Oscillatory Integrals and Euler Equations

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Summary

- S.I. (Singular Integrals) vs O.I. (Oscillatory Integrals)
  - Real Analysis
  - PDE's
  - Fourier Analysis
- O.I. $\Longleftrightarrow$ Wave Packets (Resonances)
- Euler Equations as Geometric PDE's
- Vortex Sheet; Vortex Patch; Vortex Filaments
- Numerics

S.I. vs. O.I. (Real Analysis)

- Approximations of the identity: $\int_{\mathbb{R}^n} \Phi \, dx = 1 \qquad f \in L^1$

$$u(x,t) = f * \Phi_t = \int_{\mathbb{R}^n} f(y)\Phi_t(x-y)dy \qquad \Phi_t(x) = \frac{1}{t^n}\Phi\left(\frac{x}{t}\right)$$

Q $\lim_{t \downarrow 0} u(x,t) = f(x)$ a.e. x ?

S.I. $\Phi \geq 0$ $\quad \Phi \downarrow$ Maximal function (H-L)

$$Mf(x) = \sup_t |u(x,t)|$$

YES Weak 1.1 inequality C-Z decomposition

Example: $\Phi(x) = e^{-\pi|x|^2}$

O.I.: $\Phi(x) = e^{i\pi|x|^2}$ $\displaystyle\int_{\mathbb{R}^n} \Phi dx = e^{in\pi/4}$

$$\lim_{t \downarrow 0} u(x,t) = u(x,0) = f \qquad \mathcal{S}, L^2, \ldots$$

$$= f \qquad \text{a.e. } x \qquad \textbf{NO}$$

Q Maximal function ?

Q C-Z decomposition ?

S.I. vs. O.I. (PDE's)

		S.I.		O.I.
$(x,y) \in \mathbb{R}^2$	C-R	$\partial_x u + i\partial_y u = 0$	Transport	$\partial_x u + \partial_y u = 0$
$(t,x) \in \mathbb{R} \times \mathbb{R}^n$	Laplace	$\partial_t^2 u + \partial_x^2 u = 0$	Wave	$\partial_t^2 u - \partial_x^2 u = 0$
$(t,x) \in \mathbb{R} \times \mathbb{R}^n$	Heat	$\partial_t u = \Delta u$	Schrödinger	$\partial_t u = i\Delta u$
$x \in \mathbb{R}^n$		$\Delta u - u = 0$	Helmholtz	$\Delta u + u = 0$
		←	Airy	$\partial_t u + \partial_x^3 u = 0$
			Klein-Gordon	
			Dirac	

S.I. vs O.I. (Fourier Analysis)

		S.I.		O.I.
$\xi, \eta \in \mathbb{R}$	C-R	$m(\xi,\eta) = \xi + i\eta$	Transport	$m(\xi,\eta) = \xi - \eta$
$(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n$	Laplace	$m(\tau,\xi) = \tau^2 + \lvert\xi\rvert^2$	Wave	$m(\xi,\eta) = \tau^2 - \lvert\xi\rvert^2$
$(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^n$	Heat	$m(\tau,\xi) = i\tau - \lvert\xi\rvert^2$	Schrödinger	$m(\tau,\xi) = \tau - \lvert\xi\rvert^2$
$\xi \in \mathbb{R}^n$		$m(\xi) = -\lvert\xi\rvert^2 - 1$	Helmholtz	$m(\xi) = \lvert\xi\rvert^2 - 1$
			Airy	$m(\tau,\xi) = \tau - \xi^3$

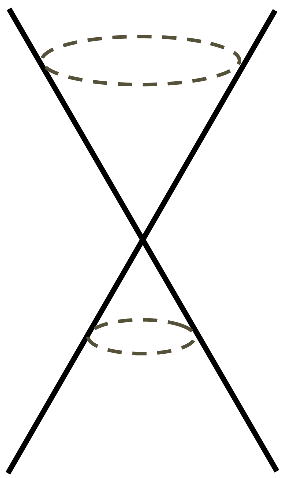
$$\mathcal{L}u = f \qquad m(\tau,\xi)\widehat{u} = \widehat{f}$$

$$m(\tau,\xi) = 0 \;!!$$

O.I.

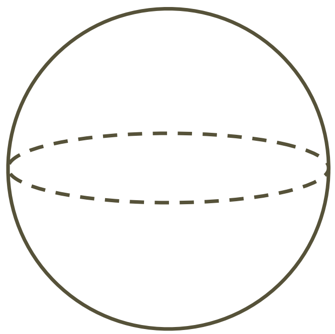
$$\tau = \Phi(\xi)$$

$$|\xi| = |\tau|$$



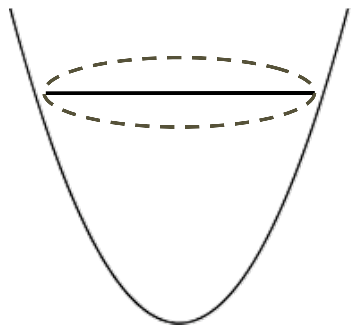
Wave

$$|\xi|^2 = 1$$



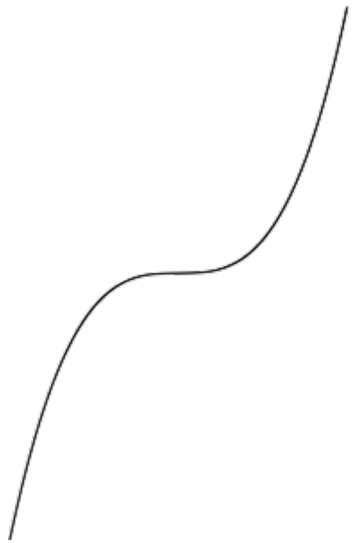
Helmholtz

$$\tau = |\xi|^2$$



Schrödinger

$$\tau = \xi^3$$



Airy

O.I.

$$u(x,t) = \int e^{it\Phi(\xi)+ix\xi}\widehat{f}(\xi)d\xi$$

$$\Phi(\xi) = |\xi| \quad ; \quad |\xi|^2 \quad ; \quad \sqrt{1-|\xi|^2} \quad ; \quad \xi^3 \quad ; \quad \sqrt{g|\xi|\tanh(h|\xi|)}$$

water wave $\quad$ g gravity, h height

- $\widehat{f} \equiv 1 \qquad \int e^{it\Phi(\xi)+ix\xi}d\xi.$ **Fundamental solution**

Stationary Phase:

$$t\nabla\Phi + x = 0$$

$$D^2\Phi \neq 0$$

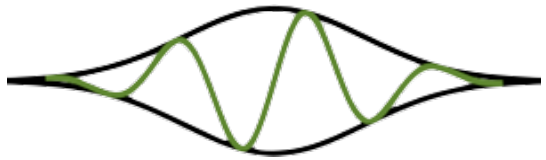
$$\Phi(\xi) = |\xi|^2 \qquad \int e^{it\Phi(\xi)+ix\xi}d\xi = \frac{C_n}{t^{n/2}}e^{i\frac{|x|^2}{4t}}$$

Wave Packets (Physical Space)

$$u(x,t) = \int e^{it\Phi(\xi)+ix\xi}\psi(\xi-\xi_0)d\xi$$

ψ bump

Stationary Phase:

$u(x,t) =$ 

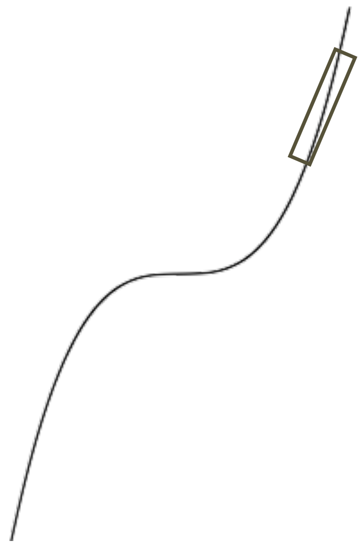
- Center: $t\nabla\Phi(\xi_0) + x = 0$
- Amplitude: $t^{-n/2}\left|D^2\Phi(\xi_0)\right|$
- Width : L^2 has to be preserved

CONCLUSION: O.I. describe wave propagation

- **DISPERSION:** the velocity is $\nabla\Phi(\xi_0)$, and depends on ξ_0.

Wave Packets (Phase Space)

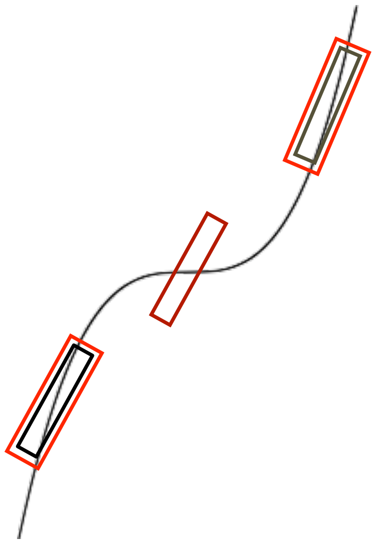
$\widehat{u}(\xi, \tau)$

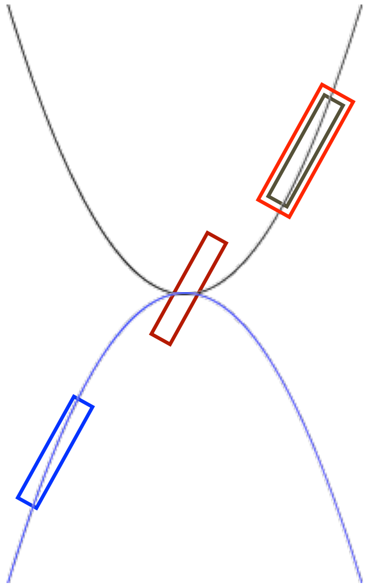


- Córdoba-Fefferman

Wave Packets (Resonances)

$$\partial_t u + \mathcal{L}u = F(u) \qquad u: \text{ Wave Packet}$$





mkdV $\quad F(u) = uuu_x$

$\dot{H}^{1/4}$ (KPV '93)

$NLS \quad F(u) = \overline{u}\,\overline{u}\,\overline{u} \quad \dot{H}^s \quad s < 0$ (KPV '96)

$= u\,\overline{u}\,u \quad \dot{H}^s \quad s < 0$ Ill posed

(KPV '01)

- **Bonus** of proving LWP by Picard iteration.

- Tao '04, Germain, Masmoudi, Shatah '08, '09: Time resonances/Space resonances

Euler Equations

- $\vec{u}$ velocity field
- Particles of fluid:

$$\dot{x}(t) = \vec{u}\big(x(t), t\big)$$

- p: pressure
- ϱ: density
- External (conservative) forces

- $$D_t \vec{u} = -\nabla p + \vec{g} \qquad\qquad D_t = \partial_t + u^j \partial_j$$
  $$\operatorname{div} \vec{u} = 0$$
- Vorticity

$$\vec{w} = \nabla \wedge \vec{u}$$

$$D_t \vec{w} = \vec{w} \cdot \nabla \vec{u}$$

$$\nabla \wedge \vec{w} = \nabla \wedge \nabla \wedge \vec{u} = -\Delta \vec{u}$$

Biot-Savart Law
(S.I.)

- $\vec{w} = \text{constant} \quad \Longrightarrow \quad \Delta \vec{u} = 0$

Euler Equations/Geometric PDE's

$\vec{u}$

$\nabla \wedge \vec{u} = \vec{w}$

- Helmholtz Laws

  1. If $\vec{w}(x_0, 0) = 0 \implies \vec{w}\big(x(t), t\big) = 0$

  2. Vortex lines travel with the fluid

  3. Strength of vortex tubes remain constant

Geometric Objects

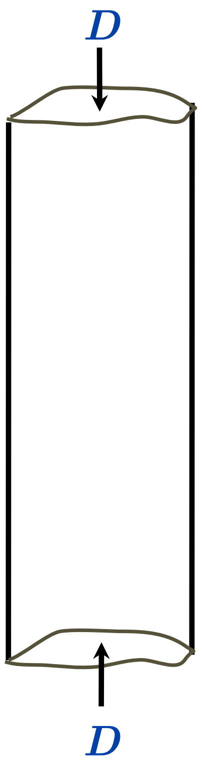
$\vec{u}$

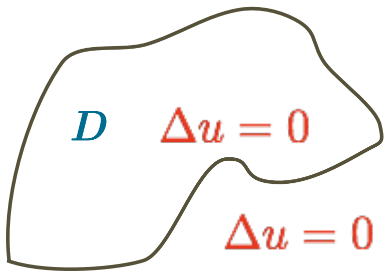
$\nabla \wedge \vec{u} = \vec{w}$

- Vortex patches $\vec{w} = \chi_\Omega$ Ω straight cylinder
- Vortex sheet $\vec{w} = \delta_{\{\mathrm{surface}\}}$
- Vortex filaments $\vec{w} = \delta_{\{\mathrm{curve}\}}$

$$-\Delta \vec{u} = \nabla \wedge \nabla \wedge \vec{u} = \nabla \wedge \vec{w}$$

Vortex Patches (Planar Flow)





$\partial \boldsymbol{D}$ travels with the fluid

Geometric flow of plane curves

Examples: Circles

Ellipse (Kirchoff)

V-State (Kelvin-Deem-Zabusky)

Euler; S.I, O.I.

- Vortex Patch Chemin’91 $\nabla u \sim \frac{1}{z^2} = K$

$$\int_{\mathbb{S}_1^+} K = 0$$

NO SINGULARITIES

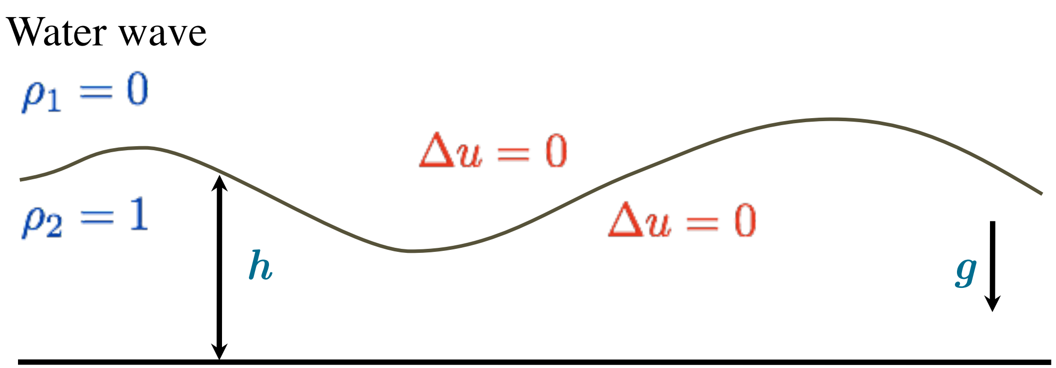
Q 1. theory $\mathcal{C}^{1^+}$

Q 2. All the upper bounds for the geometric quantities are e^{e^t}

Q 3. Dynamics of the fingering

Q 4. Stability of V-states **O.I. ?**

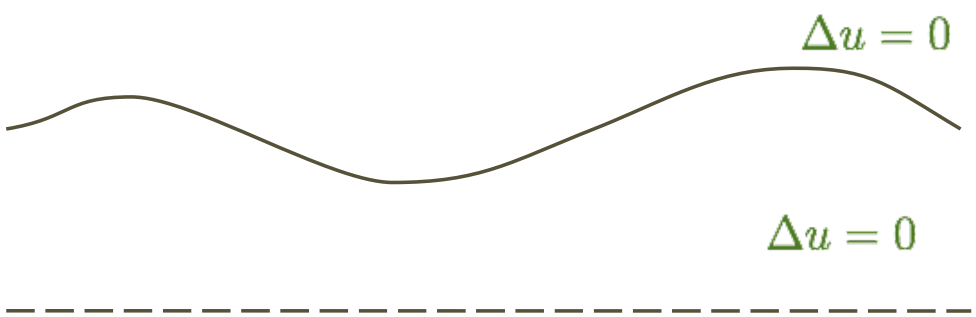
Vortex Sheet



Examples:

- Straight line
- Traveling wave/solitary wave (Levi-Civita, Nekrasov, conformal mapping)
- Stokes wave (extreme form)
- KdV

- Water waves



- S. Wu: Local existence result (smooth solutions, ‘97 2D, ‘99 3D) **(S.I.)**
- D. Lannes: Dirichlet-Neumann map (smooth solutions, ‘05 3D) **(S.I.)**
- S. Wu: Almost Global existence (smooth solutions, ‘09 2D) **(O.I.)**

Dispersive estimate: $$\int e^{it|\xi|^{1/2}+ix\cdot\xi}\psi(\xi)d\xi.$$

- Córdoba, Córdoba, Gancedo (smooth solutions) ’08 Local existence **(S.I.)**
- Germain, Masmoudi, Shatah (smooth solutions) ’09 Global existence theory for small data **(O.I.)**
- S. Wu (smooth solutions) ’09 Global existence theory for small data **(O.I.)**
- D. Lannes: Vortex sheet with surface tension ’10 **(S.I.)**
- STOKES WAVE (rough solutions) $H^{3/2}$? **(O.I.?)**

Vortex Filaments

- Circles (Smoke rings)
- Straight lines
- Helix
    - "No result" working directly on Euler equations
    - Approximation of the binormal flow (Da Rios, 1906)

$$X = X(s,t) \qquad X_t = cb$$

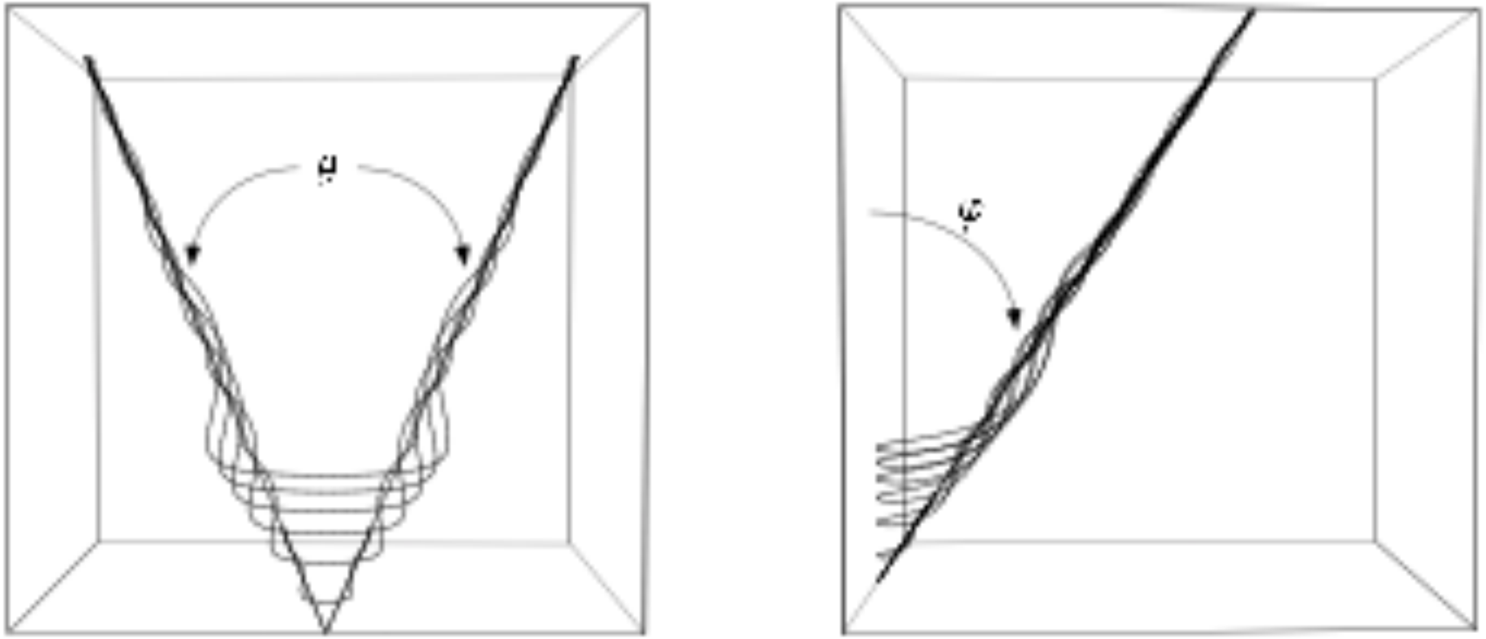
c: curvature

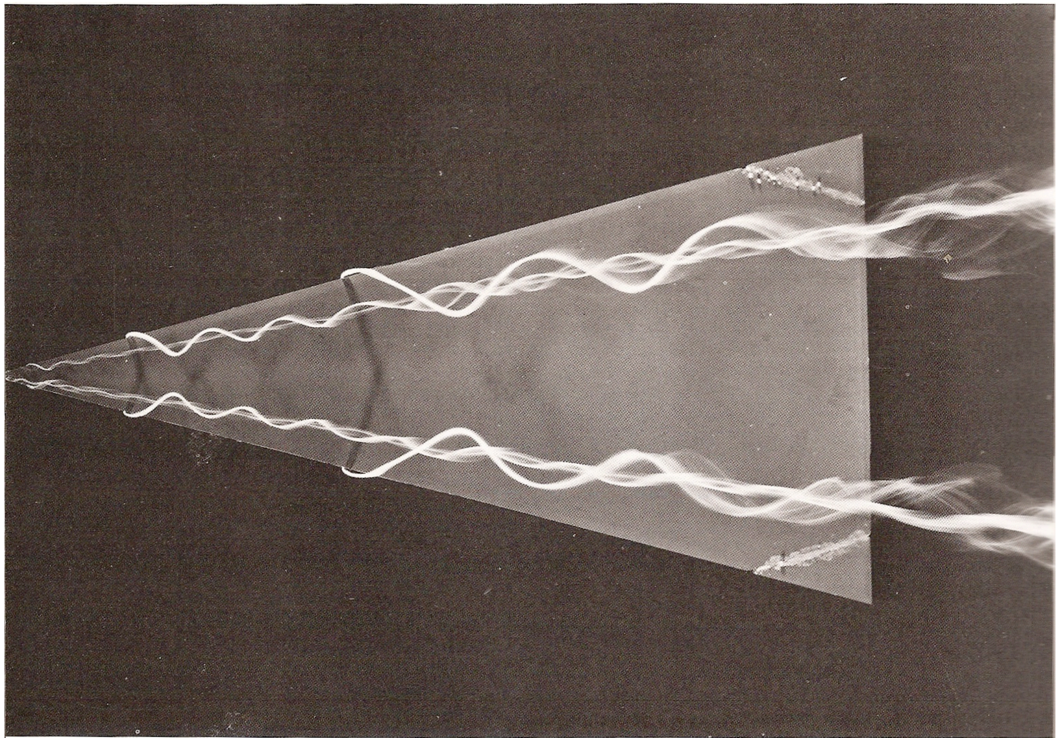
b: binormal

$$X_t = X_s \wedge X_{ss}$$

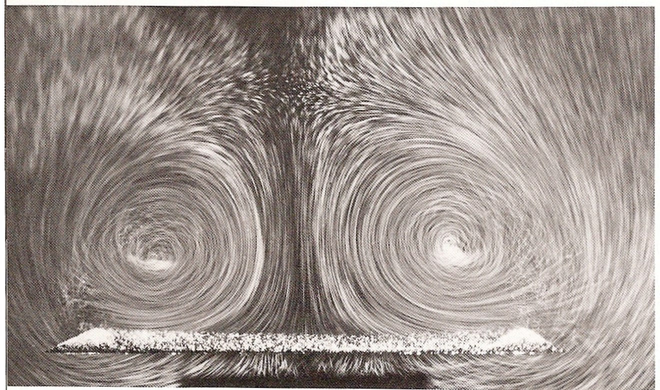
$$X_s = T \quad T_t = T \wedge T_{ss} = J D_s T_s$$ **Schrödinger Map**

$$|T| = 1$$

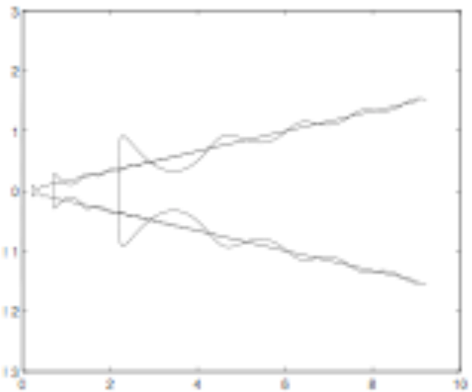




90. Vortices above an inclined triangular wing. Lines of colored fluid in water show the symmetrical pair of vortices behind a thin wing of 15° semi-vertex angle at 20° angle of attack. The Reynolds number is 20,000 based on chord. Although the Mach number is very low, the flow field is practically conical over most of the wing, quantities being constant along rays from the apex. *ONERA photograph, Werlé 1963*



91. Cross section of vortices on a triangular wing. Tiny air bubbles in water show the vortex pair for the flow above in a section at the trailing edge of the wing. *ONERA photograph, Werlé 1963*


3
2
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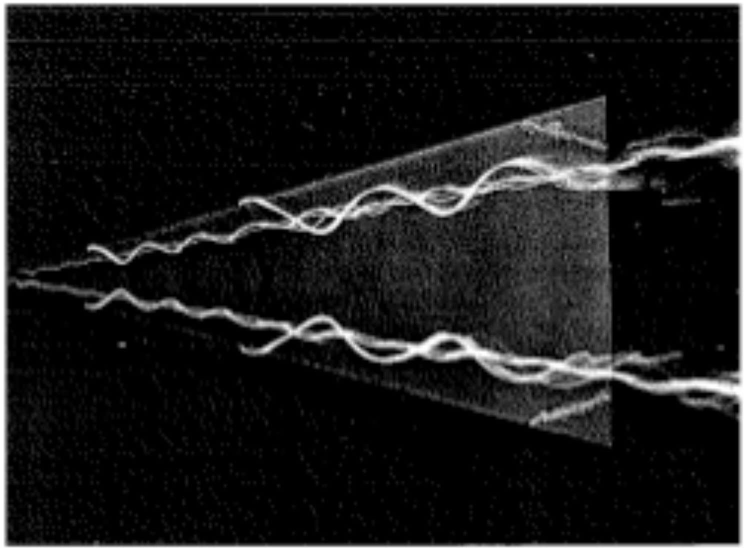
2
4
6
8
10

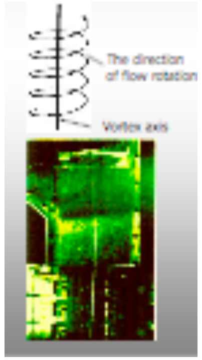


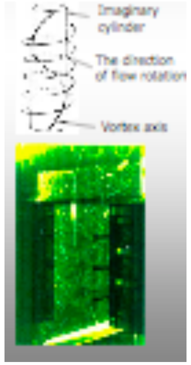
Theorem.- **(with V. Bánica)**
The selfsimilar solutions of the binormal flow are stable.

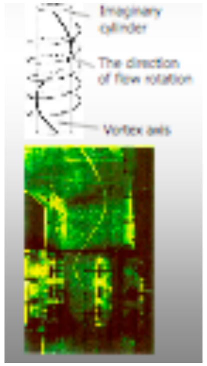
- Classification of selfsimilar solutions (GV '03-'04, L '03-'03)
- Use of the conformal transformation: Scattering problem:
    - Existence of the wave operator '09 (Stability of the "corner")
    - Asymptotic completeness '10 (Blow up of the zero Fourier mode)

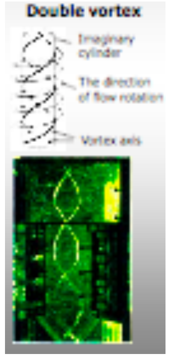
Open Problems

- Helix (Hardin)
- Stability of more than one helix (Klein, Majda, Damodaran, KPV'03)





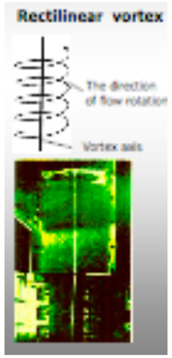


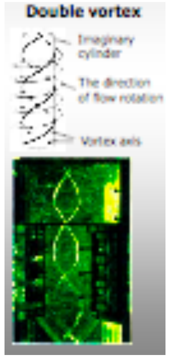


S.V. Alekseenko, P.A. Kuibin, V.L. Okulov

Theory of Concentrated Vortices

03/07


Rectilinear vortex
The direction
of flow rotation
Vortex axis


Double vortex
Imaginary
cylinder
The direction
of flow rotation
Vortex axis

MUCHAS GRACIAS POR SU ATENCIÓN