Blow up for the1d Cubic NLS and related systems

Luis Vega

eman ta zabal zazu

UPV/EHU

basque center for applied mathematics

F. Merle's conference, IHES, May 24, 2023

Summary

$$\partial_t u = i\big(\partial_x^2 u + |u|^2 u\big) \qquad \textbf{(NLS)}$$

- "Theorem" **(V. Banica, R. Luca, N. Tzvetkov, LV)**

  (i) If
  $$e^{-i\frac{|x|^2}{4}} u(x,1) \text{ is } 4\pi\text{–periodic in } H^{0^+}(2\mathbb{T})$$
  then $u(x,t)$ "blows up" at $t=0$.

  (ii) If
  $$e^{i\xi^2}\widehat{u}(\xi,1) \text{ is } 2\pi\text{–periodic in } H^{0^+}(\mathbb{T})$$
  then $u(x,t)$ "blows up" at $t=0$.

Nevertheless, u can be continued for $t \leq 0$ as a **"**geometric**"** solution.

Two (universal?) examples:

(i) $$E_a(x,t) = \frac{a}{\sqrt{it}} e^{i\frac{|x|^2}{4t}} e^{i|a|^2 \lg t}$$

- $e^{-i\frac{|x|^2}{4}} E_a(x,1) = a$
- Observe that $\lambda E_a(\lambda x, \lambda^2 t) = e^{i|a|^2 \lg \lambda} E_a$
- $\lim_{t \downarrow 0} e^{-i|a|^2 \lg t} E_a(x,t) = a\delta$ **(Kenig – Ponce – V)**

- **Ozawa's** (long range) scattering result

  - If $f \in \mathcal{S}(\mathbb{R})$ (small) then

$$\frac{1}{\sqrt{it}} e^{i\frac{|x|^2}{4t}} f\left(\frac{x}{t}\right) e^{i\left|f\left(\frac{x}{t}\right)\right|^2 \lg t} + R(x,t)$$

    solves **NLS** with $R(x,t) \to 0$ when $t \to \infty$.

  - Take $f_\lambda(\xi) = f(\lambda\xi)$, the corresponding solution converges at least formally to E_a with $a = \int_{-\infty}^{\infty} \check{f}(x)dx$ when $\lambda \downarrow 0$.

  - The ode

$$\frac{d}{dt} v(\xi,t) = \frac{i}{t}|v|^2 v$$

    gives the leading profile.

(ii) $$D_{c_M} = c_M \sum_{j=-\infty}^{\infty} e^{-itj^2+ijx} e^{i\Phi_{c_M}(t)}$$

$$\Phi_{c_M} = c_M^2 \left(\frac{\lg t}{4\pi} + \int_t^\infty \sum_{m\neq 0} \# r(m) e^{-im\tau} \frac{d\tau}{4\pi\tau} \right).$$

(Banica – Bravin – V)

Observe that

$$\begin{aligned} \frac{1}{2\pi} \sum_j e^{-4\pi^2 itj^2+2\pi ijx} &= e^{it\partial_x^2} \sum_j \delta(x-j) \\ &= \sum_j \frac{1}{\sqrt{it}} e^{i\frac{(x-j)^2}{4t}} \\ &= \frac{1}{\sqrt{it}} e^{i\frac{|x|^2}{4t}} \sum e^{-i\frac{x}{2t}j} e^{i\frac{j^2}{4t}}. \end{aligned}$$

Hence at $t=1$

$$\frac{2\pi}{c_M} e^{-i\frac{|x|^2}{4}} D_{c_M}(2\pi x, 4\pi^2) = \sum_j e^{-i\frac{x}{2}j+i\frac{j^2}{4}} \;.$$

$\#\mathbf{r(m)} = \ ?$

- $u(x,t) = \sum_j A_j(t) e^{i\Phi_j(t)} e^{it\partial_x^2} \delta(x-j) \qquad 0 < t \leq 1$

$$\Phi_j(t) = |a_j|^2 \lg t \quad A_j = a_j + R_j(t) \qquad R_j(t) \to 0 \quad \text{if} \quad t \to 0$$

- $v(y,\tau) = \frac{1}{\sqrt{i\tau}} e^{i\frac{|y|^2}{4\tau}} \overline{u}\left(\frac{y}{\tau}, \frac{1}{\tau}\right)$

$$iv_\tau + v_{yy} + \frac{1}{\tau}\left(|v|^2 - |a_j|^2\right) v = 0$$

- $v(y,\tau) = \sum e^{i\tau j^2} B_j(\tau) e^{ijy}$

$$B_j(\tau) = \overline{A_j}\,(1/\tau)\,.$$

- $$i\partial_t B_j = -\frac{1}{\tau}\sum_{m\neq 0}\sum_{r(m)} e^{-imt} e^{-i\wedge_m \lg t} B_{j_1}\overline{B_{j_2}} B_{j_3} + \frac{1}{\tau}\left(|B_j|^2 - |a_j|^2\right) B_j$$

$$\wedge_m = |a_{j_1}|^2 - |a_{j_2}|^2 + |a_{j_3}|^2 - |a_j|^2$$

$r_m = \{k \in \mathbb{Z} \,:\, 2k/m\}$ $\quad m = 2(j_1 - j_2)(j - j_1)$ $\quad \mathbf{\# r(m) = card\, r(m)}$

Hence the **"**Theorem**"** is a result that interpolates example **(i)** and example **(ii)**

$$e^{-i\frac{|x|^2}{4}} u(x,1) = \sum_j A_j(1) e^{ij\frac{x}{2} + i\frac{j^2}{4}} \quad .$$

Theorem For initial data

$$u_0(x) = e^{i\frac{x^2}{4}} f(x),$$

at time $t=1$, with f a periodic function in $H^s(0,4\pi)$ and $s \in (0, \frac{1}{2})$ there exists a unique solution of the 1D cubic NLS equation that blows up in finite time in the following sense: the solution belongs for $t \in (0,1]$ to the functional framework

$$w(t,\xi) := e^{it\xi^2} \hat{u}(t,\xi) \text{ is } 2\pi\text{-periodic},$$

$w \in C\big((0,1]; H^s(\mathbb{T})\big)$, and falls out of this framework at $t=0$. More precisely, if $A_j(t)$ are the Fourier coefficients of $w(t)$, then

$$\sup_{t\in(0,1]} \|\{A_j(t)\}_{j\in\mathbb{Z}}\|_{l^{2.s}} \leq C(\|f\|_{H^s(0,4\pi)}),$$

and there exists a sequence $\{a_j\}_{j\in\mathbb{Z}} \in l^{2,s}$ such that

$$|A_j(t) - e^{i(|a_j|^2 - 2\|f\|^2_{L^2(0,4\pi)})\log t} a_j| \leq C(\|f\|_{H^s(0,4\pi)})\, t, \quad \forall j \in \mathbb{Z}, t \in (0,1).$$

Finally the solution can be written as

$$u(t,x) = \sum_{j\in\mathbb{Z}} A_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}} = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{4t}} \overline{v}\left(\frac{x}{t}, \frac{1}{t}\right).$$

Hence the **“**Theorem**”** is a result that interpolates example **(i)** and example **(ii)**

$$e^{-i\frac{|x|^2}{4}}u(x,1) = \sum A(1)e^{ij\frac{x}{2}+i\frac{j^2}{4}}$$

- $a_j \in l^p \quad p < +\infty$ at least assuming smallness. $(\mathbf{Bravin - V})$

Observe that for solutions of the free Schrödinger equation

$$iu_t + u_{xx} = 0,$$

it is immediate to see that if $e^{i\xi^2}\widehat{u}(1,\xi)$ is $2\pi-$periodic then $e^{it\xi^2}\widehat{u}(t,\xi)$ is also $2\pi-$periodic, for every $t \in \mathbb{R}$. In fact

$$e^{it\xi^2}\widehat{u}(t,\xi) = \widehat{u}_0(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{ik\xi}.$$

It is not so obvious that the periodicity of

$$w(t,\xi) := e^{it\xi^2}\widehat{u}(t,\xi),$$

holds, at least formally, for solutions of the nonlinear problem. However let us notice that the equation for ω is:

$$w_t(\eta,t) = \frac{i}{8\pi^3} e^{-it\eta^2} \int e^{it(\xi_1^2-\xi_2^2+(\eta-\xi_1+\xi_2)^2)} w(\xi_1,t)\bar{w}(\xi_2,t)w(\eta-\xi_1+\xi_2,t)\, d\xi_1 d\xi_2.$$

The phase can be factorized as $(\xi_1-\xi_2)(\xi_1-\eta)$, so it is invariant under translations. Thus the equation of ω is compatible with periodicity; the solutions we construct are in this framework.

Functional Setting

Symmetries $(x_0, t_0, \xi_0, \lambda)$

- Translations (x_0, t_0) $\quad t_0 \to e^{it_0\xi^2}$
- $\xi_0 :$ Galilean Invariance
- $\lambda :$ scaling

We look for a functional setting that remains invariant under (x_0, ξ_0, λ).

In particular "ξ_0" implies that translations in Fourier space should be harmless.

Previous Results

- Sobolev spaces:

$s \geq 0$ **Cazenave – Weisler, Ginibre – Velo, Tsutsumi, Bourgain.**

$s < 0$ Ill posedness: **Kenig – Ponce – V, Christ– Coliander – Tao, Carles – Kappelev, Kishimoto, Oh.**

$-1/2 < s < 0$ Well posedness: **Koch – Tataru, Killip– Visan – Zhang, Harrop-Griffiths – Killip – Visan**

- Fourier–Lebesgue $(\widehat{u}_0 \in L^p, p < \infty)$:

Vargas – V, Grunrock – Herr.

- At the critical level:

Banica – V, Bravin – V, Guérin.

About the proof

- $$i\partial_t B_j = -\frac{1}{\tau}\sum_{m\neq 0}\sum_{r(m)} e^{-imt}e^{-i\wedge_m \lg t} B_{j_1}\overline{B_{j_2}}B_{j_3} + \frac{1}{\tau}\left(|B_j|^2 - \mathbf{?}\right)B_j$$

$$\wedge_m = |\mathbf{A_{j_1}(t)}|^{\mathbf{2}} - |\mathbf{A_{j_2}(t)}|^{\mathbf{2}} + |\mathbf{A_{j_3(t)}}|^{\mathbf{2}} - |\mathbf{A_j(t)}|^{\mathbf{2}}$$

$$r_m = \{k \in \mathbb{Z} \,:\, 2k/m\} \qquad m = 2(j_1 - j_2)(j - j_1)$$

- $$\frac{d}{d\tau}|B_j(t)|^2 = 2\mathrm{Re}\,\frac{i}{\tau}\sum_{m\neq 0}\sum_{r(m)} e^{-imt}e^{-i\wedge_m \lg t} B_{j_1}\overline{B_{j_2}}B_{j_3}\overline{B_j}$$

Assuming $\sum |B_j|^2\langle j\rangle^{\epsilon} < +\infty$ RHS is (conditionally) integrable.
Hence:

$$|B_j(t)|^2 \to |a_j|^2$$

$$e^{i|a_j|^2 \lg t} B_j \to a_j$$

$$? \quad = 2M - |a_j|^2 \qquad M = \sum_j |A_j(t)|^2 = \sum |a_j|^2.$$

- Intermittency: $a_j = \dfrac{1}{N^{1/2}} \qquad |j| \leq N.$

The related systems

- $\chi_t = \chi_x \wedge \chi_{xx}$ **(Binormal Flow)**

  Related to the evolution of vortex filaments (LIA)

Example (i) One corner **(Gutierrez – Rivas – V)**

$$\chi_a(0,x) = \begin{cases} A^+ x & x \geq 0 \\ A^- x & x \leq 0 \end{cases}$$

Example (ii) Regular polygons (M–sides)

(Jerrard – Smets, De la Hoz – V, Banica – Bravin – V)

$\chi_x = T \qquad |T| = 1$ **Landau–Lipschitz equation**

- $T_t = T \wedge T_{xx}$ Ferromagnetism (Heisenberg chain)

The conection (Hasimoto'71)

If

$$\begin{aligned} T_x &= \qquad\qquad \alpha e_1 + \beta e_2 \\ e_{1x} &= -\alpha T \\ e_{2x} &= -\beta T, \end{aligned}$$

then

$$u = (\alpha + i\beta) \qquad \text{satisfies}$$

$$\partial_t u = i\big(\partial_x^2 u + (|u|^2 - A(t))u\big)$$

$$A(t) \in \mathbb{R} \qquad\qquad \text{(Gauge invariance).}$$

- The geometric structure implies that the singularities of u are "harmless" and the solution can be continued as a geometric object.

- There is however a blow up of the frame of the normal plane $N(x,t) = (e_1 + ie_2)(x,t)$.

- The selfsimilar solutions associated to E_a are crucial in this step:

The selfsimilar solutions associated to E_a are crucial in this step:

$$\chi_a = \sqrt{t}\, G_a\left(x/\sqrt{t}\right)$$

$$\xi^2 \widehat{G_a} = \widehat{Z}(\xi^2)$$ **(De la Hoz – V)**

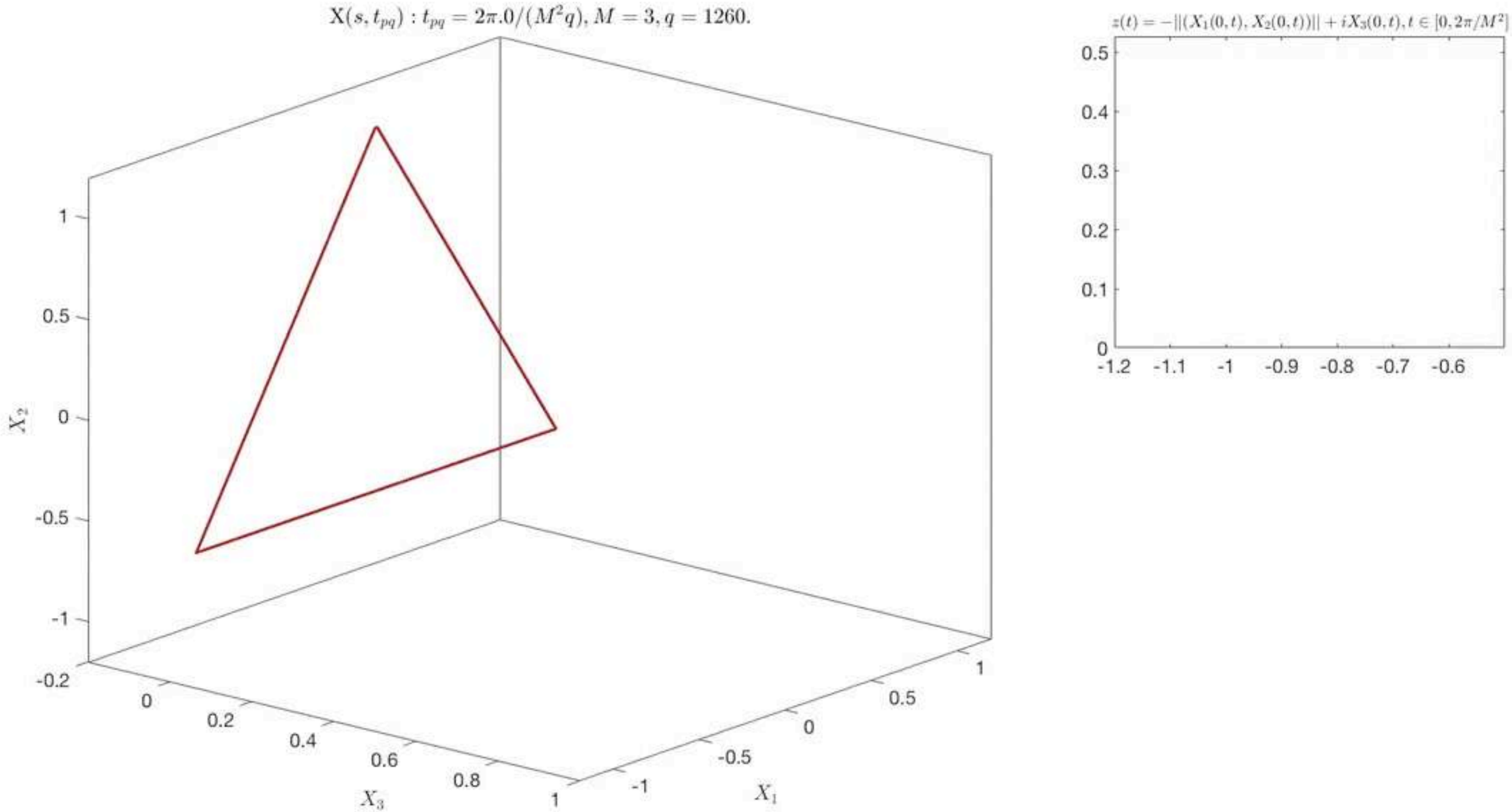
$$\widehat{Z}'' + \left(1 - \frac{c_0^2}{\eta}\right)\widehat{Z}(\eta) = 0, \quad \eta > 0.$$

$$\widehat{\partial_x T_a}(\xi, t) = \widehat{Z}(t\xi^2)$$

$\widehat{T_a}(\xi, 1)$:

- Jump $(T(+\infty) - T(-\infty))$ at $\xi = 0$
- Re $\left(e^{-ia^2 \lg|\xi| + i\xi^2}\widetilde{N}(\pm\infty)\right) \quad \xi \to \pm\infty$

The matching problem: complete integrability.



$X(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
X_2
1
0.5
0
-0.5
-1
-0.2
0
0.2
0.4
0.6
0.8
1
X_3
-1
-0.5
0
0.5
1
X_1
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$
0.5
0.4
0.3
0.2
0.1
0
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6

Resonances and cascade of energy for T

The equation for T (SM) is

$$T_t = T \wedge T_{xx}$$

$$\partial_t |T_x|^2 = \partial_x(\text{flux})$$

Hence $|T_x|^2 dx$ is a natural energy density.

Moreover

$$\int_0^{2\pi} |v(\xi,t)|^2 d\xi = \lim_{n\to\infty} \int_{2\pi n}^{2\pi(n+1)} \left|\widehat{T_x}(\xi,t)\right|^2 d\xi$$

Theorem **(Banica – V 2021)**

Assume

$$\begin{cases} a_{-1} = a = a_{+1} & a \neq 0 \\ a_j = 0 & \text{otherwise} \end{cases}$$

Then there exists $c > 0$

$$\sup_{\xi} \left| \widehat{T_x}(\xi, t) \right|^2 \geq c |\lg t| \qquad t > 0.$$

- **Colliander – Keel – Staffilani – Takaoka – Tao**
- **Hani – Pausader – Tzvetkov – Visciglia**

- This cascade can be understood associated to a linear problem.

$$(T, e_1, e_2) \quad \text{orthonormal frame}$$

$$\begin{aligned} T_x &= \alpha e_1 + \beta e_2 \\ e_{1x} &= -\alpha T \\ e_{2x} &= -\beta T \end{aligned}$$

$$\begin{aligned} T_t &= - \beta_x e_1 + \alpha_x e_2 \\ e_{1t} &= \beta_x T + (|u|^2 - M(t)) e_2 \\ e_{2t} &= -\alpha_x T - (|u|^2 - M(t)) e_1 \end{aligned}$$

$$\alpha + i\beta = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{4t}} \overline{v}\left(\frac{x}{2t}, \frac{1}{t}\right) = u(x,t)$$

(Chevillard et al.)

About the proof

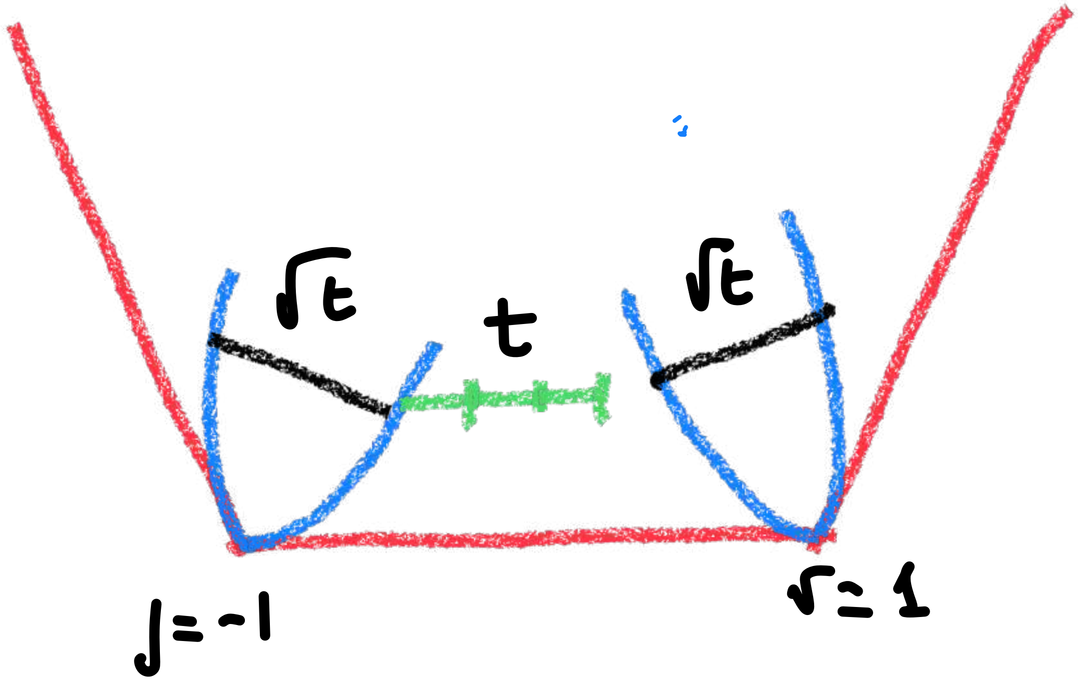
$$T_x = \mathrm{Re}\,\overline{u}N$$

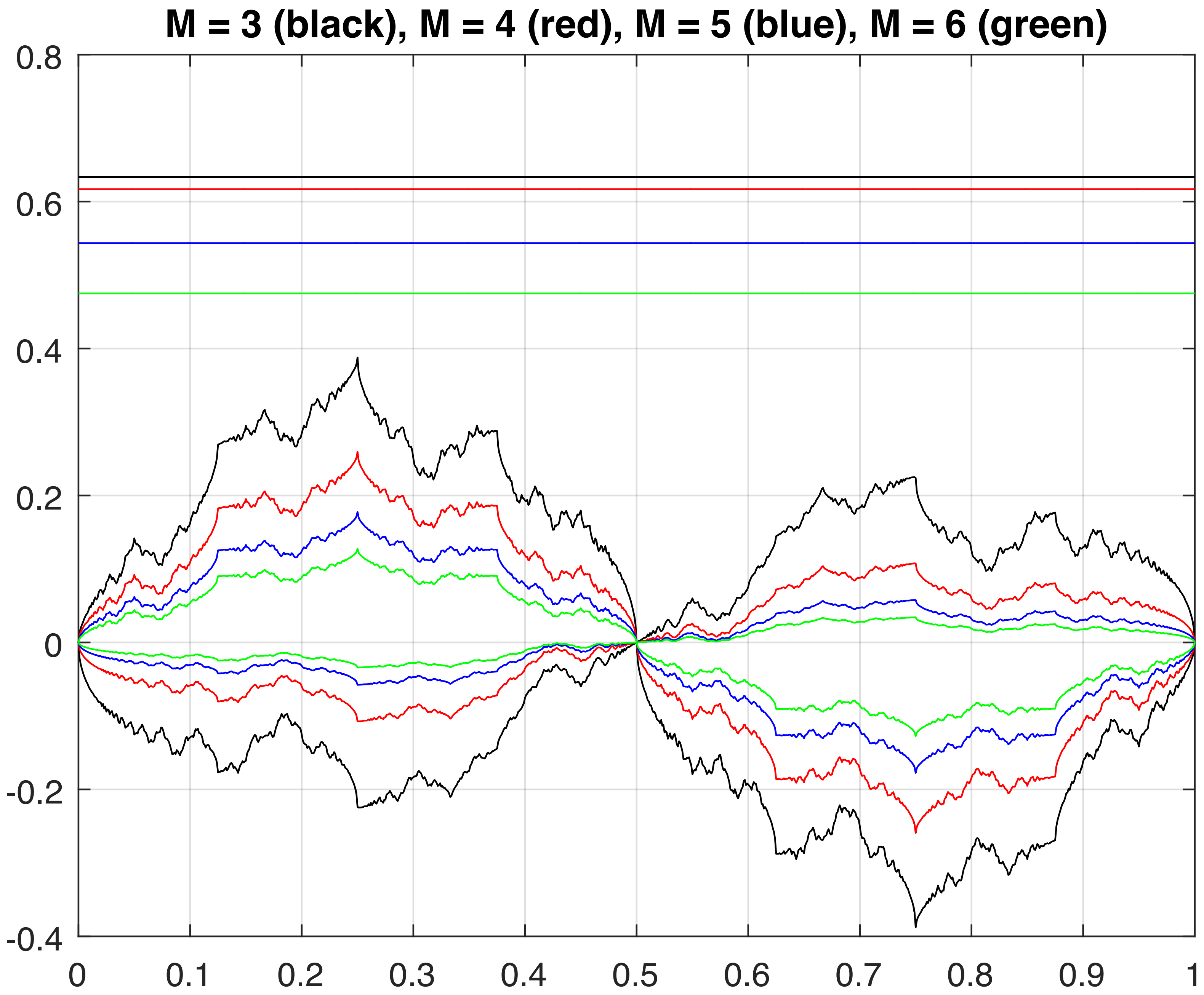
$$N_x = uT$$

$$u = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{4t}} \text{ nice,} \quad T \sim T^{\pm\infty} \quad x \to \pm\infty, \quad i\frac{2t}{x-r} \quad \partial_x u \sim u.$$

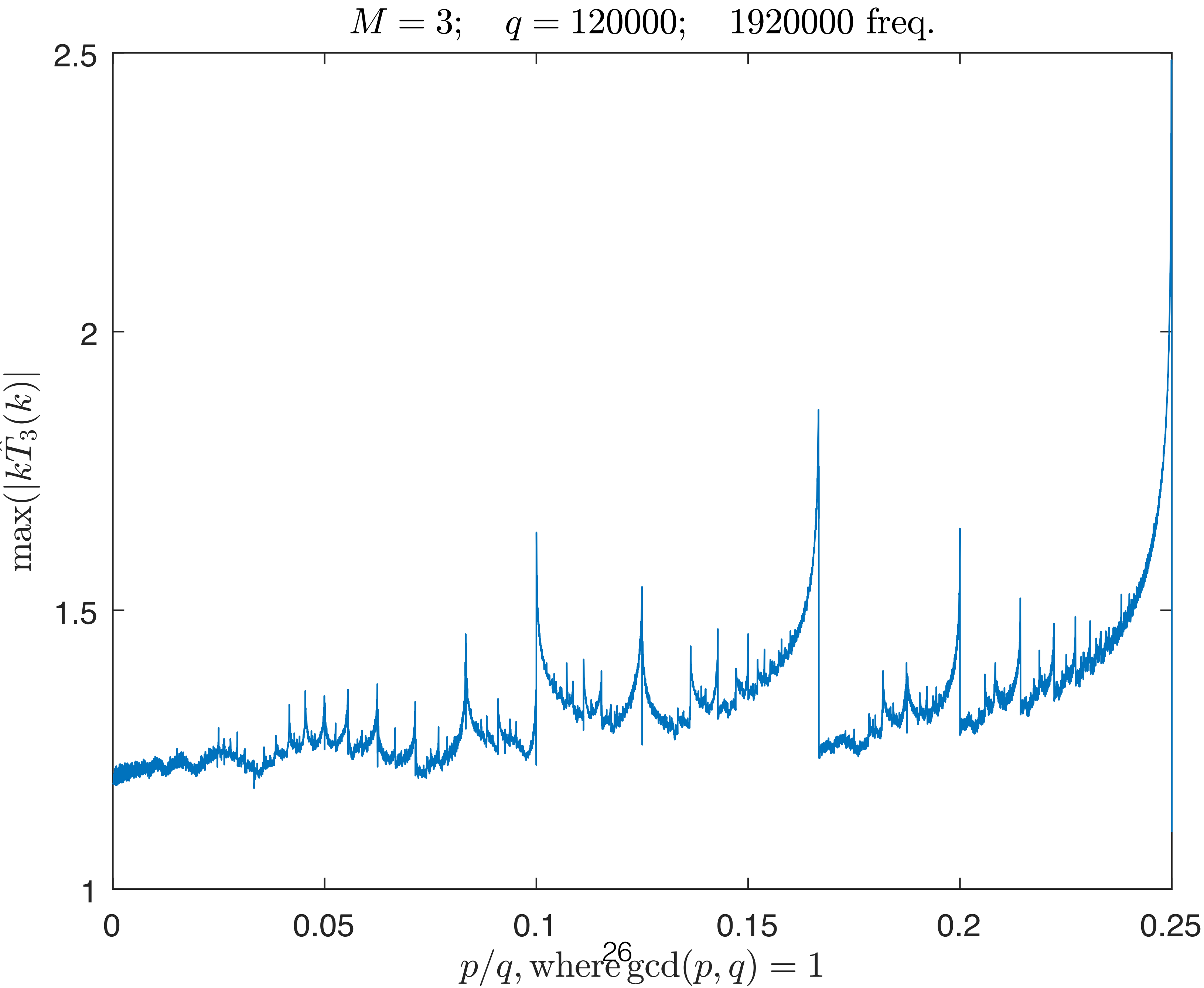
$$N \simeq \sqrt{t} \sum a_r \frac{i}{x-r} e^{i\frac{(x-r)^2}{4t}} T(x,0) \quad \text{if} \quad x-r > \sqrt{t}$$

$$T_x \simeq \sum_{j,r} \overline{a_j} a_r \frac{i}{x-r} e^{i\frac{x(r-j)}{2t}} T(r,0) \quad \text{for} \quad \frac{1}{2} > x-r > \sqrt{t}$$


√t
t
√t


M = 3 (black), M = 4 (red), M = 5 (blue), M = 6 (green)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Energy Transfer (with De la Hoz)



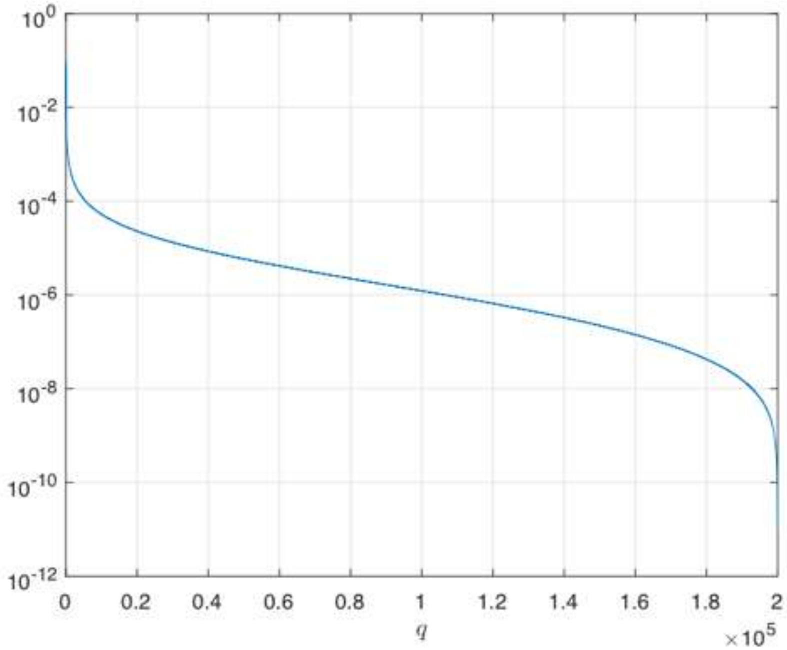
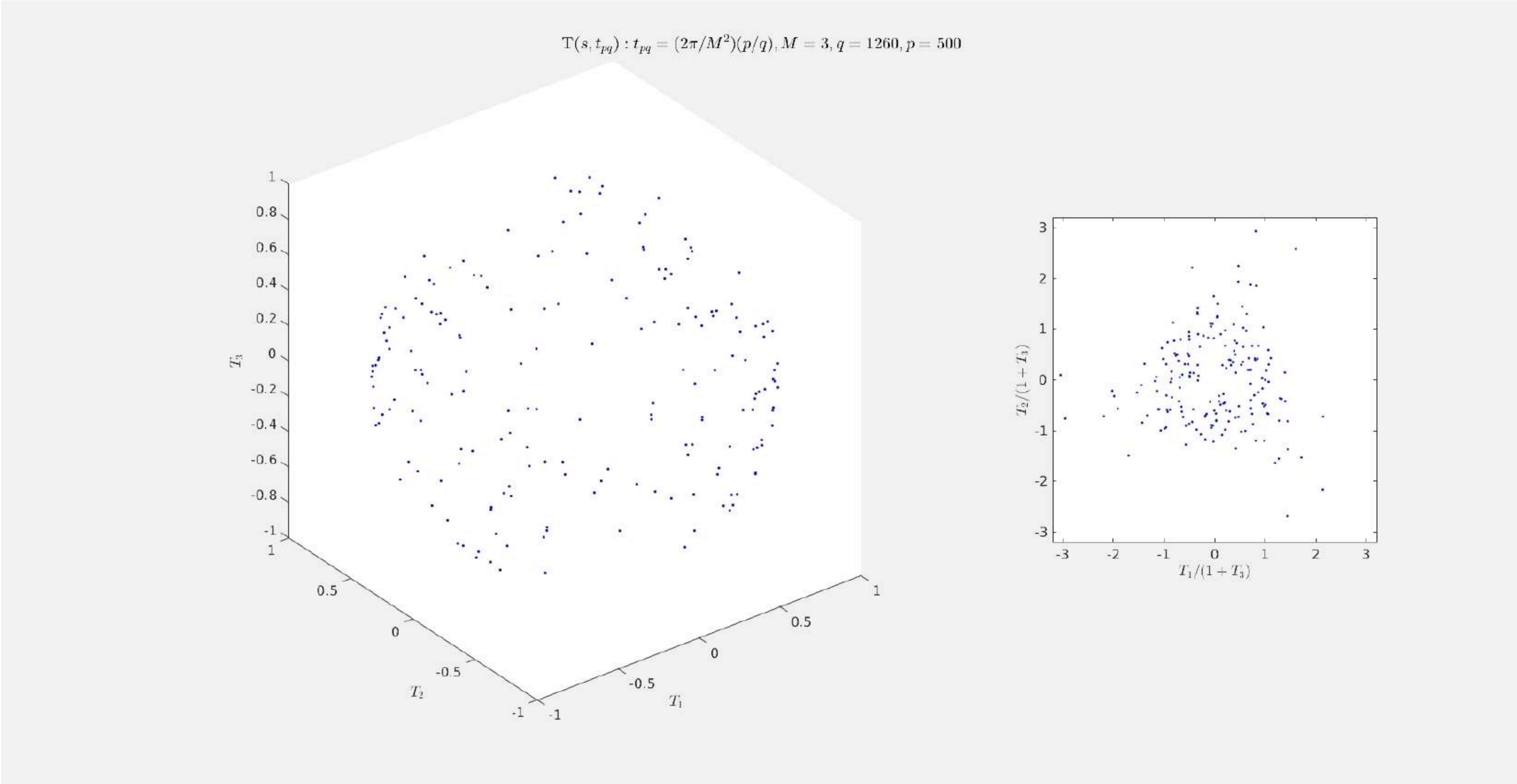


Figure 10: $|\sqrt{2}\max_{t_{pq}} \|\widehat{T_{1,s}}(t_{pq})\|_\infty - a\ln(q) - b|$, for $a = 0.258039752572419$, $b = 0.152992510344641$.


$\mathrm{T}(s, t_{pq}) : t_{pq} = (2\pi/M^2)(p/q), M = 3, q = 1260, p = 500$
T_3
T_2
T_1
$T_2/(1+T_3)$
$T_1/(1+T_3)$

THANK YOU FOR YOUR ATTENTION