SHELL INTERACTIONS FOR DIRAC OPERATORS: AN ISOPERIMETRIC-TYPE INEQUALITY

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Free Dirac operator in $\mathbf{R}^3$: $\quad H = -i\alpha\cdot\nabla + m\beta$

$(m = \text{mass} > 0)$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \left(\begin{pmatrix} 0 & \widehat{\sigma}_1 \\ \widehat{\sigma}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_2 \\ \widehat{\sigma}_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_3 \\ \widehat{\sigma}_3 & 0 \end{pmatrix} \right)$$

$$\widehat{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \alpha_i^2 = \beta^2 = I_d & i = 1,2,3 \\ \{\alpha_i, \beta\} = \{\alpha_i, \alpha_j\} = 0 & i \neq j \end{cases}$$

(Clifford Algebra Structure)

$$H^2 = (-\Delta + m^2) I_d \quad \Longrightarrow \quad \begin{cases} H \text{ local version of } \sqrt{-\Delta + m^2} \\ \text{1st order symmetric differential operator} \\ \text{Introduced by Dirac (1928)} \longrightarrow \textbf{electron} \end{cases}$$

Electrostatic Shell Interactions:

$\Omega \subset \mathbb{R}^3$ bounded smooth domain

$\sigma =$ surface measure on $\partial\Omega$

$N =$ outward unit normal vector field on $\partial\Omega$

Electrostatic shell potential $V_\lambda = \lambda\delta_{\partial\Omega}$:

$$\lambda \in \mathbb{R}, \qquad V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$$

$\varphi_\pm =$ non–tangential boundary values of $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ when approaching from Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$

Electrostatic shell interaction for H: $H + V_\lambda$

$a \in (-m, m)$

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}\,|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2 - a^2}\,|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right]$$

$$= \text{ fundamental solution of } H - a$$

$$\mathcal{D}(H + V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (Gdx + gd\sigma),\ G \in L^2((R)^3)^4 \ \ g \in L^2(\partial\Omega)^4,$$

$$\lambda\left(\phi^0 * (Gdx)\right)\big|_{\partial\Omega} = -\left(1 + \lambda C^0_{\partial\Omega}\right) g)\Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \phi^a(x-y)g(y)d\sigma(y)\ ,\ x \in \partial\Omega.$

If $\lambda \neq \pm 2 \implies H + V_\lambda$ is self–adjoint on $\mathcal{D}(H + V_\lambda)$.

(**[AMV, 2014]**, more general **[Posilicano,2008]**
Ω ball $\longrightarrow$ **[Dittrich, Exner, Seba,1989]**)

Point spectrum on $(-m,m)$ for $H+V_\lambda$:

Birman–Schwinger principle: $a \in (-m,m)$, $\lambda \in \mathbb{R}\setminus\{0\}$,

$$\ker(H+V_\lambda - a) \neq 0 \iff \ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0$$

(problem in $\mathbb{R}^3$) **(problem in $\partial\Omega$)**

Properties of $C^a_{\partial\Omega}$, $a \in [-m,m]$**:**

(a) $C^a_{\partial\Omega}$ bounded self–adjoint operator in $L^2(\partial\Omega)^4$.

(b) $\left[C^a_{\partial\Omega}(\alpha\cdot N)\right]^2 = -\frac{1}{4} I_d.$ $\left(\alpha\cdot N = \sum_{j=1}^{3}\alpha_j N_j \text{ multiplication operator}\right)$

$$\ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0 \begin{cases} \overset{(a)}{\Longrightarrow} & |\lambda| \geq \lambda_l(\partial\Omega) > 0 \quad \text{and} \quad \lambda_l(\partial\Omega) \leq 2 \\ \overset{(b)}{\Longrightarrow} & |\lambda| \leq \lambda_u(\partial\Omega) < +\infty \quad \text{and} \quad \lambda_u(\partial\Omega) \geq 2 \end{cases}$$

Therefore, $\ker(H+V_\lambda - a) \neq 0 \implies |\lambda| \in [\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$

Main result:

Question: How small can $\left[\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)\right]$ be?

(Isoperimetric–type statement w.r.t. Ω)

(Find optimizers)

Examples: $\Omega \subset \mathbb{R}^3$ bounded smooth domain

- Isoperimetric inequality: $\mathrm{Vol}\,(\Omega)^2 \leq \frac{1}{36}\,\mathrm{Area}\,(\partial\Omega)^3$.
- Pólya–Szegö inequality:

$$\mathrm{Cap}\,(\overline{\Omega}) = \left(\inf_{\nu} \iint \frac{d\nu(x)d\nu(y)}{4\pi|x-y|}\right)^{-1}$$

ν probability Borel measure $\mathrm{supp}\,\nu \subset \overline{\Omega}$

$$\mathrm{Cap}\,(\overline{\Omega}) \geq 2\big(6\pi^2\mathrm{Vol}\,(\Omega)\big)^{1/3}. \quad \longleftarrow \quad \text{[Pólya, Szegö, 1951]}$$

In both cases, $=$ holds $\Longleftrightarrow$ Ω is a ball.

Theorem [AMV2015].– $\Omega \subset \mathbb{R}^3$ bounded smooth domain. If

$$m \frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})} > \frac{1}{4\sqrt{2}},$$

then

$$\sup\{|\lambda| : \ker(H + V_\lambda - a)) \neq 0 \ \text{ for some } \ a \in (-m, m)\}$$
$$\geq 4\left(m\frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}\,(\partial\Omega)^2}{\text{Cap}\,(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

and

$$inf\,\{|\lambda| : \ker(H + V_\lambda - a)) \neq 0 \ \text{ for some } \ a \in (-m, m)\}$$
$$\leq 4\left(-m\frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}\,(\partial\Omega)^2}{\text{Cap}\,(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Ingredients of the proof:

(1) The monotonicity of $\lambda(a)$ in $\ker\left(\dfrac{1}{\lambda(a)} + C^a_{\partial\Omega}\right)$ reduces the study of $(*)$ to $a = \pm m$.

(2) The quadratic form inequality relates $\sup\{|\lambda| : \ker(H + V_\lambda - a)) \neq 0 \text{ for some } a \in (-m, m)\}$ in $(*)$ with the optimal constant of an inequality involving the single layer potential K and a SIO. (Here appears the $1/4\sqrt{2}$)

(3) Isoperimetric type statement for K in terms of Area $(\partial\Omega) \setminus$ Cap $(\overline{\Omega})$.

Proof:

(1) $\ker\left(\frac{1}{\lambda(a)} + C^a_{\partial\Omega}\right) \neq 0 \quad \Longrightarrow \quad C^a_{\partial\Omega} g_a = \frac{1}{\lambda(a)} g_a \;, \; \|g_a\| = 1$

$$\Longrightarrow \quad \frac{1}{\lambda(a)} = \frac{1}{\lambda(a)}\langle g_a, g_a\rangle = \langle C^a_{\partial\Omega} g_a, g_a\rangle$$

$$C^a_{\partial\Omega} \hookrightarrow (H-a)^{-1} \quad \Longrightarrow \quad \frac{d}{da} C^a_{\partial\Omega} \hookrightarrow (H-a)^{-2}$$

$$\Longrightarrow \quad \frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a\rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

(2)

$$\left.\begin{aligned} Kf(x) &= \frac{1}{4\pi}\int \frac{f(y)}{|x-y|}d\sigma y \qquad \begin{pmatrix}\text{compact}\\ \text{positive}\end{pmatrix} \\ Wf(x) &= \frac{1}{4\pi}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} i\cdot\widehat{\sigma}\cdot\frac{x-y}{|x-y|^3}f(y)\,d\sigma(y) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(SIO)} \end{aligned}\right\} C^a_{\partial\Omega} = \begin{pmatrix} 2mK & W \\ W & 0 \end{pmatrix}$$

$$\left(\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right)$$

Then,

$$[C^m_{\partial\Omega}(\alpha\cdot N)]^2 = -\frac{1}{4} \quad \Longrightarrow \quad \begin{cases} \{(\widehat{\sigma}\cdot N)K, (\widehat{\sigma}\cdot N)W\} = 0 \\ [(\widehat{\sigma}\cdot N)W]^2 = -\dfrac{1}{4} \end{cases} \qquad (**)$$

$$\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0 \quad\Longrightarrow\quad C^m_{\partial\Omega}g=\frac{1}{\lambda}g \qquad g=\begin{pmatrix}\mu\\ h\end{pmatrix}$$

$$\Longrightarrow\quad \begin{cases} 2mK\mu+Wh &= -\dfrac{1}{\lambda}\mu \\[2ex] W\mu &= -\dfrac{1}{\lambda}h \end{cases}$$

$$\overset{(**)}{\Longrightarrow}\quad \exists f\in L^2(\partial\Omega)^2,\ f\neq 0\ \text{ such that }\ \left(-\frac{8m}{\lambda}K+1-\frac{16}{\lambda^2}W^2\right)f=0$$

Multiply by $\overline{f}$ and integrate on $\partial\Omega$:

$$\begin{pmatrix}\text{decreasing}\\ \text{on } \lambda>0\end{pmatrix} \qquad \left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\underbrace{\int_{\partial\Omega}Kf\cdot\overline{f}}_{\geq 0}=\int_{\partial\Omega}|f|^2$$

Quadratic form inequality:

$$\lambda_\Omega=\inf\left\{\lambda>0:\left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\int_{\partial\Omega}Kf\cdot\overline{f}\leq\int_{\partial\Omega}|f|^2 \qquad \forall f\in L^2(\partial\Omega)^2\right\}$$

[Esteban,Séré, 1997]

Theorem [AMV2015].–

(a) $$4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\frac{1}{4}}\right)$$

$$\leq \lambda_\Omega \leq 4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\|W\|^2_{\partial\Omega}}\right).$$

(b) If $\lambda > 0$ and $\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0 \quad \Longrightarrow \quad \lambda \leq \lambda_\Omega$.

(c) If $\lambda_\Omega > 2\sqrt{2}$ $\left(\hookrightarrow (\frac{1}{4\sqrt{2}}\right)$, equality is attained and minimizers $\hookrightarrow \ker\left(\frac{1}{\lambda_\Omega}+C^m_{\partial\Omega}\right)\neq 0$.

Review :

(3) $\longrightarrow$ Isoperimetric–type result for λ_Ω.

(2) $\longrightarrow$ Theorem **(b)** and **(c)** ensure

$$\lambda_\Omega = \sup\{|\lambda| : \ker(1/\lambda + C^m_{\delta\Omega}) \neq 0\}.$$

(1) $\longrightarrow$ Use monotonicity to replace "for some $a \in (-m, m)$" by $a = m$.

(3) Ω ball $\quad\Longrightarrow\quad \|W\|^2_{\partial\Omega} = \dfrac{1}{4}$

$\left(\text{``}\Leftarrow\text{''}\right.$ **[Hofmann, Marmdejo–Olea, Mitrea, Pérez–Esteva, Taylor, 2009]**)

$$\Longrightarrow \quad \lambda_\Omega = 4\left(m\|K\|_{\partial\Omega} + \sqrt{m^2\|K\|^2_{\partial\Omega} + \|W\|^2_{\partial\Omega}}\right)$$

Ω general,

$$\|K\|_{\partial\Omega} = \sup_{f\neq 0} \frac{1}{\|f\|^2_{\partial\Omega}} \int_{\partial\Omega} Kf\cdot\overline{f} \geq \iint \overbrace{\frac{d\sigma(y)}{4\pi|x-y|}}^{\inf \mathbf{D}} \frac{d\sigma(x)}{\sigma(\partial\Omega)} \geq \frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})}$$

(f = 1)

(“ = ” $\quad\Longleftrightarrow\quad$ Ω is a ball: **Gruber's conjecture**)

THANK YOU FOR YOUR ATTENTION

- Neumann eigenvalue problem $(\Omega \subset \mathbb{R}^2)$

$$\begin{cases} -\Delta u = \mu u \ \text{ in } \ \Omega \\ \dfrac{\partial u}{\partial N} = 0 \ \text{ on } \ \partial\Omega \end{cases} \implies \mu_1(\Omega) \leq \frac{C}{\text{Area}\,(\Omega)}$$

[Szegö, 1954]

(disks give "=")

- Steklov eigenvalue problem $(\Omega \subset \mathbb{R}^2)$

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \Omega \\ \dfrac{\partial u}{\partial N} = \omega u \ \text{ on } \ \partial\Omega \end{cases} \implies \omega_1(\Omega) \leq \frac{2\pi}{\text{Length}\,(\partial\Omega)}$$

[Weirstock, 1954]