Desingularization of the Biot-Savart integral and the Localised Induction Approximation (LIA)

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Summary

1.- The setting

2.- The movies

3.- The binormal flow **(LIA)** $\left\{ \begin{array}{l} \text{Intermittency} \\ \\ \text{Multifractality} \end{array} \right.$

$$\psi(x,t) = c_M \sum_k e^{itk^2+ikx}$$

4.- The desingularization of the **Biot-Savart** integral

1.- The Setting

One of the most fundamental elements in fluid mechanics are the fluid filaments. These are structures around which vorticity ω tends to concentrate. Their formation, structure, and evolution has been a central theme since the early days of fluid dynamics and the main results bear the names of Helmholtz, Lord Kelvin, Kirchhoff, etc.

A vortex filament may be considered as a singular initial data for the vorticity in the form

$$\omega(\mathbf{x},0) = \frac{\Gamma}{2\pi}\delta_\chi \mathbf{T}(s,0)$$

where δ_χ is the Dirac delta measure supported in the curve χ and $\mathbf{T}(s,0)$ the tangent vector.

When a vortex filament is curved and its motion affected by the velocity field created by itself,

$$\mathbf{v} = \nabla \times \omega \qquad \nabla \cdot \mathbf{v} = 0,$$

one could in principle think of a closed-form equation to describe the evolution of the filament with the velocity field given by the Biot-Savart law applied to the vorticity.

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{4\pi} \int \omega(\mathbf{x}', t) \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}'.$$

This yields a singular integral and an infinite velocity field at the filament. By replacing the zero-thickness filament by a filament of radius ε, the singularity is removed and the velocity is given by

$$\mathbf{v} = -\frac{\Gamma}{4\pi} \kappa(s, t)(\log \varepsilon)\mathbf{b}(s, t) + O(\varepsilon^0)$$

- Γ is the velocity circulation around the vortex filament (or vortex filament strength),

- $\kappa(s,t)$ is the local curvature with s being the arc-length parameter,

- $\mathbf{b}(s,t)$ is the binormal vector.

By neglecting the $O(\varepsilon^0)$ one introduces the celebrated localized induction approximation **(LIA)** and to the so-called binormal flow dynamics for the vortex filament **(Da Rios in 1906)**

A natural question is then what is the later evolution of the filament according to **Navier-Stokes** equations and whether there is a connection (or not) with the binormal flow. Although solutions are smooth at later times (**Giga and Miyakawa, 1989**; Γ is sufficiently small), one can expect them to be concentrated around a curve at least for short enough times. **What is the evolution of such a curve?** A natural candidate (joint work with **M. A. Fontelos**) is the solution of the (normalized) binormal flow:

$$\frac{d\chi(s,t)}{dt} = -\frac{\Gamma}{4\pi}\kappa(s,t)\log(\nu t)^{\frac{1}{2}}\mathbf{b}(s,t)$$

where Γ is the vortex filament strength, $\kappa(s,t)$ the local curvature and $\mathbf{b}(s,t)$ the binormal vector.

Using Frenet equations and after renormalizing the time we get the binormal flow **(BF/LIA)**

$$\chi_t = \chi_s \times \chi_{ss} = \kappa \mathbf{b}.$$

3.- The binormal curvature flow

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb \quad c$: curvature $\quad b$: binormal

(SM) • $\chi_s = T$ Schrödinger map $\qquad T_t = T \wedge T_{ss}$

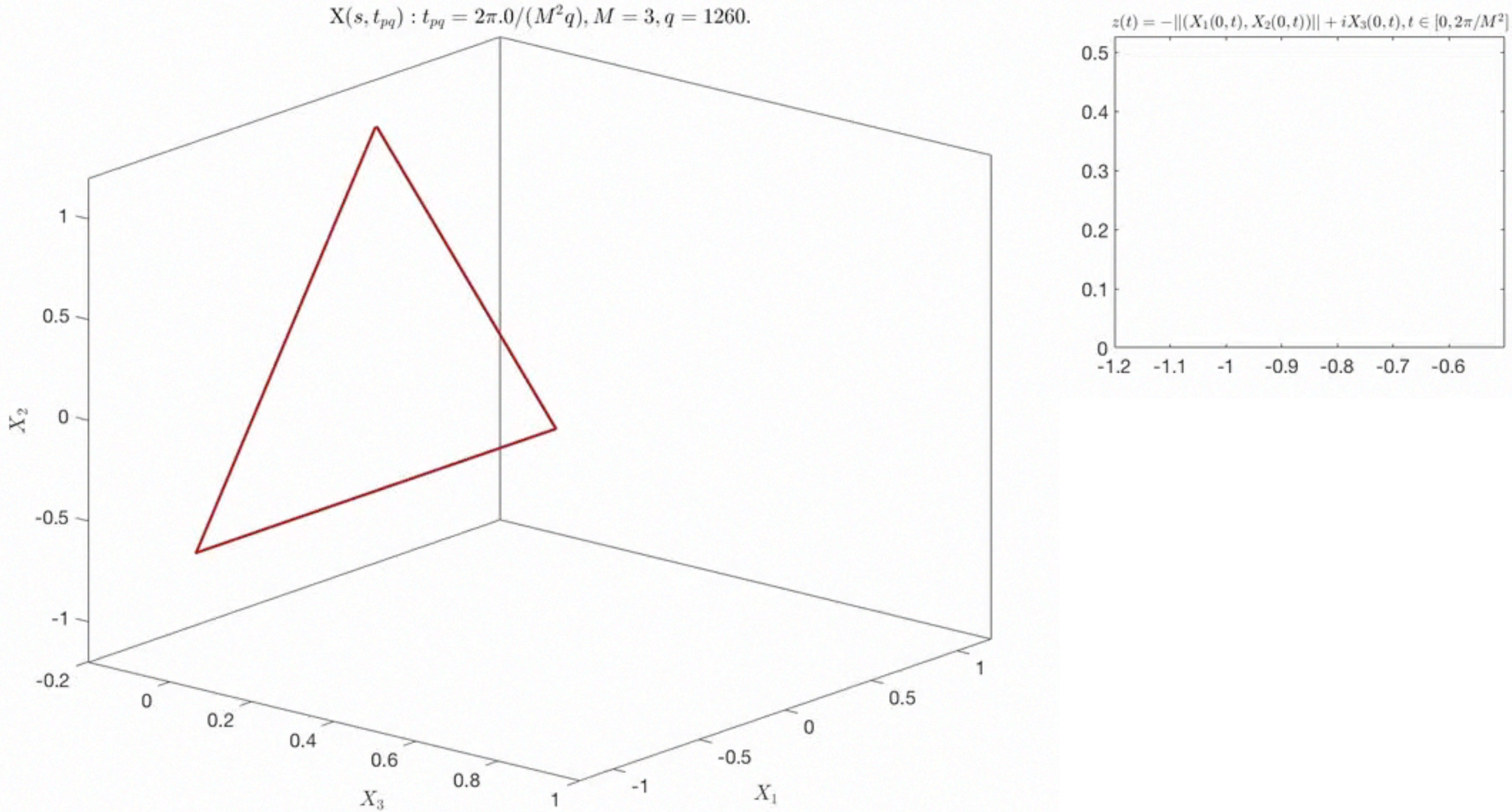
$$
\begin{aligned}
T_s &= \quad cn \\
n_s &= -cT + \tau b \\
b_s &= -\tau n
\end{aligned}
$$

Examples:

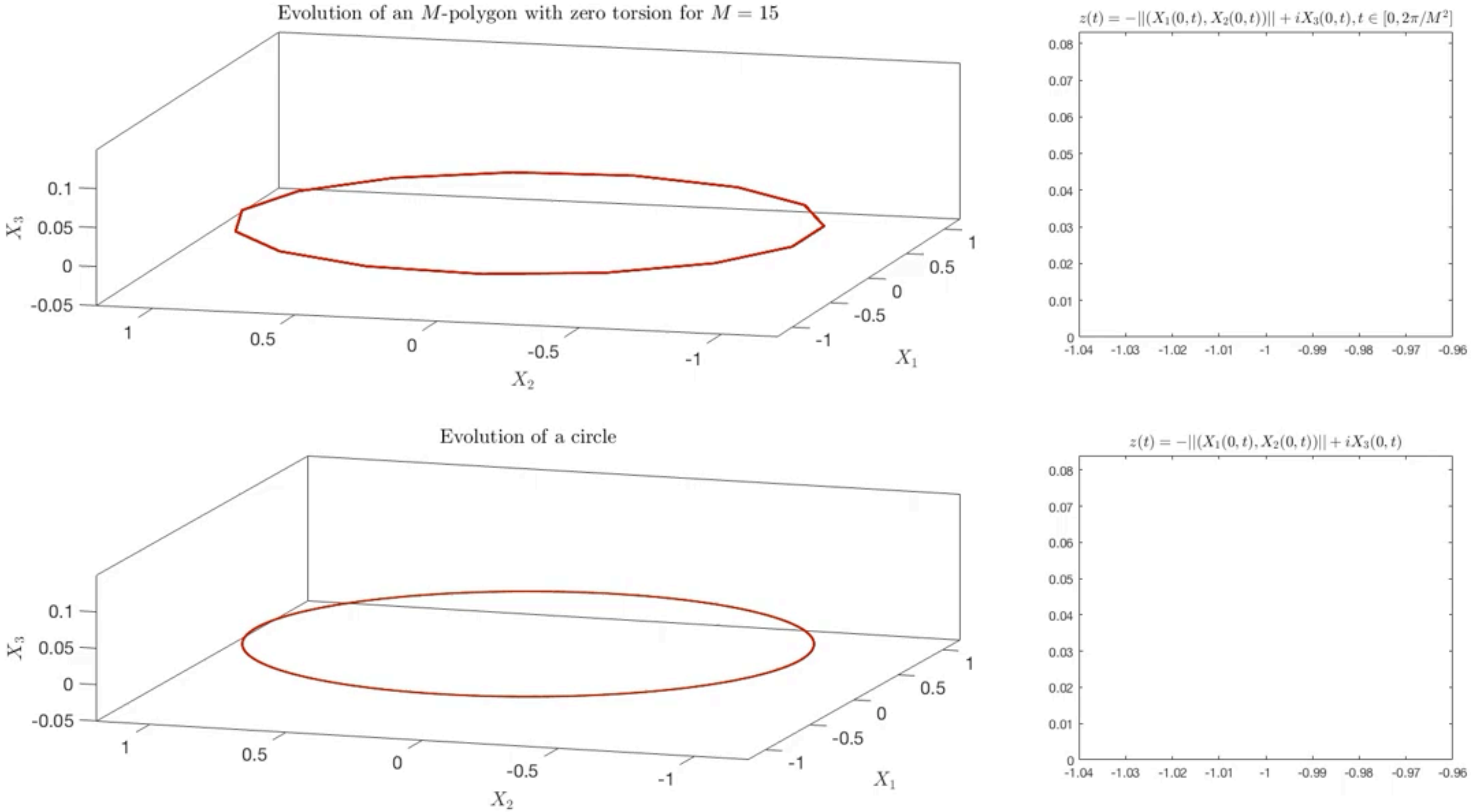
(i) Straight lines.

(ii) Circles.

(iii) Helices.



$X(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
X_2
1
0.5
0
-0.5
-1
-0.2
0
0.2
0.4
0.6
0.8
1
X_3
-1
-0.5
0
0.5
1
X_1
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$
0.5
0.4
0.3
0.2
0.1
0
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6


Evolution of an M-polygon with zero torsion for $M = 15$
$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t), t \in [0, 2\pi/M^2]$
Evolution of a circle
$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t)$
X_1
X_2
X_3

Riemann's function

$$\varphi_R(t) = \sum_{j=1}^{\infty} \frac{\sin(tj^2)}{j^2} \qquad (\sim 1860)$$

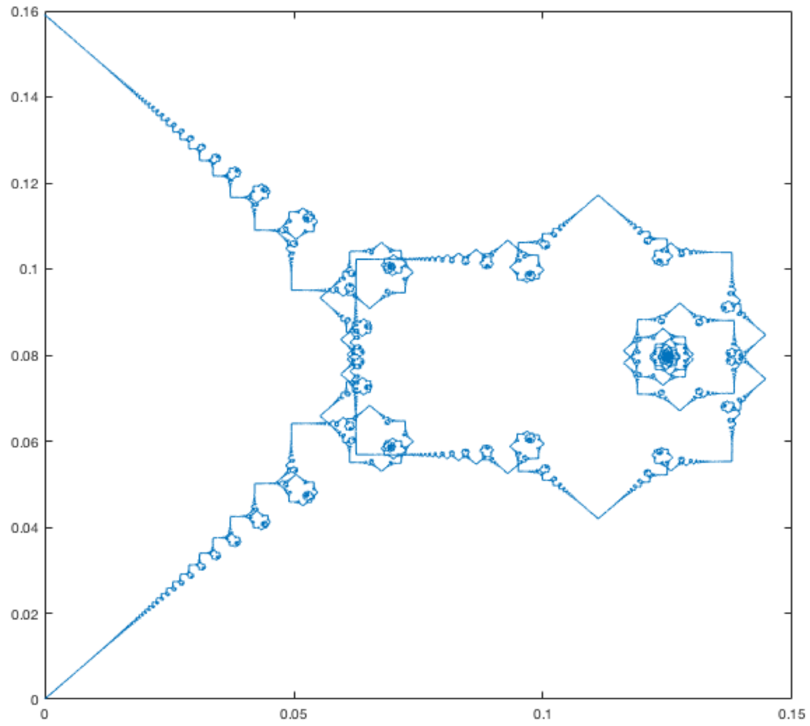
- Hardy 1915 (H–Littlewood circle method)
- Gerver 1960 (Riemann was wrong)

At $t_{p,q} = \pi p/q$ $\quad p, q$ odd, the derivative exists and is $-1/2$

$$\varphi_D(t) = \sum_{j=1}^{\infty} \frac{e^{itj^2}}{ij^2} \qquad \text{Duistermaat 1991}$$

Fractal behavior of the graph.

- Graph on $[0, 2\pi]$ of Riemann's function $\mathfrak{R}(t) = \sum_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$:



- $\mathfrak{R}$ satisfies the multifractal formalism of Frisch-Parisi (Jaffard 96) is intermittent (Boritchev-Eceizabarrena-Da Rocha 19), its graph has no tangents (at the end Riemann was right!!) and has Haussdorf dimension $\leq \frac{4}{3}$ (Eceizabarrena 19)

- The theorem gives a non-obvious non-linear geometric interpretation for Riemman's function.

Multifractal formalism

Spectrum of singularities (of a function f):

$$d_f(\beta) = H\text{-dim } E_\beta$$

$$E_\beta = \{t_0 \in [0,2] \ : \ f \text{ is } \beta\text{-Hölder at } t_0\}$$

i. e. $$\sup\{\alpha \ : \ f \in \mathcal{C}^\alpha(t_0)\}$$

$$\left|f(t) - P(t-t_0)\right| < C|t-t_0|^\alpha$$

Example: Weierstrass functions

$$\mathcal{W}_{a,b}(t) = \sum_{n \neq 0} a^n \cos(b^n t) \qquad a < 1 < ab$$

- Nowhere differentiable
- $\alpha = -\lg a / \lg(b)$ (monofractal)

- Antonia, Hopfinger, Gagne, and Anselment experiment 1984

- Frisch–Parisi: multifractal model

- Multifractal formalism (Frisch–Parisi conjecture)

$$d_f(\beta) = \inf_p \big(\beta p - \eta_f(p) + 1\big)$$
$$\eta_f(p) = \sup \big\{ s \; : \; f \in B_p^{s/p,\infty} \big\}$$

- Jaffard: (1996)

$$d_{\varphi_R}(\beta) = 4\beta - 2 \qquad \tfrac{1}{2} \le \beta \le \tfrac{3}{4}$$

Multifractal formalism is true for φ_R

- $\displaystyle\sum_k \frac{\sin t(ck+d)^2}{(ck+d)^2}$ $\qquad c, d \in \mathbb{Z}$ $\qquad$ Oskolkov 2013, Chamizo–Ubis 2013

Evolution of polygonal lines by the binormal flow

Theorem (Banica-V. 18).– Let $\chi_0(x)$ be a polygonal line parametrized by arc length with corners located at $x = k \in \mathbb{Z}$, of angles θ_k s.t. $\{a_k\}$ defined by $\sin(\frac{\theta_k}{2}) = e^{-\pi\frac{a_k^2}{2}}$ belongs to $l^{2,3}$. Then there exists $\chi(t)$ smooth solution of the binormal flow on $\mathbb{R}^*$,

solution in the weak sense on $\mathbb{R}$ with

$$|\chi(t,x) - \chi_0(x)| \leq C\sqrt{t}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

Refined analysis for some families of polygonal lines

Let $n \in \mathbb{N}^*$, $\nu \in]0,1]$, $\Theta > 0$.

From now on we focus on particular classes of initial data: polygonal lines $\chi_n(0)$ with finite but many corners located at $j \in \mathbb{Z}$ with $|j| \le n^\nu$, of same torsion ω_0 and angles θ_n such that

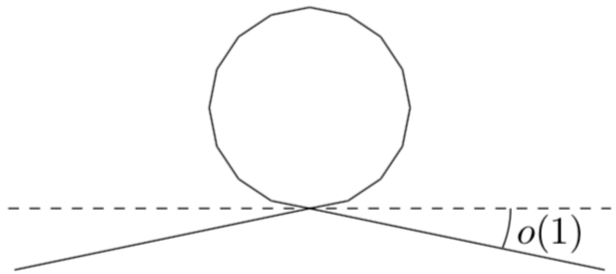
$$\lim_{n\to\infty} n(\pi - \theta_n) = \Theta,$$

and we suppose without loss of generality

$$\chi_n(0,0) = (0,0,0), \quad \partial_x \chi_n(0,0^\pm) = (\sin\frac{\theta_n}{2}, \pm\cos\frac{\theta_n}{2}, 0).$$

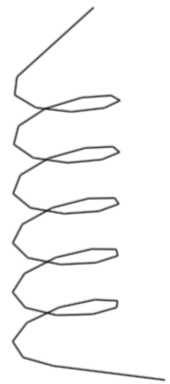
The initial data are the polygonal lines:

$\chi_n(0)$ planar approximation of a line



$\chi_n(0)$ planar approximation of a (multi-)loop

$\chi_n(0)$ non-planar approximation of a line



$\chi_n(0)$ approximation of multi-turns of helices

Theorem (a Frish-Parisi multifractal behaviour)

For the previous solutions with torsion $\omega_0 \in \pi\mathbb{Q}$ we have the following description of the trajectory of the corner $\chi_n(t,0)$, uniformly on $(0,T)$:

$$n\,\chi_n(t,0) - (0, \Re(\tilde{\mathfrak{R}}(t)), \Im(\tilde{\mathfrak{R}}(t))) \overset{n\to\infty}{\longrightarrow} 0.$$

The function $\tilde{\mathfrak{R}}$ is multifractal, and its spectrum of singularities $d_{\tilde{\mathfrak{R}}}$ satisfies the multifractal formalism of Frisch-Parisi:

$$d_{\tilde{\mathfrak{R}}}(\beta) := \dim_{\mathcal{H}}\{t, \tilde{\mathfrak{R}} \in \mathcal{C}^{\beta}(t)\} = \inf_p (\beta p - \eta_{\tilde{\mathfrak{R}}}(p) + 1),$$
$$\eta_{\tilde{\mathfrak{R}}}(p) := \sup\{s,\, \tilde{\mathfrak{R}} \in B_p^{\frac{s}{p},\infty}\},$$

a model for predicting the structure function exponents in turbulent flows.

In the torsion-free case $\tilde{\mathfrak{R}}(t) = -\Theta\frac{\mathfrak{R}(4\pi^2 t)}{4\pi^2}$, where $\mathfrak{R}(t) = \sum_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$ is a complex version of Riemann's non-differentiable function.

4.- NS: The Assymptotics

Remember that $\chi(t)$ is the (closed) curve at time t. Under suitable smoothness conditions on $\chi(0)$ we can find a tube of radius R sufficiently small ($R < \min \kappa/2$) around the filament $\chi(t)$, and define at any given time (cilindrical) coordinates (s, ρ, θ) for a point $\mathbf{x}$ where

$$\rho = \frac{dist(\mathbf{x}, \chi(\mathbf{t}))}{(\nu t)^{\frac{1}{2}}},$$

the coordinate s is the arclength at which the minimum distance between $\mathbf{x}$ and $\chi(s,t)$ is achieved, and θ is such that

$$\mathbf{e}_r \cdot \mathbf{n}(s,t) = \cos\theta.$$

Here $\mathbf{n}$ is the normal vector and

$$\mathbf{e}_r = \frac{\mathbf{x} - \chi(s,t)}{|\mathbf{x} - \chi(s,t)|}, \qquad \mathbf{e}_\theta = \mathbf{T} \times \mathbf{e}_r.$$

We claim the following asymptotics

Claim. Let $\chi(s,0) \in C^3$. If $\frac{\Gamma}{\nu}$ is sufficiently small, then there exists $T > 0$ such that for any given $t < T$, there exists $R > 0$ independent of t and sufficiently small such that for any $\mathbf{x}$ with $dist(\mathbf{x}, \chi(s,t)) < \frac{R}{2}$ the vorticity can be written as

$$\begin{aligned} \omega(x,t) &= \frac{1}{(\nu t)} \frac{\Gamma}{4\pi} e^{-\frac{\rho^2}{4}} \mathbf{T}(s,t) - \frac{1}{(\nu t)^{\frac{1}{2}}} \frac{\Gamma\kappa}{8\pi} \rho e^{-\frac{\rho^2}{4}} (\cos\theta)\, \mathbf{T}(s,t) \\ &+ \frac{1}{(\nu t)^{\frac{1}{2}}} \left(\Omega_1^{c(2)}(\rho) (\cos\theta) + \Omega_1^{s(2)}(\rho) (\sin\theta) \right) \mathbf{T}(s,t) \\ &+ \widetilde{\omega}(\mathbf{x}, t) \end{aligned}$$

with

$$\left| \Omega_1^{s,c(2)}(\rho) \right| \le C \frac{\Gamma^2}{\nu} (\rho + \rho^2) e^{-\frac{\rho^2}{4}}$$

and

$$\|\widetilde{\omega}\|_{L^2}^2 (t) + (\nu t)^{1-2\delta} \int_0^T (\nu t')^{2\delta - 1} \|\nabla \widetilde{\omega}\|_{L^2}^2 (t') dt' \le C (\nu t)^{1-2\delta}$$

where $0 < \delta \ll 1$.

Desingularization of the Biot-Sarvat Integral

Recall that the distance of a given point $\mathbf{x} \in \mathbb{R}^3$ to the filament is uniquely well defined in a sufficiently small neighborhood of the filament.

We introduce then the vortex tube

$$\omega_0(\mathbf{x}, t) = \frac{1}{(\nu t)} \Omega_0\left(\rho\right) \eta_R(\mathbf{x}) \mathbf{T} \qquad \rho = \frac{r}{(\nu t)^{1/2}},$$

with

$$\begin{aligned} \Omega_0\left(\rho\right) &= \frac{\Gamma}{4\pi} e^{-\frac{\rho^2}{4}} \\ V_0\left(\rho\right) &= \frac{\Gamma}{2\pi} \frac{1}{\rho} \left(1 - e^{-\frac{\rho^2}{4}}\right), \end{aligned}$$

and η_R a cut off.

Lemma. Let $\omega_0(\mathbf{x}, t)$ be as before. Assume in addition that the (closed) filament is smooth. Then, the induced velocity field given by the Biot-Savart law is, in the tube of radius R around the filament, given by

$$\begin{aligned}\mathbf{v}_0(\rho, \theta, s, t) &= \frac{1}{(\nu t)^{\frac{1}{2}}} V_0(\rho)\mathbf{e}_\theta(s, t) \\ &\quad -\frac{\Gamma}{4\pi}\kappa(s, t)\left(\log(\nu t)^{\frac{1}{2}} + \frac{1}{2\pi}F(\rho)\right)\mathbf{b}(s, t) + O(1),\end{aligned}$$

with $F(\rho)$ a smooth function such that $F(s) \sim 2\pi \log s$, as $s \to \infty$ and $F(0) = \pi(2\log 2 - \gamma)$ (γ being Euler's constant). At the center of the filament we have

$$\begin{aligned}\mathbf{v}_0(0, s, t) &= \frac{\Gamma}{4\pi}\kappa(s, t)\left(\log(\nu t)^{-\frac{1}{2}} + \frac{1}{2}(\gamma - 2\log 2)\right)\mathbf{b}(s, t) \\ &\quad +\mathbf{v}_0^*(s, t) + O((\nu t)^{\frac{1}{2}}\log(\nu t))\end{aligned}$$

with

$$\mathbf{v}_0^*(s, t) = \frac{\Gamma}{4\pi}\lim_{\varepsilon \to 0}\left(\int_{|s'-s|>\varepsilon}\mathbf{T}(s', t) \times \frac{\mathbf{x}(s, t) - \mathbf{x}'(s', t)}{|\mathbf{x}(s, t) - \mathbf{x}'(s', t)|^3} ds' + \kappa(s, t)\mathbf{b}(s, t)\log\varepsilon\right).$$

THANK YOU FOR YOUR ATTENTION