Singular solutions of the Binormal Flow: transfer of energy and momentum

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Binormal Flow

$$X_t = X_s \wedge X_{ss} \qquad X(s,t) \in \mathbb{R}^3 \qquad \begin{array}{l} s \in \mathbb{R} \\ s \in [0, 2\pi] \quad \text{periodic} \end{array} \qquad \textbf{(BF)}$$

Recall that if

$$X_s = T$$

then

$$T_t = T \wedge T_{ss} = J D_s T_s$$

Hence $T_t \cdot T = 0$ and if $|T|^2 = R$ at time zero then this property holds for all times. Notice that formally we recover the free Schrödinger equation at the limit $R = \infty$.

In the rest of the talk we will assume that $R = 1$. Then

$$\begin{aligned} T_s &= cn \\ n_s &= -cT + \tau b \\ b_s &= -\tau n, \end{aligned}$$

and **(BF)** becomes

$$X_t = cb \qquad X(s,t) \in \mathbb{R}^3 \qquad \begin{array}{l} s \in \mathbb{R} \\ s \in [0, 2\pi] \ \text{ periodic.} \end{array} \qquad (1)$$

The equation was first obtained by Da Rios in 1906 as an approximation to the evolution of vortex filaments according to Euler equations. Jerrard-Smets'15, Jerrard-Seis'16.

The motivation of the talk is some numerical experiments done for initial data given by a regular polygon. We can think in non–circular jets as the corresponding problem in real fluids. In this case the dynamics seem to be much more complicated than the one of vortex rings. At the qualitative level two relevant facts are observed in real experiments:

- Axis switching phenomena.
- Symmetries that are a multiple of the starting symmetry appear. (Grinstein et al., '96).

It is interesting that in all these references **(BF)** is used as a justification of the seen dynamics.

In a recent work with de la Hoz we showed that these phenomena appear in the **(BF)** and that in fact was nothing but a non–linear Talbot effect (Olver '10, Erdogan-Tsirakis '13, De la Hoz-Vega '13)

THE TALBOT EFFECT

$$\sum_k e^{itk^2+iks}.$$

$$t_{pq} = (2\pi/M^2)(p/q)$$

$$\psi(s,0) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta(s - \tfrac{2\pi k}{M}).$$

$$\psi(s,t_{pq}) = \frac{2\pi}{Mq} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{q-1} G(-p,m,q)\delta(s - \tfrac{2\pi k}{M} - \tfrac{2\pi m}{Mq})$$

"Non-linear version":

$$\begin{aligned} T_s &= \alpha e_1 + \beta e_2 \\ e_{1s} &= -\alpha T \\ e_{2s} &= -\beta T \end{aligned}$$

The generalized quadratic Gauß sums are defined by

$$\sum_{l=0}^{|c|-1} e^{2\pi i(al^2+bl)/c},$$

for given integers a, b, c, with $c \neq 0$.

$$G(-p, m, q) = \begin{cases} \sqrt{q}e^{i\theta m}, & \text{if } q \text{ is odd}, \\ \sqrt{2q}e^{i\theta m}, & \text{if } q \text{ is even and } q/2 \equiv m \mod 2, \\ 0, & \text{if } q \text{ is even and } q/2 \not\equiv m \mod 2, \end{cases}$$

for a certain angle θ_m that depends on m (and, of course, on p and q, too).

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FLOW CONTROL WITH NONCIRCULAR JETS[1]

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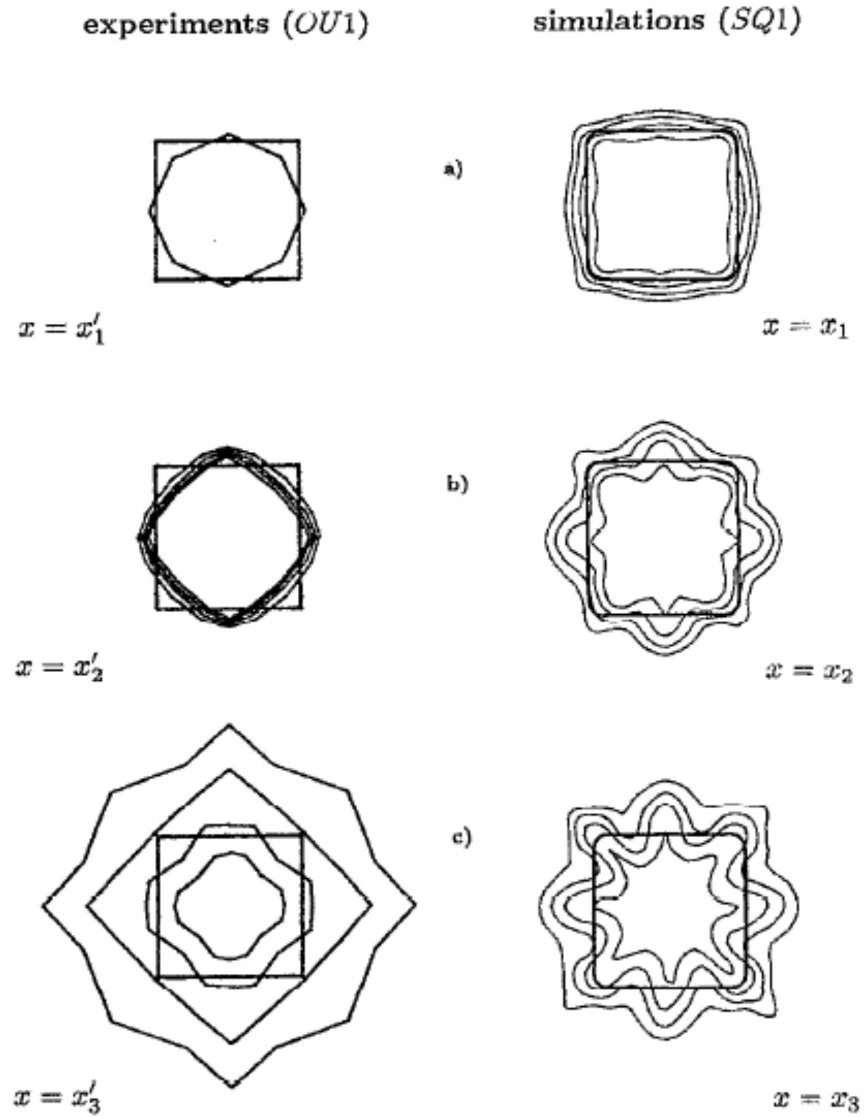
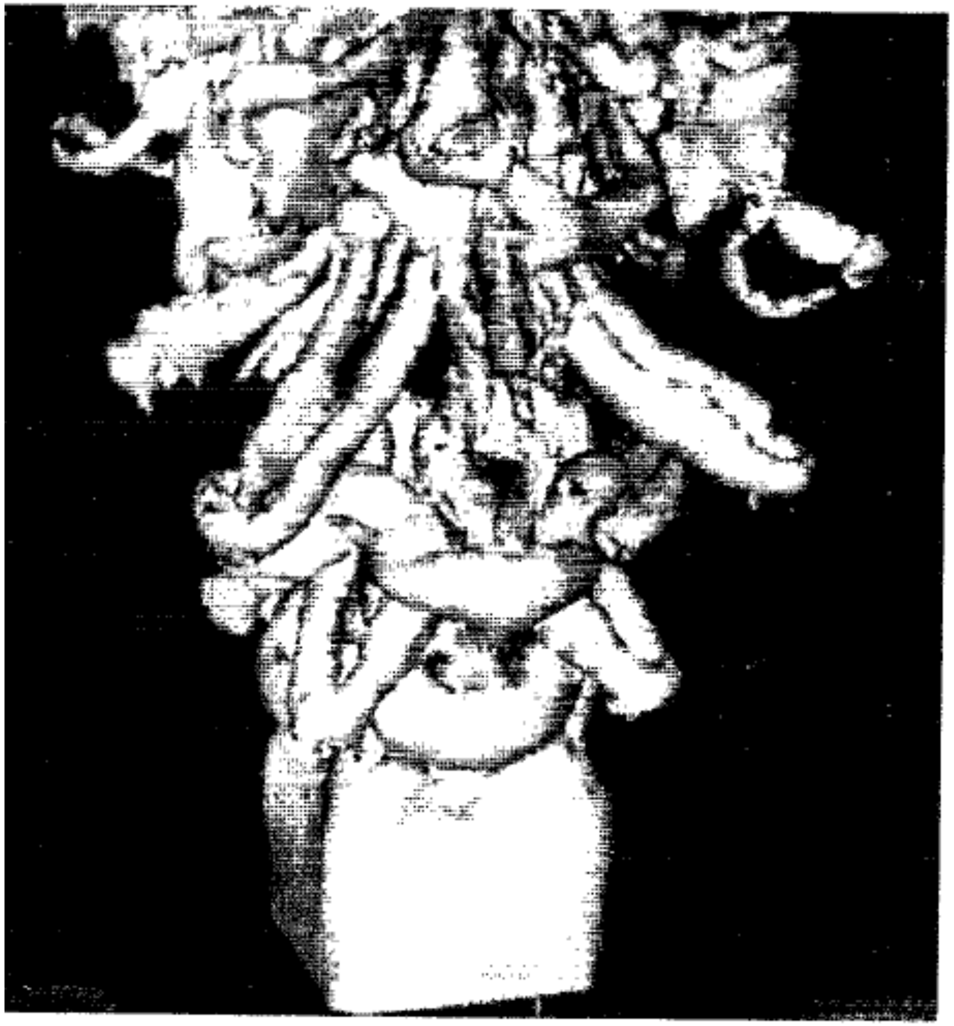
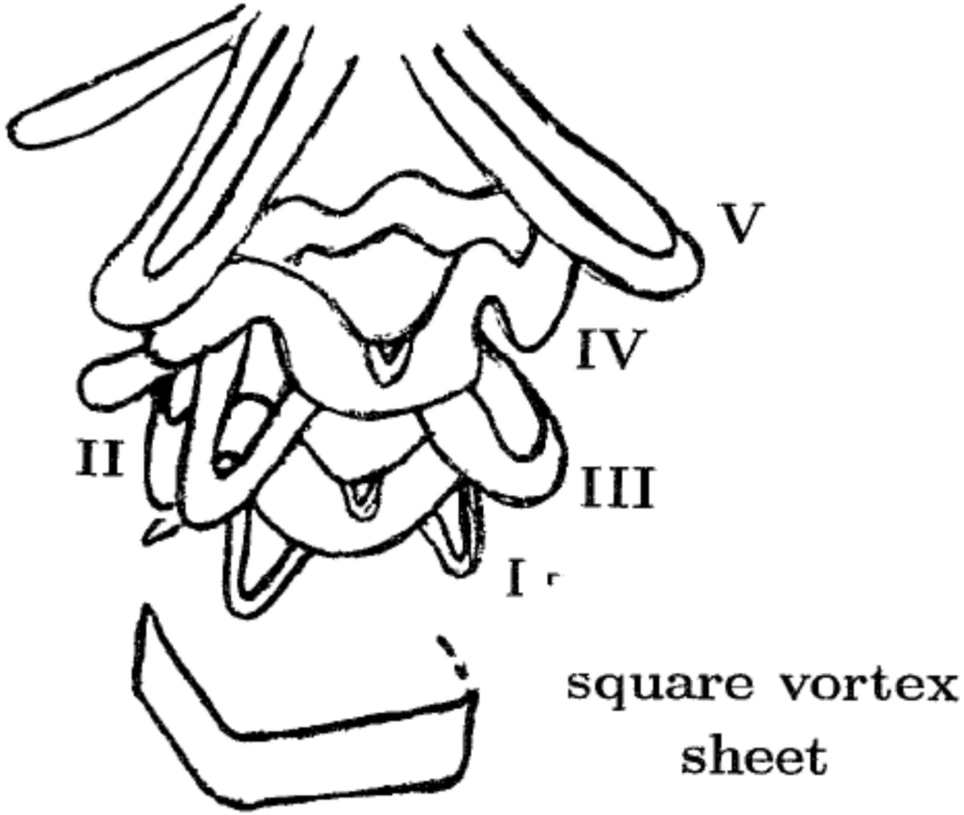


FIG. 10. Axis switching of the jet cross section in terms of isocontours of time-averaged streamwise velocity scaled with its local centerline value (u/u_{cl}) for experimental (OU1) and simulated (SQ1) jets. Contour levels are 0.2, 0.4, 0.6, and 0.8. The geometry of the experimental nozzle is superimposed on each slice on the left; the initial half-width velocity cross section of the simulated jets is superimposed on each slice on the right. The stream-



I, III, V : hairpin (braid) vortices
II, IV : deformed vortex rings



Motivation

New numerical simulations suggest that the dynamics at time 0^+ of any of the corners of the regular polygon is the one of the self–similar solution that is determined by the angle and location of the corner.

Hence we want to revisit what is known about the self–similar solutions focusing in three main aspects:

(i) Continuation after the singularity has been formed (Joint work with V. Banica).

(ii) Behavior of some conservation laws that are also present in Euler equations (linear momentum).

(iii) Transfer of energy: Lack of continuity of some appropriate norm.

Function Spaces

Let us start from **(iii)**. We are dealing with singular curves as polygons. Therefore the Frenet system does not seem to be the appropriate one. It is better to use the parallel frame:

$$\begin{aligned} T_s &= \alpha e_1 + \beta e_2 \\ e_{1s} &= -\alpha T \\ e_{2s} &= -\beta T \end{aligned}$$

Then (Hasimoto transformation) $\psi = \alpha + i\beta$ solves

(NLS)

$$\psi_t = i\left(\psi_{ss} + \frac{1}{2}\Big(|\psi|^2 - A(t)\Big)\psi\right) \qquad A(t) \in \mathbb{R}.$$

Function Spaces

We want to deal with corners. That amounts to consider

$$\psi(s,0) = c_0\delta$$

which is critical for scaling but supercritical with respect the first positive conservation law

$$\int |\psi(s,t)|^2 = \int |\psi(s,0)|^2$$

Results "below" L^2 were first obtained by Vargas–V'01, and then extended by Grünroch'05; Christ'07. More recently there has been plenty of activity Koch-Tataru'16, Kappeler-Molnar'16, Killip-Visan-Zhang'16. All the results are subcritical from the scaling point of view (in the Sobolev class is $\dot{H}^{-1/2}$).

Function Spaces

The "natural" spaces are given as weighted Lebesgue spaces (and variants) of $\widehat{\psi}_0$. From this point of view a natural space would be

- $\widehat{\psi}_0 \in L^\infty$.

Similarly for T we will consider $\Big(T_s = \mathrm{Re}\left(\overline{\psi}(e_1 + ie_2)\right)\Big)$.

- $\widehat{T}_s \in L^\infty$.

Selfsimilar Solutions

In order to find the self similar solutions of **(BF)** is better not to use **(NLS)** but the **(BF)** itself. Hence we look for solutions that can be written as $\sqrt{t}\ G(s/\sqrt{t})$

From **(BF)** we get

$$\frac{1}{2}G - \frac{s}{2}G' = G' \wedge G'';$$

Calling $T = G'$,

$$-\frac{s}{2}T' = T \wedge T''.$$

Hence $c = c_0 \quad \tau = s/2$ and

$$\psi(s,t) = \frac{c_0}{\sqrt{t}} e^{is^2/4t} = \frac{1}{\sqrt{t}}\psi_0(s/\sqrt{t}) \qquad \psi_0 = c_0 e^{is^2/4}$$

Selfsimilar Solutions

In a joint work with Gutiérrez and Rivas we characterized all the possible G. First observe that

$$\left(\frac{G}{s}\right)' = \frac{G'_s - G}{s^2} = -2c_0\frac{b}{s^2}$$

Hence $\lim\limits_{s\to\pm\infty} \dfrac{G}{s} = A^{\pm}$ and

$$\lim_{t\to 0}\sqrt{t}G(s/\sqrt{t}) = \begin{cases} A^+ s & s > 0 \\ A^- s & s < 0 \end{cases}$$

Selfsimilar Solutions

Moreover,

$$\cos\frac{\theta}{2} = e^{-\pi\frac{c_0^2}{2}}$$

Using similar ideas we can prove that there exist B^+, B^- such that

$$\begin{aligned} T &= A^+ + 2\frac{c_0}{s}b + O(1/s^2) \\ (n+ib) &= c_0 B^{\pm} e^{i\frac{s^2}{4} + i\frac{c_0}{2}\lg s} + O(1/s) \qquad s \to \infty \end{aligned}$$

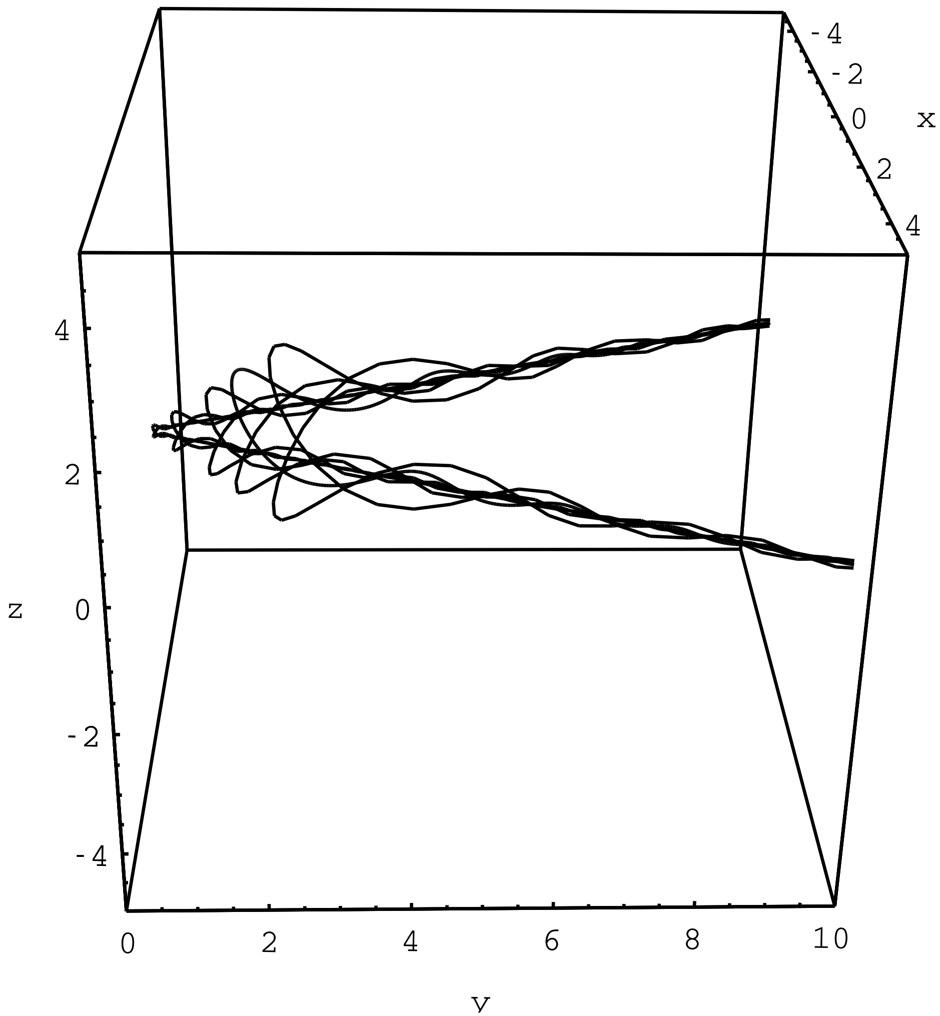
Continuation for $t < 0$

Recall that **(BF)** is a flow of oriented curves. In fact to change the orientation is equivalent to change the direction of time, because of the symmetry:

“$X(-s,-t)$ is a solution of **(BF)** if $X(s,t)$ is a solution”

It is also rotation invariant. Hence we can construct the following “artificial” solution:

$$\tilde{X}(s,t) = \begin{cases} \sqrt{t}G(s/\sqrt{t}) & t \geq 0 \\ \\ \sqrt{|t|}\rho \cdot G(-s/\sqrt{|t|}) & t \leq 0 \end{cases}$$



-4
-2
0
x
2
4
4
2
z
0
-2
-4
0
2
4
6
8
10
y

Continuation for $t < 0$

Theorem.– (with V. Banica) $\tilde{X}$ is a stable (in an appropriate sense) solution. As a consequence the process of creating/anhilating a corner is stable.

Remarks.–

(1) This is not true at the level of **(NLS)**

$$t = 1 \quad \psi_0 + \epsilon_0 \longrightarrow \frac{1}{\sqrt{t}}\psi_0\left(s/\sqrt{t}\right) + \epsilon(s,t)$$

$$\text{for } 0 < t < 1 \quad \text{and } \nexists \lim_{t\downarrow 0^+} \epsilon(\cdot, t)$$

(2) $T(s,t) \xrightarrow{t\downarrow 0^+} T_0(s); A^{\pm}\, B^{\pm}$

About (ii)

Linear momentum density

$$\vec{x} \wedge \vec{w} \qquad \begin{array}{l} \vec{u} \ \text{velocity} \\ \vec{w} \ \text{vorticity} \end{array}$$

In the **(BF)** setting the invariant is given by the density

$$X \wedge T \, ds$$

In our case

$$t > 0 \quad \int \sqrt{t} G \left(s/\sqrt{t}\right) \wedge T \left(s/\sqrt{t}\right) ds = t \int G \wedge T =$$

$$= 2t \int c_0 b \wedge T = 2t \int T_s = 2t(A^+ - A^-)$$

and something similar for $t < 0$.

Hence **Linear Momentum is not preserved!!**:

$$\int \widetilde{X} \wedge \widetilde{T} = 2|t|(A^+ - A^-)$$

About (iii)

Theorem.– (with V. Banica)

$$\left\| \widehat{T}_s \right\|^2_{L^\infty} \begin{cases} < 4\left(1 - e^{-\pi c_0^2}\right) + \epsilon & t = 0 \\ \\ > 4\pi c_0^2 - \epsilon & t > 0 \end{cases}$$

Hence $\left\| \widehat{T}_s(t) \right\|^2_{L^\infty} > \left\| \widehat{T}_s(0) \right\|^2_{L^\infty}$ if $c_0 > 0$ and ϵ small enough.

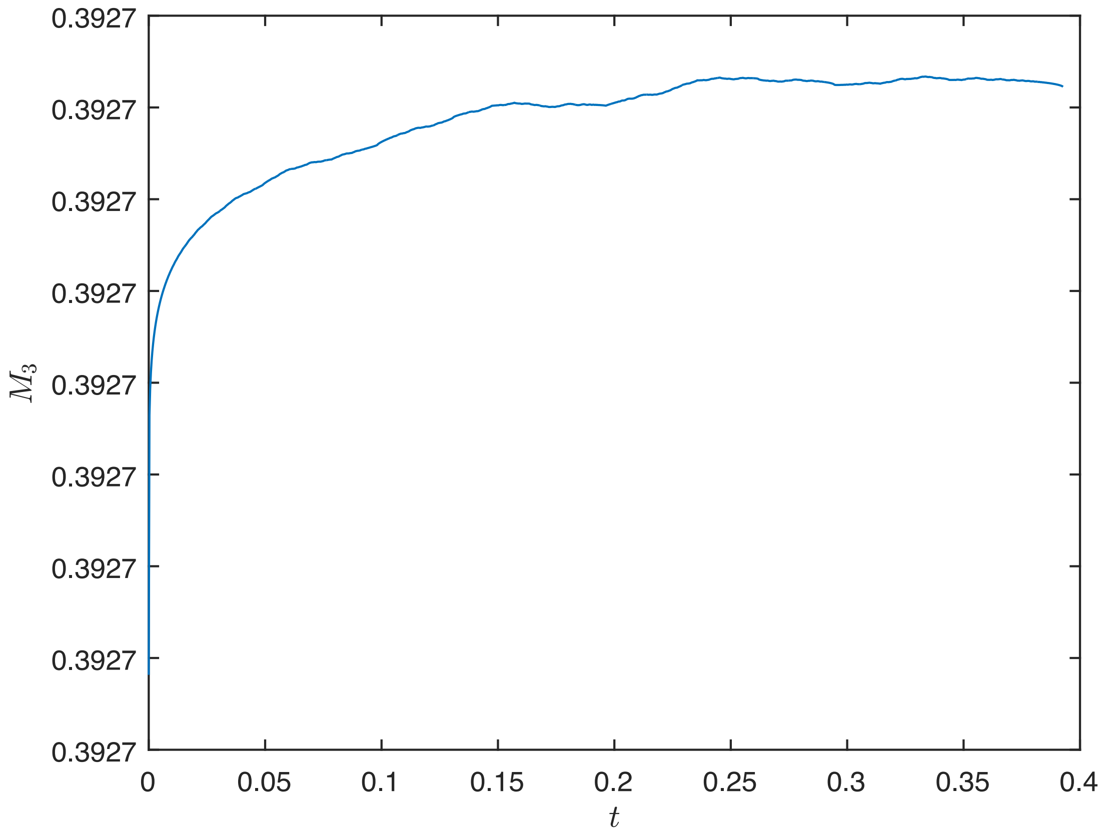
Two Numerical Experiments

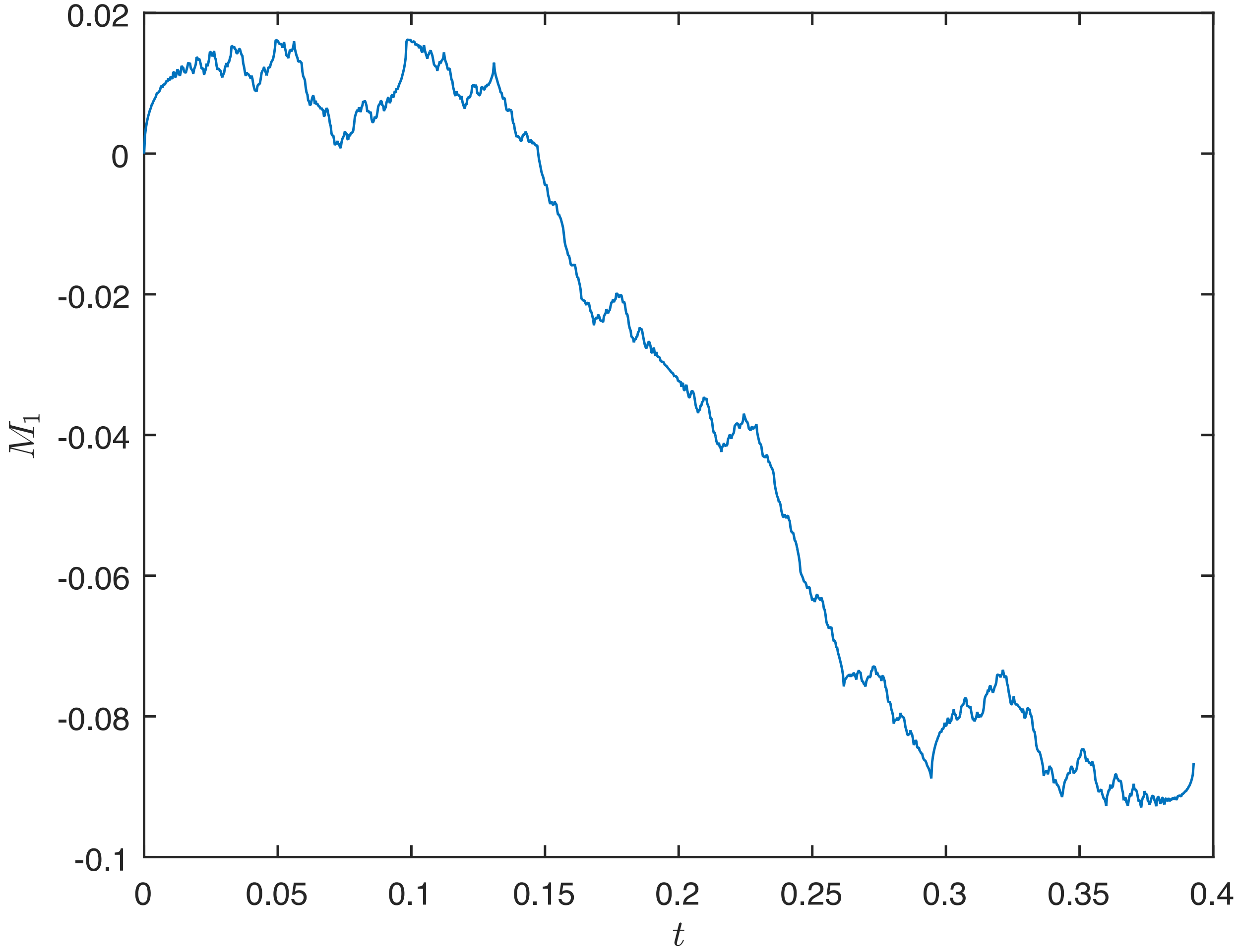
- The first one is that the linear momentum seems to be preserved in the case of a regular polygon. However locally it seems to have an "intermittent" behavior.

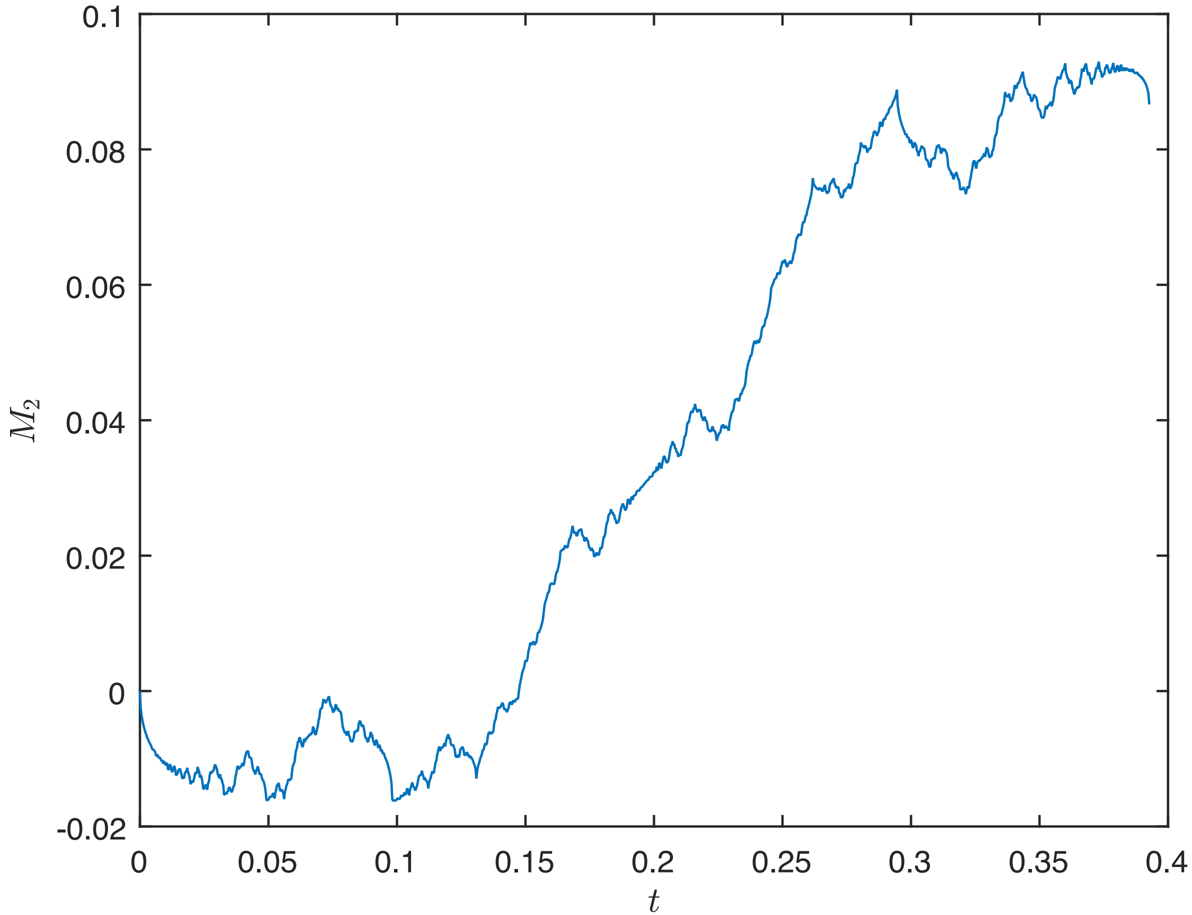
- The second one is about the size of $\left\|\widehat{T}(\cdot,t)\right\|_{L^\infty}^2$ for a regular polygon

  $$\widehat{T}_3 \qquad \widehat{T}_1 + i\widehat{T}_2$$

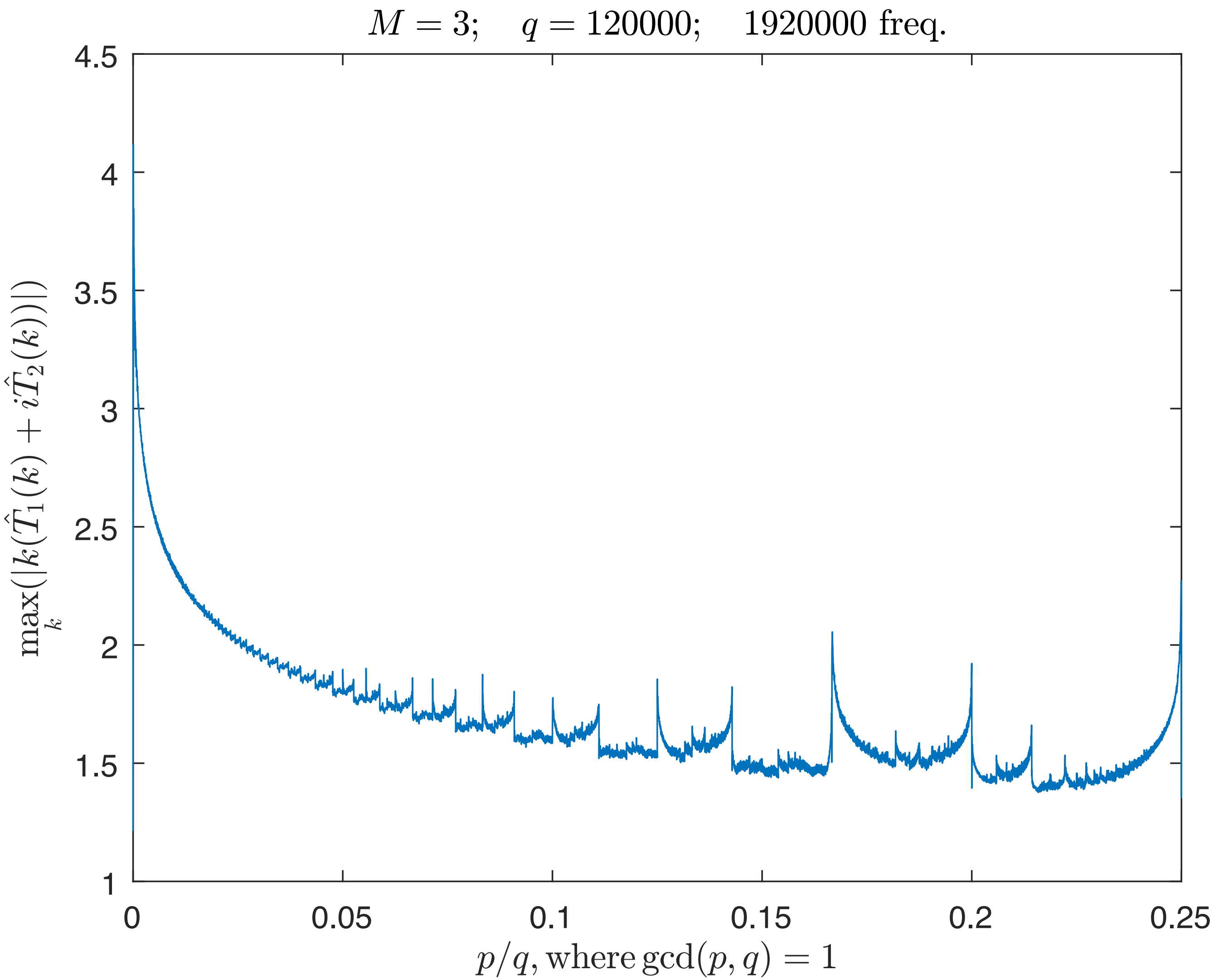
  (they have some symmetry: when one is zero the other one is not).


0.3927
0.3927
0.3927
0.3927
0.3927
0.3927
0.3927
0.3927
0.3927
M_3
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
t

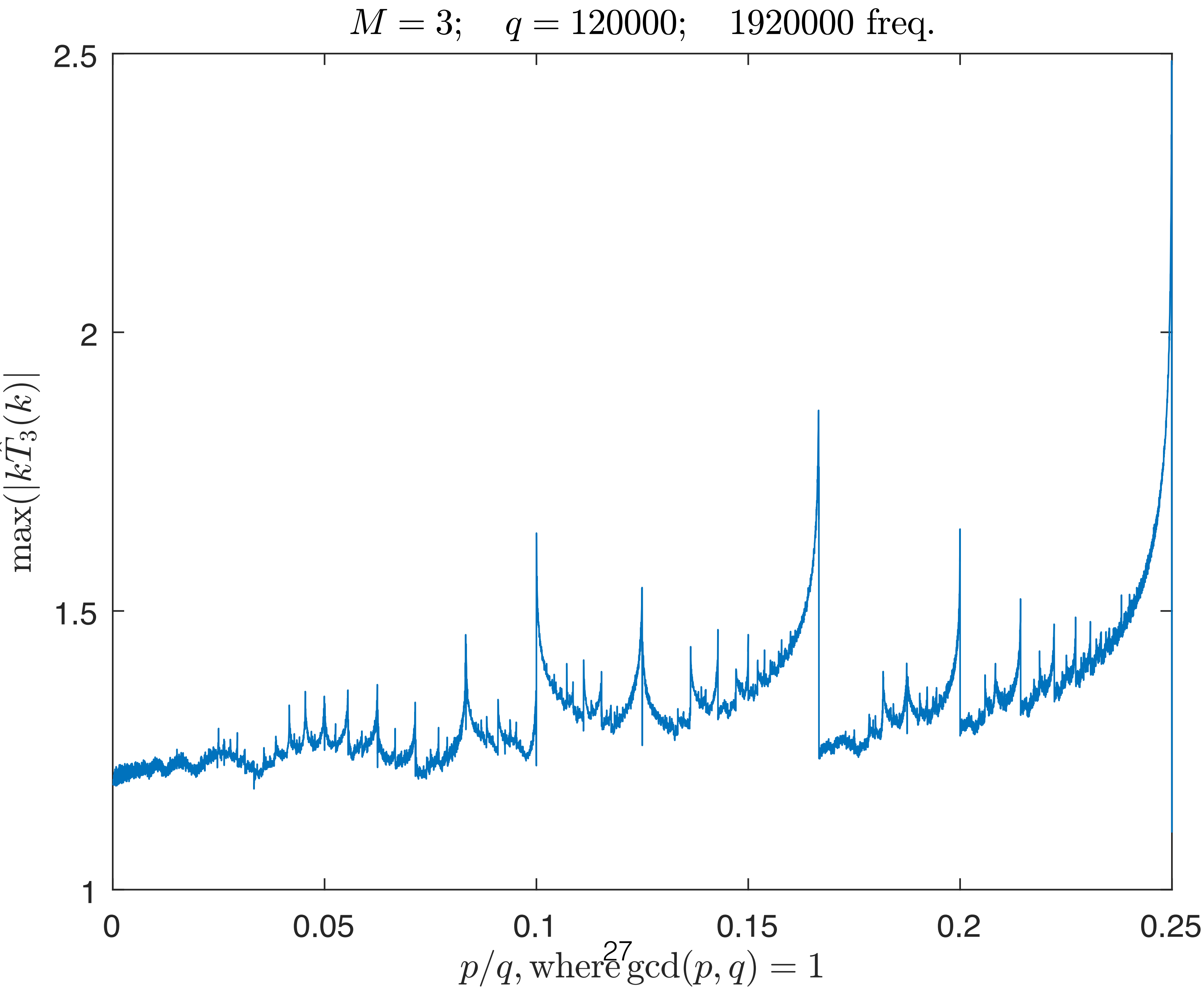

0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
M_1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
t



0.1
0.08
0.06
0.04
0.02
0
-0.02
M_2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
t



$M = 3; \quad q = 120000; \quad 1920000$ freq.
4.5
4
3.5
3
2.5
2
1.5
1
0
0.05
0.1
0.15
0.2
0.25
$\max_k(|k(\hat{T}_1(k) + i\hat{T}_2(k))|)$
p/q, where $\gcd(p, q) = 1$


$M = 3; \quad q = 120000; \quad 1920000$ freq.
2.5
2
1.5
1
$\max(|k\hat{T}_3(k)|)$
0
0.05
0.1
0.15
0.2
0.25
p/q, where $\gcd(p,q) = 1$

$(\mathbf{a = c_0})$

Since

$$T_x(0,x) = (A^- - A^+)\delta_0 + \Re\left(\hat{f}_+(\frac{x}{2})e^{-ia^2\log|x|}\tilde{N}(0,x)\right),$$

we have

$$\left|\widehat{T_x}(0,\xi) - (A^- - A^+)\right| \leq C\int|\hat{f}_+(\frac{x}{2})|dx \leq C\|f_+\|_{H^1}.$$

Recalling that

$$|A^- - A^+|^2 = 4(1 - A_1^2) = 4(1 - e^{-\pi a^2}),$$

we get

$$\left||\widehat{T_x}(0,\xi)|^2 - 4(1 - e^{-\pi a^2})\right| \leq C\|f_+\|_{H^1}.$$

As $4\pi a^2 > 4(1 - e^{-\pi a^2})$, if $\|f_+\|_{H^1}$ is small enough we obtain the second inequality in the statement.

To get the lower bound $4\pi a^2$ for $\|\widehat{T_x}(t)\|_{L^\infty}$ we look at large frequencies. We have

$$\widehat{T_x}(t,\xi) = \widehat{\Re\overline{\psi}N}(t,\xi).$$

Recall that if we denote

$$\tilde{N}(t,x) = N(t,x)e^{i\Phi(t,x)} \quad , \quad \Phi(t,x) = a^2 \log\frac{|x|}{\sqrt{t}},$$

then

$$\exists \lim_{x\to\infty} \tilde{N}(t,x) = N^\infty,\, \Re N^\infty, \Im N^\infty \in \mathbb{S}^2.$$

We can then write

$$\widehat{T_x}(t,\xi) = \int e^{-ix\xi}\, \Re\left(\frac{e^{-i\frac{x^2}{4t}}}{\sqrt{t}}(a+u)\left(\frac{1}{t},\frac{x}{t}\right)e^{-i\phi(t,x)}\,(N^\infty - g(t,x))\right)dx, \qquad (*)$$

with the function $g(t,x)$ defined by $g(t,x) := N^\infty - \tilde{N}(t,x)$ satisfying

$$g(t) \in L^\infty,\, g(t,x) \overset{x\to\infty}{\longrightarrow} 0 \qquad (**)$$

The leading term in (*) is the same one as for the self-similar solutions, with N^∞ instead of B, so computations on it go the same. We get

$$\lim_{|\xi|\to\infty}\left|\int e^{-ix\xi}\,\Re\left(\frac{e^{-i\frac{x^2}{4t}}}{\sqrt{t}}ae^{-ia^2\log\frac{|x|}{\sqrt{t}}}N^\infty\right)dx-2\sqrt{\pi}a\Re\left(e^{i\xi^2t-ia^2\log 2|\xi|\sqrt{t}-i\frac{\pi}{4}}N^\infty\right)\right|=0.$$

Then we notice the following orthogonality relation. By construction $\Re N(t,x)\perp\Im N(t,x)$, that writes

$$\Re\tilde{N}(t,x)e^{-i\phi(t,x)}\perp Im\tilde{N}(t,x)e^{-i\phi(t,x)},$$

and implies

$$\Re\tilde{N}(t,x)\perp Im\tilde{N}(t,x).$$

From this together with (**) it follows that

$$\Re N^\infty\perp\Im N^\infty. \qquad (***)$$

Using the orthogonality relation $(***)$ we have

$$\left| \Re \left(e^{i\xi^2 t - ia^2 \log 2|\xi|\sqrt{t} - i\frac{\pi}{4}} N^\infty \right) \right| = 1,$$

and therefore

$$\lim_{|\xi|\to\infty} \left| \left| \int e^{-ix\xi} \, \Re \left(\frac{e^{-i\frac{x^2}{4t}}}{\sqrt{t}} a e^{-i\phi(t,x)} \, N^\infty \right) dx \right|^2 - 4\pi a^2 \right| = 0.$$

The three term left to estimate in $(*)$ are

$$\int e^{-ix\xi} \frac{e^{-i\frac{x^2}{4t}}}{\sqrt{t}} e^{-i\phi(t,x)} m(t,x) dx, \quad m(t,x) \in \{u\left(\frac{1}{t}, \frac{x}{t}\right), g(t,x), u\left(\frac{1}{t}, \frac{x}{t}\right) g(t,x)\}.$$

These integrals tend to zero as $|\xi|$ goes to infinity. So we have obtained

$$\|\widehat{T_x}(t)\|^2_{L^\infty} \geq 4\pi a^2.$$

periodicity

+

dispersion

=

Talbot Effect

THANK YOU FOR YOUR ATTENTION