# Fluctuations of $\delta$-moments of the free Schrödinger equation


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HADE

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# Summary

- $\delta-$wave packets: How do they disperse?
- Motivation:

  **(a)** Unique continuation

  **(b)** Multifractality/Intermittency
- Talbot effect
- $\delta = 1$ : Heisenberg UP
- $0 < \delta < 1$ : Fractional UP
- Fluctuations

# $\delta$-wave packets

$$\begin{cases} \partial_t u = \frac{i}{2}\Delta u & x \in \mathbb{R}^n \\ u(x,0) = f(x), \end{cases}$$

We measure regularity using the space

$$\Sigma_\delta(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \mid \|f\|_{\Sigma_\delta}^2 := \left\| |x|^\delta f \right\|_2^2 + \left\| D^\delta f \right\|_2^2 < \infty \right\},$$

where $D^\delta f := |\xi|^\delta \hat{f}(\xi)$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x)\, dx.$$

- $(x_0, t_0)$ Translations in space time

$$u(x_0 + x, t_0 + t)$$

- $\lambda > 0$ Dilations

$$u(\lambda x, \lambda^2 t)$$

- $\xi_0$ Translations in phase space

$$e^{-i\frac{t}{2}|\xi_0|^2 + ix_0\xi} u\left(x - t\xi_0, t\right)$$

Hence, if $u$ "remains concentrated" up to time one close to the origin by "tuning" the parameters $\lambda$, $x_0$, $t_0$, $\xi_0$ we create a wave packet that is "concentrated" around $x - t\xi_0$ in a box $\lambda^{-1} \times \cdots \times \lambda^{-1} \times \lambda^{-2}$.

Beyond that time, the wave packet starts to disperse.

**Q.– How does it disperse?**

$$h_\delta(t) = \int |x|^{2\delta} \, |u(x,t)|^2 \, dx \qquad 0 < \delta \le 1$$

$$x_0 = 0$$

$$\xi_0 = 0$$

$$\lambda : \quad \int |x|^2 \, |u(x,0)|^2 \, dx = \int |\xi|^2 \, |\widehat{u}(\xi,0)|^2 \, d\xi = a_\delta^2$$

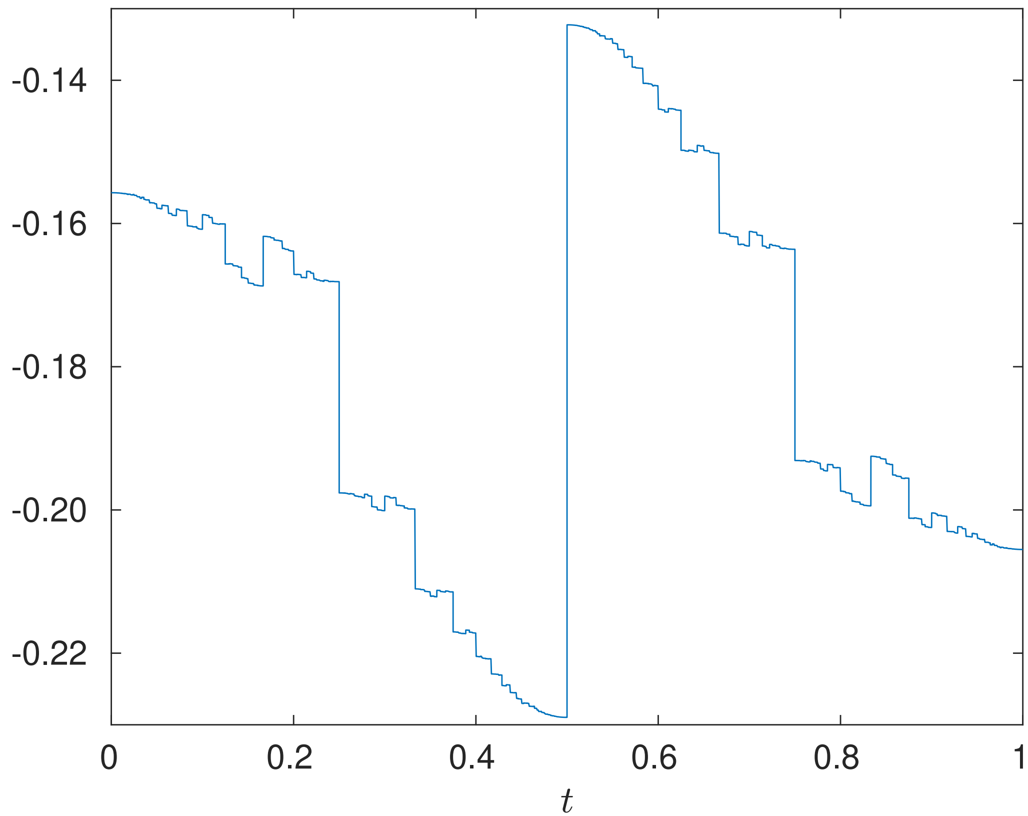
$t = 0$ is a minimum of $h_\delta$

$$\int |u_0(x)|^2 \, dx = 1$$

- Motivation:

**(a)** Unique continuation

**(b)** Multifractality/Intermittency




*8.2 Self-similar and intermittent random functions* 123

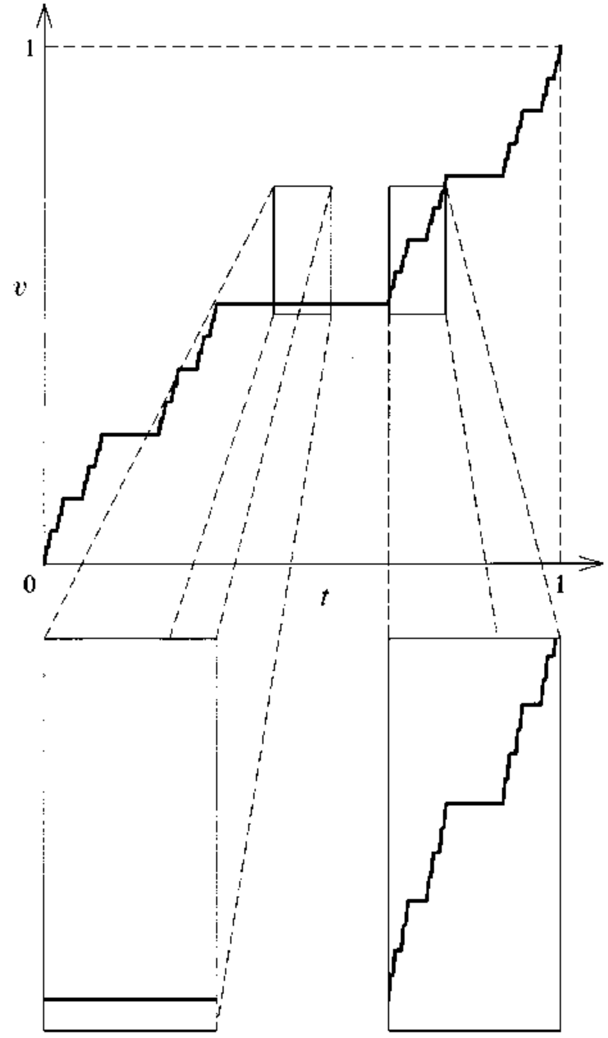
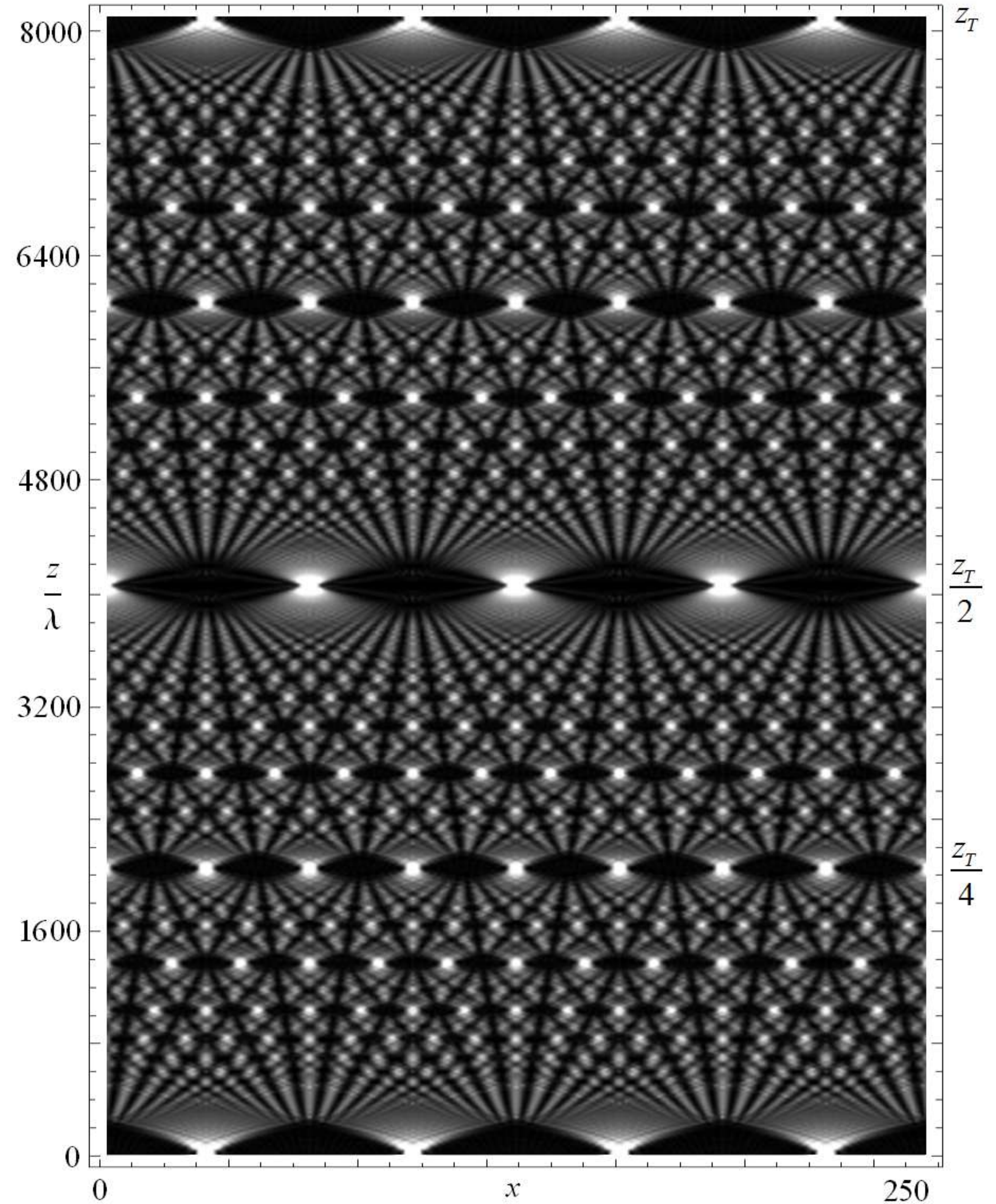



Fig. 8.2. The Devil's staircase: an intermittent function.


8000
6400
4800
3200
1600
0
$\frac{z}{\lambda}$
0
$x$
250
$z_T$
$\frac{z_T}{2}$
$\frac{z_T}{4}$

# The Talbot effect

$$t_{pq} = \frac{\pi p}{q}$$

$$u(x,0) = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k)$$

$$u(x,t_{pq}) = \sum_{k=-\infty}^{\infty} e^{-ik^2\pi p/q + ikx}$$

$$u(x,t_{pq}) = \frac{2\pi}{q} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{q-1} G(-p,m,q)\delta\left(x - 2\pi k - \frac{2\pi m}{q}\right)$$

$G$ is the Gauss sum $\quad |G| \sim \sqrt{q}$

# $\delta = 1$

$$\ddot{h}_1 = 2a^2 \qquad \dot{h}_1(0) = 0$$

$$h_1(t) = a^2(1 + t^2)$$

How small is $a^2 = \int |x|^2 \, |u_0(x)|^2 \, dx = \int |\xi|^2 \, |\widehat{u}_0(\xi)|^2 \, d\xi$ ?

Heinsenberg uncertainty principle

$$a^2 \geq \frac{n}{4\pi}$$

Minimizers are Gaussians **!!**

# $h_\delta(t) =$?

- Upper bound: persistence

$$\left\| e^{i|x|^2} f \right\|_{\Sigma_\delta} \le c_+ \|f\|_{\Sigma_\delta}$$ **(Nahas-Ponce 2009)**

Scaling gives

$$h_\delta(t) \le c_+ (1 + t^2)^\delta$$

Example $\qquad u(x,0) = e^{-\pi/2|x|^2}$

$$h_\delta(t) = c_G (1 + t^2)^\delta$$

- Is this a generic behaviour?

# Lower bound

**Theorem 1 (Static, Fractional Uncertainty Principle)**

*There exists a constant* $a_\delta > 0$*, for* $0 < \delta < 1$*, such that*

$$\inf_{\|f\|_2=1} \left\| |x|^\delta f \right\|_{L^2(\mathbb{R}^n)} \left\| D^\delta f \right\|_{L^2(\mathbb{R}^n)} = a_\delta^2.$$

*Equality is attained and a minimizer* $Q_\delta$ *can be chosen strictly positive and satisfying* $\left\| |x|^\delta Q_\delta \right\|_2 = \left\| D^\delta Q_\delta \right\|_2$*. Any other minimizer* $f$ *is of the form* $f(x) = c\lambda^{n/2} Q_\delta(\lambda x)$ *for some* $\lambda > 0$ *and* $|c| = 1$*. Furthermore,* $Q_\delta(x) \simeq |x|^{-n-4\delta}$ *for* $|x| \gg 1$*.*

**Theorem 2 (Dynamical, Fractional Uncertainty Principle)**

*If* $f \in \Sigma_\delta(\mathbb{R}^n)$, *for* $0 < \delta < 1$, *and* $\|f\|_2 = 1$, *then*

$$h_\delta[f](t) \geq \left( \frac{a_\delta^2}{\| |x|^\delta f \|_2 \, \| D^\delta f \|_2} \right)^2 \max \left( \| |x|^\delta f \|_2^2 \, , \| D^\delta f \|_2^2 \, |t|^{2\delta} \right),$$

*where* $a_\delta$ *is the constant in* Th. 1*. Furthermore, for any* $T \neq 0$

$$h_\delta[f](0) h_\delta[f](T) \geq a_\delta^4 |T|^{2\delta},$$

*with equality if and only if*

$$f(x) = c e^{-\pi i |x|^2 / T} \lambda^{n/2} Q_\delta(\lambda x)$$

*for some* $\lambda > 0$ *and* $|c| = 1$.

# Proofs

## (a) Theorem 1

**Lemma** *The class* $\Sigma_\delta(\mathbb{R}^n)$ *is a Hilbert space compactly embedded in* $L^2(\mathbb{R}^n)$*; in particular,*

$$\|f\|_2 \leq C\left(\left\||x|^\delta f\right\|_2^2 + \left\|D^\delta f\right\|_2^2\right)^{\frac{1}{2}}.$$

*Furthermore, there exists a function* $Q_\delta$ *with* $\|Q_\delta\|_2 = 1$ *such that*

$$\inf_{\|f\|_2=1} \|f\|_{\Sigma_\delta} = \|Q_\delta\|_{\Sigma_\delta}.$$

**Lemma** *If* $\|Q_\delta\|_{\Sigma_\delta} = \inf_{\|u\|_2=1} \|u\|_{\Sigma_\delta}$ *and* $\|Q_\delta\|_2 = 1$*, then*

$$D^{2\delta} Q_\delta + |x|^{2\delta} Q_\delta = 2a_\delta^2 Q_\delta.$$

Kaleta and Kulczycki proved that the ground state satisfies $Q_\delta(x) \simeq 1/|x|^{n+4\delta}$ $(0 < \delta < 1)$ for $|x| \gg 1$. **(2010)**

$$Q_\delta = \widehat{Q}_\delta \qquad \Big(\text{Long tails for } Q_\delta\,,\ \widehat{Q}_\delta\Big) \quad \textbf{!!}$$

## (b) Theorem 2

**Proof.** The solution $u$ can be represented as

$$u(x,t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\pi i|x|^2/t} \int f(y) e^{\pi i|y|^2/t - 2\pi i x\cdot y/t}\, dy, \qquad \text{where } \mathrm{Re}\sqrt{it} > 0.$$

If we define $g_t(y) := f(y)e^{\pi i|y|^2/t}$, then the solution can be written as

$$u(x,t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\pi i|x|^2/t} \widehat{g}_t(x/t).$$

By the uncertainty principle we have

$$a_\delta^2 \leq \left\| |x|^\delta g_t \right\|_2 \left\| D^\delta g_t \right\|_2 = |t|^{-\delta} h_\delta(0)^{\frac{1}{2}} h_\delta(t)^{\frac{1}{2}},$$

with equality if and only if $g_t(x) = c\lambda^{n/2} Q_\delta(\lambda x)$ for some $\lambda > 0$ and $|c| = 1$, so and referencia hold. This inequality implies the lower bound

$$h_\delta(t) \geq \frac{a_\delta^4}{\| |x|^\delta f \|_2^2} |t|^{2\delta}.$$

# Conclusion

- $c_-(1+t^2)^\delta \le h_\delta(t) \le c_+(1+t^2)^\delta$

- Gaussian $\qquad h_\delta(t) = c_G(1+t^2)^\delta$

$$c_G \ne c_- \, , \, c_+ = ?$$

**Q.–** **Are there fluctuations?**

* $n \geq 3$ $\qquad h_\delta(t)$ is convex for $\delta \geq 1/2$.

* Decay $\widehat{h}_\delta(\tau)$; $\qquad \widehat{h}_\delta(0)$

* $d = 1, 2$ are the relevant ones.

* We will focus our attention in $d = 1$.

* Dirac comb

$$F_D(x) := \sum_{m \in \mathbb{Z}} \delta(x - m)$$

  can be relevant.

* Periodic case?

# Renormalization

$h_\delta[F_D]$ does not make sense, we are able to extend, after renormalization, the functional $h_\delta$ to periodic functions and then to the Dirac comb. To approach the Dirac comb in $\mathbb{R}$ we use functions of the form

$$f_{\varepsilon_1,\varepsilon_2}(x) := N_{\varepsilon_2}^{-1}\psi(\varepsilon_2 x)F_{\varepsilon_1}/\,\|F_{\varepsilon_1}\|_2\,,$$

where $\psi$ is a smooth function with $\psi(0)=1$, $N_{\varepsilon_2}$ is chosen so that $\|f_{\varepsilon_1,\varepsilon_2}\|_2 = 1$, and

$$F_{\varepsilon_1}(x) := \sum_{m\in\mathbb{Z}} \varepsilon_1^{-1} e^{-\pi((x-m)/\varepsilon_1)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi(\varepsilon_1 m)^2} e^{2\pi i x m}.$$

We prove that in the limit $\varepsilon_2 \to 0$ ($\varepsilon_1$ fixed) the function $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ splits into a smooth background and an oscillating, periodic function that we call $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$. In Figure 1 we can see how $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$ approaches, after renormalization, $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$.

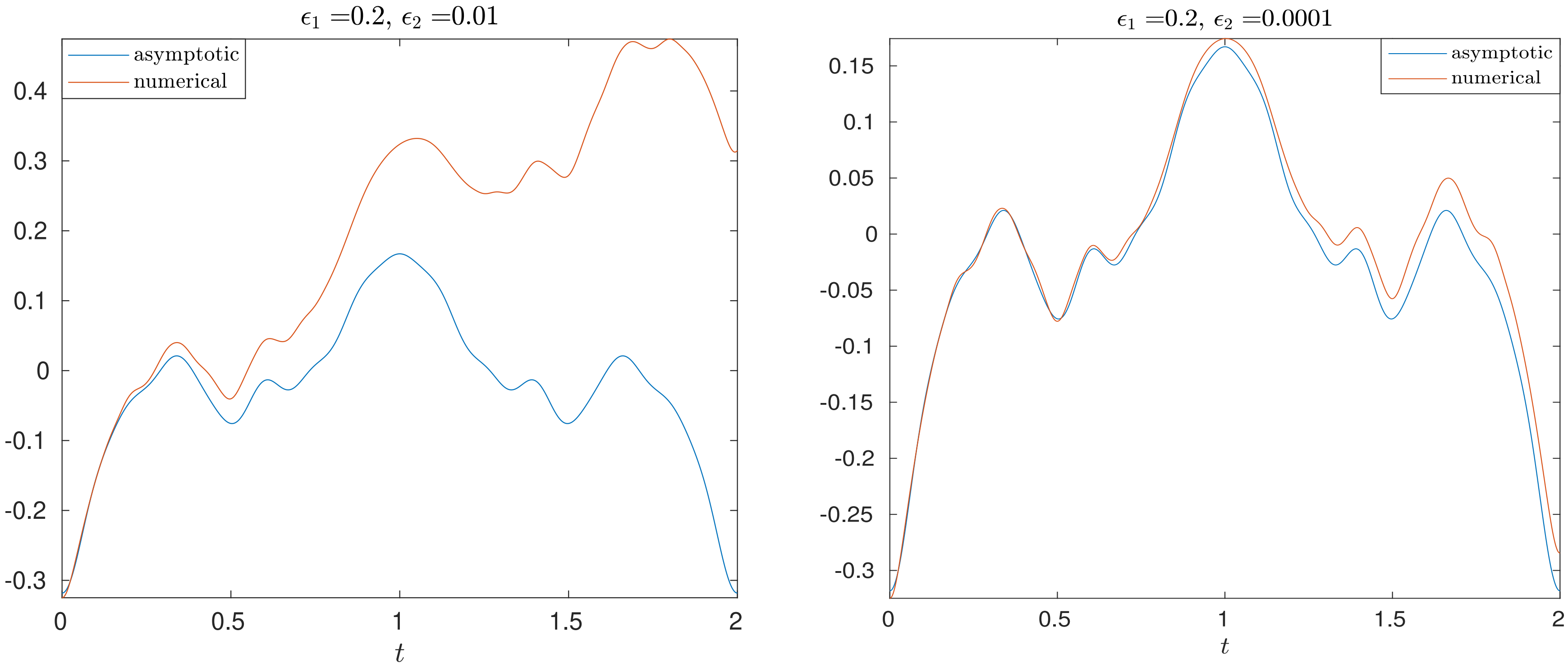



**Figure 1:** The red line is the plot of $h_\delta[f_{\varepsilon_1,\varepsilon_2}]$, for $\delta = 0.25$. The choice of $\varepsilon_1 = 0.2$ is due to the high computational cost of taking a smaller value of $\varepsilon_1$ and then to diminish $\varepsilon_2$.

## $\epsilon_2 \to 0$

**Theorem 3.**

$$h_{\mathrm{p},\delta}[F_D](2t) = -\frac{2b_{1,\delta}}{\|\psi\|_2^2}\zeta\big(2(1+\delta)\big)\Bigg[\sum_{\substack{(p,q)=1\\ q>0 \text{ odd}}} \frac{1}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t) - \\ - \sum_{\substack{(p,q)=1\\ q\equiv 2 \pmod 4}} \frac{2(2^{1+2\delta}-1)}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t) + \sum_{\substack{(p,q)=1\\ q\equiv 0 \pmod 4}} \frac{2^{2(1+\delta)}}{q^{2(1+\delta)}}\delta_{\frac{p}{q}}(t)\Bigg],$$

*where $\zeta(s)$ is the Riemann zeta function, and*

$$b_{1,\delta} = \frac{1}{(2\pi)^{2\delta}}\frac{\Gamma(2\delta)}{|\Gamma(-\delta)|\Gamma(\delta)}.$$

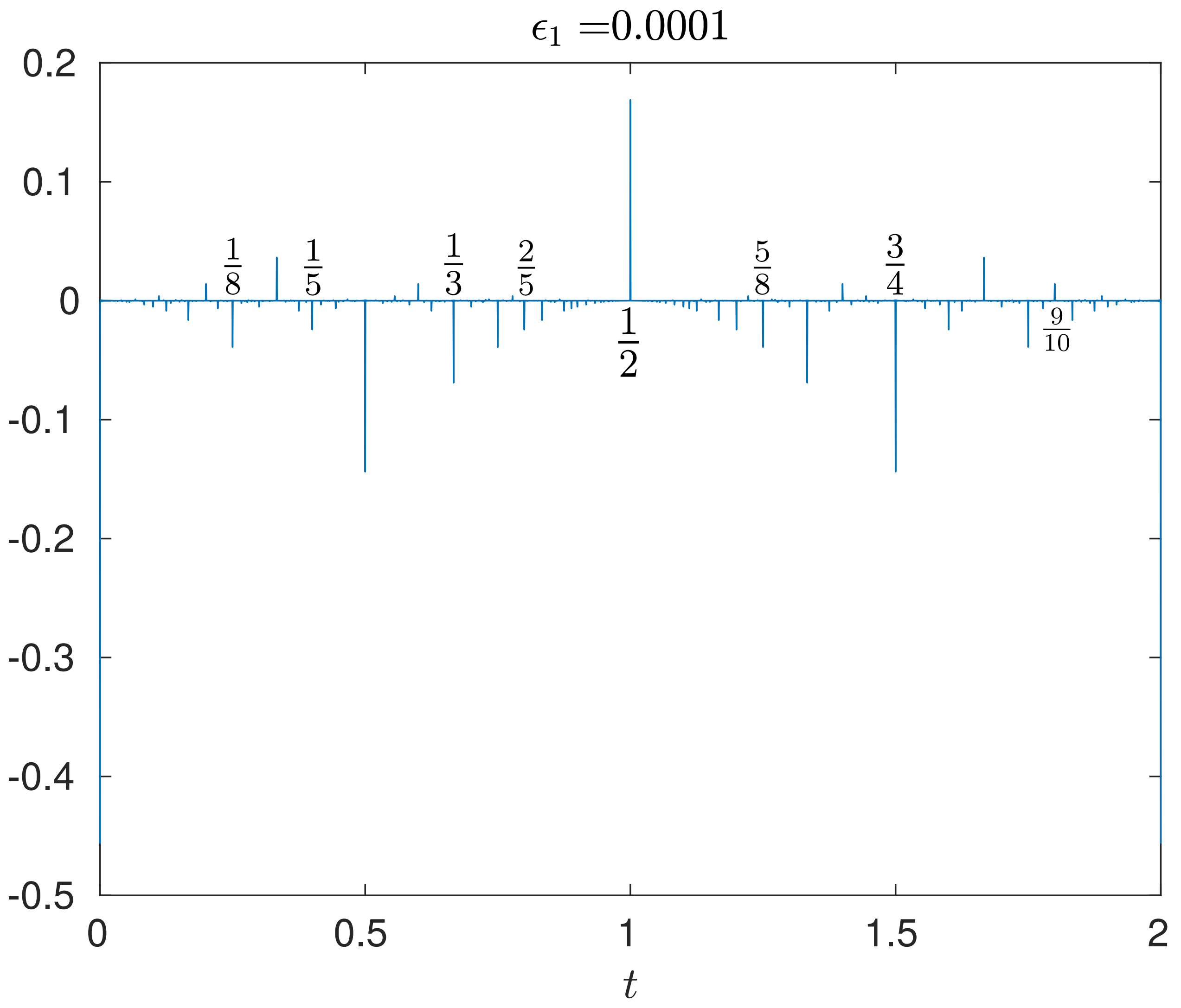



**Figure 2:** Plot of $h_{\mathrm{p},\delta}[F_{\varepsilon_1}]$ when $\delta = 0.25$.

$$H_\delta(t) := \int_{[0,t]} h_{\mathrm{p},\delta}(2s)ds.$$

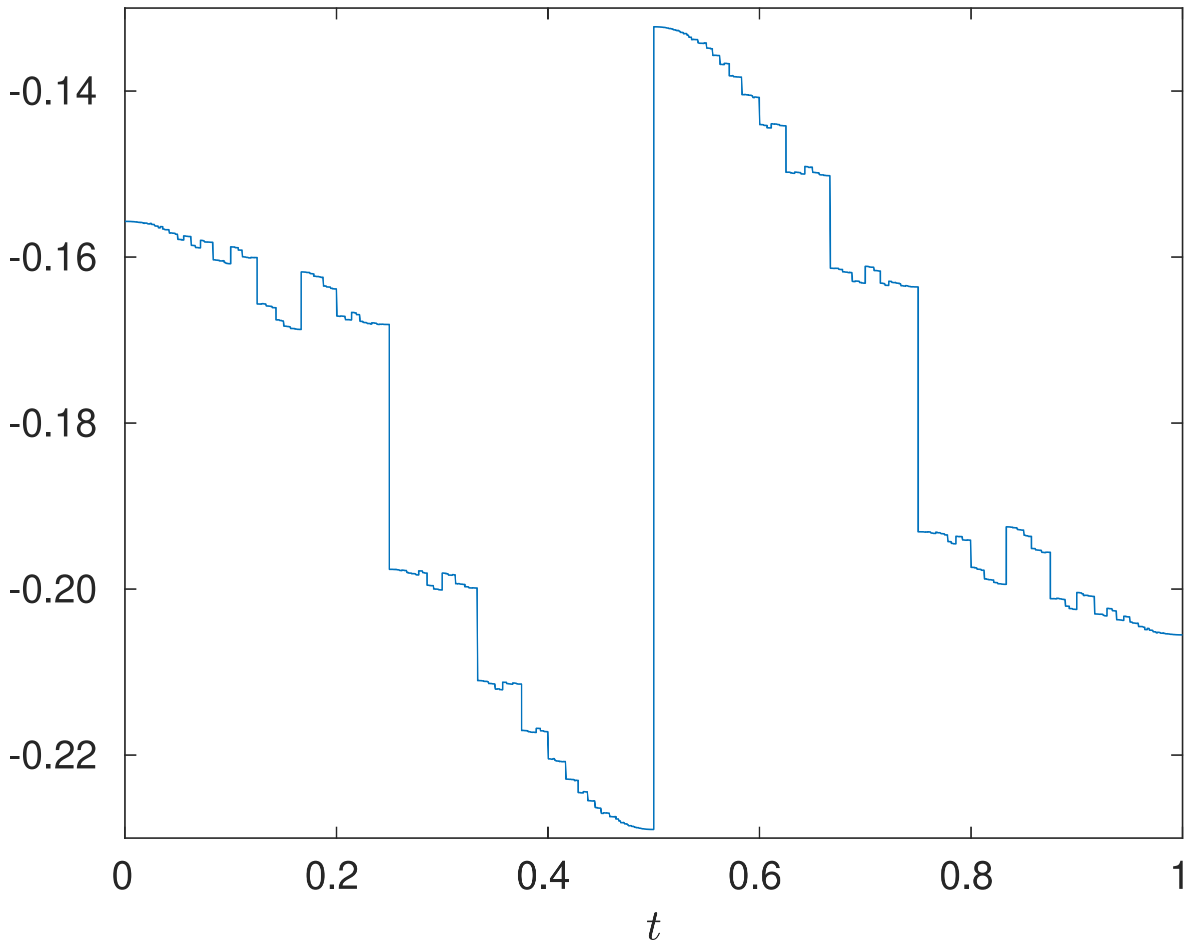



**Figure 3:** Plot of $H_\delta$. Even though $H_\delta$ has some symmetry, *e.g.* $H_\delta(1-t) = c_\delta - H_\delta(t-)$, the appearance of "unpredictable" large jumps resembles an $\alpha$-Lèvy process with small exponent $\alpha$.

We define the spectrum of singularities $d_{H_\delta}(\gamma) := \dim F_\gamma$, where

$$F_\gamma := \{t \in [0,1) \mid H_\delta \text{ has Hölder exponent } \gamma \text{ at } t\}.$$

**Theorem 4.** *Let* $\alpha := 1/(1+\delta)$, *then*

$$d_{H_\delta}(\gamma) = \alpha\gamma, \quad \textit{for } \gamma \in [0, 1/\alpha).$$

Jaffard proved **(1999)** that the spectrum of singularities of an $\alpha$-Lèvy process is almost surely

$$d_\alpha(\gamma) = \begin{cases} \alpha\gamma & \gamma \in [0, 1/\alpha] \\ -\infty & \gamma > 1/\alpha; \end{cases}$$

$d_\alpha(\gamma) = -\infty$ means that no point has Hölder exponent $\gamma$.

# About the proofs

## (a) Theorem 3

- $$\hat{h}_{\mathrm{p},\delta}[F_D](\tau) := -\frac{2b_{1,\delta}}{\|\psi\|_2^2} \sum_k \delta_{\frac{k}{2}}(\tau) \sum_{\substack{m_1 \neq m_2 \\ m_1^2 - m_2^2 = k}} \frac{1}{|m_1 - m_2|^{1+2\delta}},$$

**Lemma**

$$\sum_{\substack{m_1 \neq m_2 \\ m_1^2 - m_2^2 = k}} \frac{1}{|m_1 - m_2|^{1+2\delta}} = \begin{cases} 2 \sum_{\substack{d|k \\ d>0}} \frac{1}{d^{1+2\delta}} & \text{for } k \in \mathbb{Z} \text{ odd} \\ \frac{1}{2^{2\delta}} \sum_{\substack{4d|k \\ d>0}} \frac{1}{d^{1+2\delta}} & \text{for } k \equiv 0 \,(\mathrm{mod}\ 4) \\ 0 & \text{for } k \equiv 2 \,(\mathrm{mod}\ 4) \end{cases}$$

## (b) Theorem 4

- The point process $\quad D_{\mathrm{p}} = \mathbb{Q} \cap [0,1]$

$$p_\delta : \mathbb{Q} \cap [0,1) \to X = \mathbb{R} \setminus \{0\}.$$

- The counting function

$$N_p(I,U) := \left|\{t \in D_p \cap I \mid p(t) \in U\}\right|.$$

- $$\begin{aligned} H_\delta(t+h) - H_\delta(t) &= \int_{\mathbb{R}\setminus\{0\}} y N_{p_\delta}(I,dy) \\ &= \int_0^\infty \left[N_{p_\delta}(I,[y,\infty)) - N_{p_\delta}(I,[-y,-\infty))\right] dy. \end{aligned}$$

**Theorem** *For $I \subset [0,1)$, the function*

$$|N|_{p_\delta}(I,r) := N_{p_\delta}\big(I, \big(-\infty, -r] \cup [r,\infty)\big), \quad \textit{for } r > 0,$$

*satisfies the bounds*

$$|N|_{p_\delta}(I,r) \le C_\delta |I| r^{-1/(1+\delta)} + 1, \qquad \textit{all } r \lesssim_\delta 1,$$

$$|N|_{p_\delta}(I,r) \gtrsim_\delta \frac{|I|}{\log(c_\delta/r)} r^{-1/(1+\delta)}, \qquad \textit{all } r \lesssim_\delta |I|^{2(1+\delta)}.$$

\+ Jarnik‘s theorem about the Hausdorff dimension of the “irrationals”.

# THANK YOU FOR YOUR ATTENTION