces, proved the averaged estimate and used it to conclude uniqueness when the conductivity is Lipschitz and $\|\nabla\log\gamma\|_\infty$ is sufficiently small.
- Our contribution was to remove the *smallness condition*.

How to remove the smallness of $\|\nabla \log \gamma\|_{L^\infty}$

In order to remove the smallness condition, we need to understand why

$$\|q\psi\|_{X_\zeta^{-1/2}} \lesssim \|\nabla \log \gamma\|_{L^\infty} \|\psi\|_{X_\zeta^{1/2}}.$$

Recall that

$$\langle q\psi, \phi\rangle = \frac{1}{4}\int |\nabla \log \gamma|^2 \phi\psi\, dx - \frac{1}{2}\int \nabla \log \gamma \cdot \nabla(\phi\psi)\, dx.$$

We have

$$\big|\langle q\psi, \phi\rangle\big| \lesssim (1 + \|\nabla \log \gamma\|_{L^\infty})^2 \Big(\|\psi\|_{H^1}\|\phi\|_{L^2} + \|\psi\|_{L^2}\|\phi\|_{H^1}\Big). \qquad (1)$$

Now $\|f\|_{L^2} \le |\zeta|^{-1/2}\|f\|_{X_\zeta^{1/2}}$, and

$$\begin{aligned}
\|\nabla f\|_{L^2} &\le \Big(\int_{|\xi|<4|\zeta|} |\xi|^2 |\widehat{f}(\xi)|^2\, d\xi\Big)^{1/2} + \Big(\int_{|\xi|\ge 4|\zeta|} |\xi|^2 |\widehat{f}(\xi)|^2\, d\xi\Big)^{1/2} \\
&\lesssim |\zeta|\|f\|_2 + \Big(\int_{|\xi|\ge 4|\zeta|} \big||\xi|^2 + 2i\zeta\cdot\xi\big| |\widehat{f}(\xi)|^2\, d\xi\Big)^{1/2} \\
&\le |\zeta|^{1/2}\|f\|_{X_\zeta^{1/2}} + \|f\|_{X_\zeta^{1/2}}.
\end{aligned}$$

By plugging this into (1): $\big|\langle q\psi, \phi\rangle\big| \lesssim \|\nabla \log \gamma\|_{L^\infty} \|\psi\|_{X_\zeta^{1/2}} \|\phi\|_{X_\zeta^{1/2}}.$

Our contribution: an improved a priori estimate

- The bad behaviour in $|\zeta|$ comes from the low-frequencies of the ∇. But this is only an operator from L^2 to L^2 with bad norm.
- Our idea is then to introduce some Carleman weights (weights depending on a parameter) that provide an improved control on the L^2-part norm of X^s_ζ.
- More precisely, our contribution consists of guessing and proving the following estimate:

$$\|\psi\|_{Y^{1/2}_\zeta} \lesssim \|e^{M(\eta\cdot x)^2/2}(-\Delta+2\zeta\cdot\nabla)(e^{-M(\eta\cdot x)^2/2}\psi)\|_{Y^{-1/2}_\zeta} \quad \forall \psi \in C^\infty_0(\Omega)$$

  where the new norms are given by

$$\|f\|^2_{Y^s_\zeta} = \int \left(M^{-1}\left||\xi|^2 + 2i\zeta\cdot\xi\right|^2 + M|\zeta|^2\right)^s |\widehat{f}(\xi)|^2 \, d\xi$$

  for a large parameter M.
- The parameter M is now chosen to include the q without the smallness condition.
- We get rid of the weights because our estimates are local. This bring us to the situation of Haberman–Tataru without the smallness condition. Using the averaged estimate and the previous ideas of Sylvester–Uhlmann, Alessandrini and Brown, the uniqueness follows.

Outline

The Calderón problem as model for electrical impedance tomography

Uniqueness for Lipschitz conductivities

How to prove uniqueness (smooth conductivities)

Difficulties for Lipschitz conductivities

To keep in mind

What deserves to be kept in mind?

- The inverse Calderón problem consists of reconstructing the conductivity in a medium their corresponding Dirichlet-to-Neumann map.
- The difficulty of the Calderón problem comes up because we are trying to detect internal information from boundary measurements.
- The Calderón problem becomes much more delicate when the conductivity is not so smooth because the coefficient to be detect sits on higher order terms in the conductivity equation.
- The information on the boundary is transmitted to the interior through complex geometrical optics solutions, which asymptotically behave as highly-oscillatory and exponentially-growing harmonic functions.
- The less regular is the conductivity the harder is to construct these solutions.