The initial value problem for non-linear Schrödinger equations

LUIS VEGA

Universidad del País Vasco
Euskal Herriko Unibertsitatea

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In this talk I shall present some work done in collaboration with C.E. Kenig and G. Ponce on quasi-linear Schrödinger equations. Our first work on this topic was published in 1993 with the title: "Small solutions to non-linear Schrödinger equations". In this paper we solve locally in time the IVP

$$\begin{cases} i\partial_t u - \Delta u &= F(u, \overline{u}, \nabla u, \nabla \overline{u}) \qquad F \text{ is non–linear} \\ u(x,0) &= u_0(x). \end{cases}$$

A fundamental piece of our argument is the inequality

$$\sup_t \|D^{1/2} u(t)\|_{L^2} + \|\nabla_x u\|_{X^*} \leq c \left\|D^{1/2} u_0\right\|_{L^2} + \|F\|_X$$

$$X = L^2_t L^2_{\text{weighted}} \qquad \text{w} = \langle x \rangle^{1^+}.$$

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$$\begin{cases} i\partial_t u - \Delta u &= F \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ u(x,0) &= u_0(x). \end{cases}$$

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The work we have done since then can be summarized as follows:

- Remove the smallness conditions (Hayashi-Ozawa, Chiara)
- Extend the result to general metrics:

$$\Delta_x \hookrightarrow \partial_j a_{jk} \partial_k \qquad a_{jk}\xi_j\xi_k \sim |\xi|^2 \quad \text{(elliptic)}$$

$$\Delta_{x_1} - \Delta_{x_2} \hookrightarrow \partial_j a_{jk} \partial_k \quad |(a_{jk}\xi_j)| \sim |\xi| \quad \text{non–degeneracy}$$

Doi, Craig-Kappeler-Strauss.

In collaboration with C. Rolvung.

The rest of the talk will be mainly devoted to try to motivate the problem and to give a hint of the tools we needed and developed in order to solve it.

The free Schrödinger equation

I want to introduce it from an elemental geometric point of view. Let me recall you a plane curve known as the Euler-Cornu spiral. This curve is characterized by its curvature *k* which is proportional to arc length:

$$k(s) = c_0 s/2 \qquad s \text{ arc length} \quad c_0 \in \mathbb{R}.$$

In Euler's work this curve describes a coiled **spring**. Cornu uses it to give a geometric explanation of Fresnel diffraction (**wave phenomena**).

Using complex notation to parametrize the curve

$$z(s) = x(s) + iy(s),$$

we obtain for the tangent vector z'

$$
\begin{aligned}
ik(s)z'(s) &= z''(s) \qquad (k(s) = c_0 s/2) \\
ic_0 \tfrac{s}{2} z'(s) &= z''(s) \\
z'(s) &= e^{ic_0 s^2/4} \qquad z'(0) = 1.
\end{aligned}
$$

Now recall the 1d free Schrödinger equation

$$
\begin{cases}
i\partial_t u = \partial_s^2 u & u = u(s,t) \\
u_0 = a\delta & \text{(Fundamental solution).}
\end{cases}
$$

Look for solutions of the type $u = \partial_s v,\ v = z\left(s/\sqrt{t}\right)$ (δ is homogeneous of degree -1).

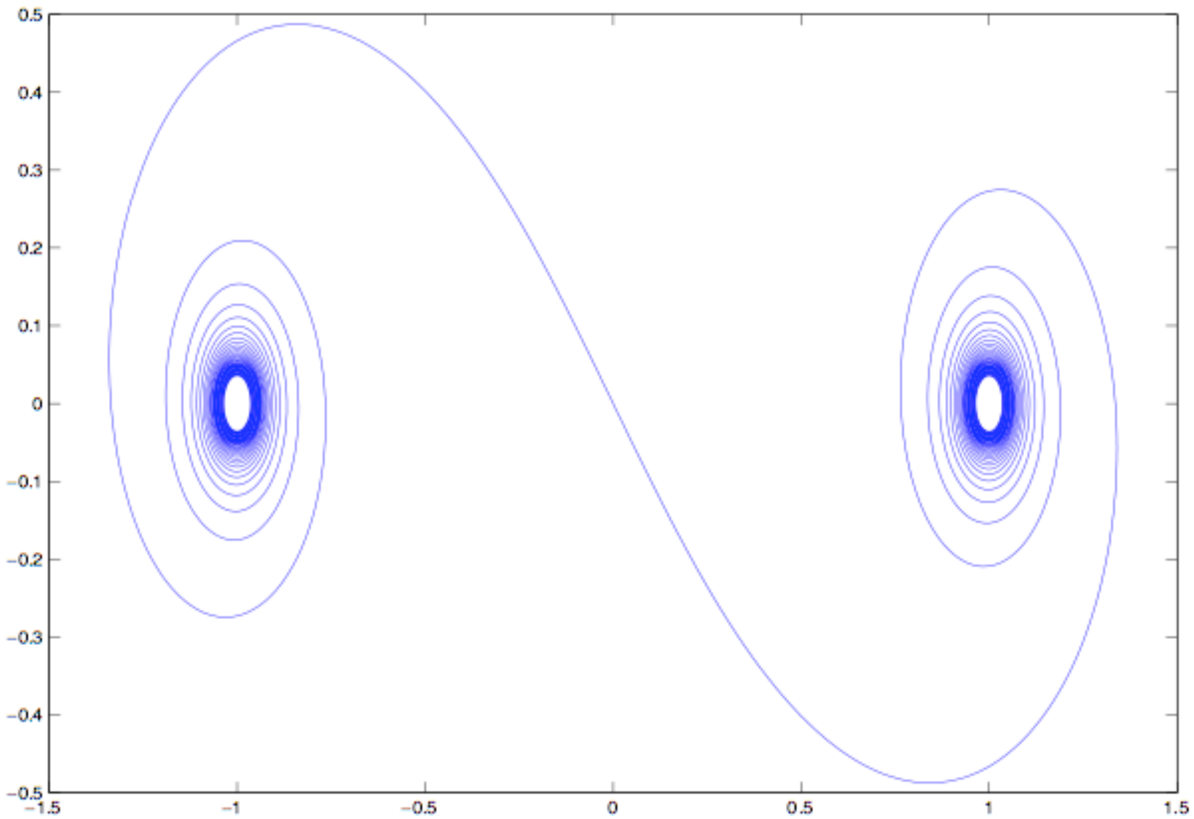
We get

$$i\frac{s}{2\sqrt{t}}\frac{1}{t}z'\left(\frac{s}{\sqrt{t}}\right) = \frac{1}{t}z''\left(\frac{s}{\sqrt{t}}\right) \qquad t=1$$

$$i\frac{s}{2}z' = z''.$$

The relation between a and c_0 is done computing the Fresnel integrals

$$\int_{-\infty}^{\infty} e^{i\pi s^2}\,ds = e^{i\frac{\pi}{4}}.$$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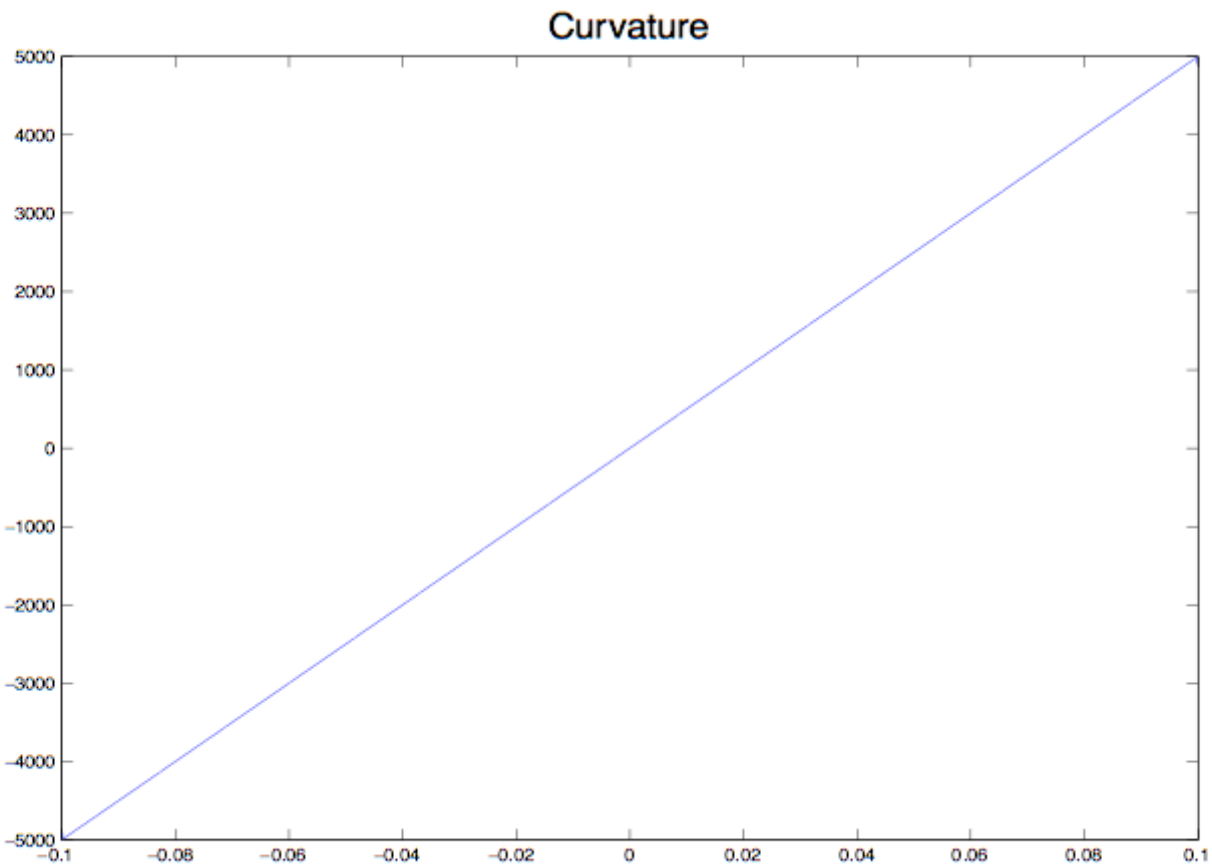
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0
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−2000
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−5000
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−0.08
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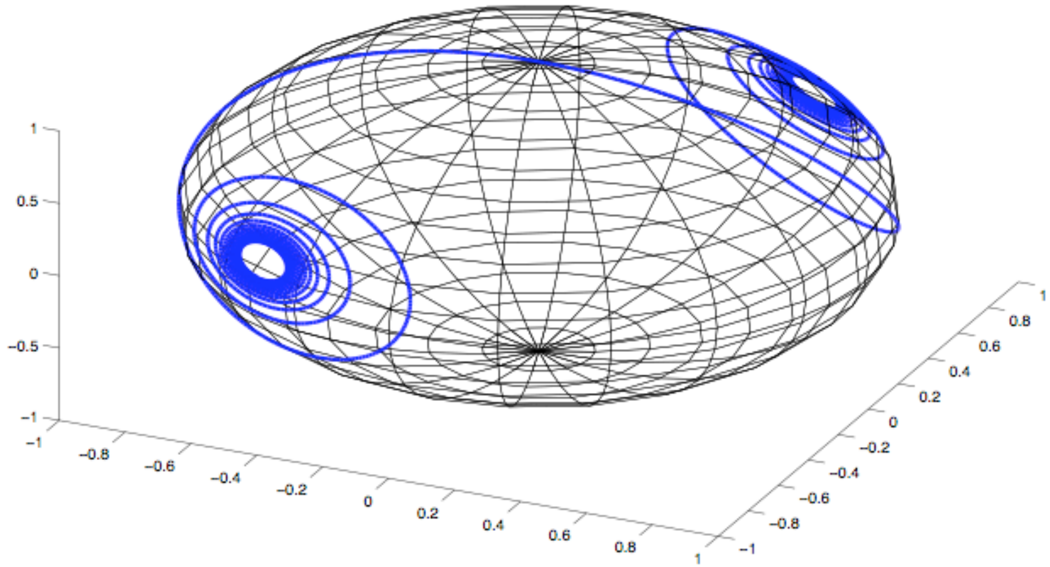
−0.04
−0.02
0
0.02
0.04
0.06
0.08
0.1

Schrödinger map

A natural extension could be to consider the Euler-Cornu spiral in the sphere:

$$z : \mathbb{R} \longmapsto \mathbb{S}^2 \qquad ; \qquad k(s) = c_0 \frac{s}{2} \qquad (2)$$



We could ask if there is a Schrödinger equation associated to (2). The answer is yes and it is given by:

$$u : \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{S}^2$$

(SM)

$$\begin{aligned} u_t &= J D_s u_s = J \Delta_s u \qquad J \text{ complex structure} \\ &= u \wedge u_{ss} \end{aligned}$$

This is a non-linear equation that appeared for the first time in 1906 in the work of Da Rios. He arrives at this model as a simplification of the **dynamics of a vortex filament**. In this case

$$u = X_s \qquad \text{the tangent vector}$$

and X solves

$$X_t = X_s \wedge X_{ss} = cb \quad (b = in).$$

Let us look at (SM), and try to understand it a little bit better by computing the stereographic projection of u on the complex plane

$$u \hookrightarrow z \qquad |u| = 1$$

$$z = x(s,t) + iy(s,t) = 2\frac{u_1 + iu_2}{1 - u_3}$$

$$\partial_t z = -i\partial_s^2 z + \frac{2i\bar{z}}{|z|^2 + 4}\left(\partial_s z\right)^2 .$$

This can be done in higher dimensions. The corresponding equation is

$$\partial_t z = -i\Delta z + \frac{2i\bar{z}}{|z|^2 + 4}\left|\nabla z\right|^2 . \qquad (\mathrm{SM}_z)$$

As we see this equation is of the type

$$\partial_t z = -i\Delta z + F\left(z, \bar{z}, \nabla z, \nabla \bar{z}\right) .$$

The cubic NLS

The difficulty of the previous PDE is that the non-linear term involves derivatives. However the equation

$$\begin{cases} i\partial_t u &= -\Delta u + |u|^2 u \\ u_0 &= u(x,0) \end{cases}$$

is much easier to solve if we don't care about the regularity of u_0. This is due to the fundamental and elemental chain of identities/inequalities:

$$\begin{aligned} i\partial_t u &= -\Delta u + F \\ \frac{d}{dt}\langle u, u\rangle &= \langle u_t, u\rangle + \langle u, u_t\rangle \\ &= \langle i\Delta u, u\rangle + \langle u, i\Delta u\rangle - [\langle iF, u\rangle + \langle u, iF\rangle] \\ &= 2\mathrm{Im}\,\langle F, u\rangle \leq 2\langle F, F\rangle^{1/2}\langle u, u\rangle^{1/2}. \\ \|u(t)\|_{L^2} &\leq \|u(0)\|_{L^2} + \int_0^t \|F\|_{L^2} dt. \end{aligned}$$

$$\|u(t)\|_{L^2} \quad \leq \quad \|u(0)\|_{L^2} + \int_0^t \|F\|_{L^2} dt$$

- If $F = |u|^2 u$; $\|F\|_{L^2} \leq \sup |u|^2 \|u\|_{L^2}$

and $\sup |u|^2$ is bounded assuming regularity on u_0 and repeating the process for $\partial_x u$, $\partial_x^2 u$, …

- If for example $F = |\nabla u|^2 u$ this method does not work.

$$\partial_t u = -i\partial_s^2 u - i|u|^2 u$$ easy

$$\partial_t z = -i\partial_s^2 z + \frac{2i\bar{z}}{|z|^2+4}\left(\partial_s z\right)^2$$ not that easy

$$\psi = \frac{\bar{z}_s}{|z|^2+4} e^{2i\int_0^s \frac{\mathrm{Im}\,(\bar{z}z_s)}{|z|^2+4}ds'}$$

$$\partial_t \psi = -i\partial_s^2 \psi - i\frac{1}{2}\left(|\psi|^2 - A(t)\right)\psi.$$

The Schrödinger map is a geometric equation and has plenty of structure. The above transformation is better understood in geometric terms. Recall that if X is a family of curves which solves

$$X_t = X_s \wedge X_{ss} = cb,$$

then the tangent vector $u = X_s$ is a solution of the Schrödinger map equation. The above transformation is in fact (Hasimoto)

$$\psi(s,t) = c(s,t)e^{i\int_0^s \tau(s',t)ds'} \qquad c = \text{ curvature} \quad \tau = \text{torsion.}$$

As a conclusion the Schrödinger map equation in 1d is solved (modulus solving Frenet equations for the curvature and the torsion). However the situation in higher dimensions is not that nice and in fact the bad "derivatives" can not be completely eliminated. This creates a very serious problem when, for example, uniqueness is considered, because one has to look at the equation satisfied by the difference of two solutions.

Landau-Lifshitz

In 1935 Landau and Lifshitz came out with the following equation

$$\text{(LL)} \begin{cases} \partial_t u &= u \wedge u_{ss} + (u \cdot e_3)\, e_3 \qquad |u| = 1 \text{ spin} \\ u(s,0) &= u_0(s). \end{cases}$$

We see we have added the extra-term $(u \cdot e_3)\, e_3$. This corresponds to the existence of a privileged direction of magnetization e_3= (0,0,1). This equation can be considered as a fundamental equation for **ferromagnetism**.

This extra term implies the existence of a solution *Q* such that

$$Q(-\infty) = -e_3 \quad \text{y} \quad Q(+\infty) = e_3$$

which plays the rôle of a switch-off/switch-on mechanism. Therefore the stability of this solution is a crucial question and as far as I know it is an open problem if no dissipation is added. In other words this extra term is by no means harmless.

Let us compute which is the stereographic projection of this equation. It is the following

$$\text{(LL)} \quad \begin{cases} \partial_t z & = & -i\partial_s^2 z + \dfrac{2i\bar{z}}{|z|^2+4}\left(\partial_s z\right)^2 - i\dfrac{|z|^2-4}{|z|^2+4} z \\ z(s,0) & = & z_0(s). \end{cases}$$

One could think this equation can not be much more complicated than the previous one. We have added just a 0-order term that, we said before, it was harmless. However this is not completely true because in this case we have non-linear terms involving derivatives. In fact, it is not a geometric equation.

A natural question is:

Is there a generalization of Hasimoto transformation so that we are reduced to a non linear equation similar to the cubic one?

The answer is **NO**.

Is there a generalization of Hasimoto transformation so that we are reduced to a non linear equation similar to the cubic one plus small first order terms?

The answer is **YES**.

Integrating factor

Before going on, let us see that Hasimoto's transformation can be seen as some kind of gauge (integrating factor) which cancels the first derivatives:

$$\partial_x^2 u + a(x)\partial_x u + bu = 0$$

$$v = \exp\left(\frac{1}{2}\int_0^x a\right) u$$

$$\partial_x^2 v = \left[\partial_x^2 u + a(x)\partial_x u + \left(\frac{1}{4}a^2 + \frac{a'}{2}\right) u\right] \exp\{\frac{1}{2}\int_0^x a\}$$

$$\partial_x^2 v + \left(b - \frac{1}{4}a^2 - \frac{a'}{2}\right) v = 0.$$

This turns out to be a fundamental trick and most of our work has been to give general conditions which assure the existence in higher dimensions of such integrating factor.

Positive commutators (Local smoothing)

We have seen that in order to understand the cubic equation and more generally equation (1) with $F = F(u, \overline{u})$ we just computed

$$\frac{d}{dt}\langle u, u\rangle = \frac{d}{dt}\int u\overline{u}\,dx \qquad \text{(Conservation of mass).}$$

In order to deal with more general non-linearities $F = F\left(u, \overline{u}, \nabla u, \nabla\overline{u}\right)$ we will rely on the elemental identity:

$$\frac{d}{dt}\langle Ku, u\rangle = i\left\langle (KH - HK)\,u, u\right\rangle \qquad (K \text{ time independent})$$

with u a solution of

$$\partial_t u = iHu \qquad H = H^* \qquad (H \text{ time independent})$$

$$\frac{d}{dt}\langle Ku, u\rangle = i\,\langle (KH - HK)\,u, u\rangle$$

Then if

(#) $$i\,(KH - HK) \geq 0$$

and

(##) $$\sup_{0<t\leq T} \|Ku\|_{L^2} < C'$$

we obtain an a priori bound.

For example if K as an operator of multiplication by a bounded function. Then (##) will follow from the conservation of mass.

More generally we could consider

(*) $$K = \phi(x)\partial_x^\alpha,$$

but unfortunately the algebra does not work and choices as (*) don't give (#). We have to extend the algebra of differential operators to the one of pseudo-differential operators.

For example if

$$H = -\partial_x^2$$
$$K = \frac{2}{\pi}\arctan x \cdot h$$

with h the Hilbert transform

$$\widehat{h}(\xi) = i\pi \operatorname{sig} \xi,$$

then

$$i\,(HK - KH) \geq \frac{c_0}{1+x^2}\sqrt{-\partial_x^2} - C_1.$$

Notice that

$$\|Ku\|_{L^2} \leq \|hu\|_{L^2} = \|u\|_{L^2}.$$

Therefore both conditions (#) and (##) are satisfied.

Quasi-linear equations. Elliptic case

Consider the model problem

$$\partial_t u = i\Delta_a u + i\Delta_b \overline{u} + \overrightarrow{b_1}\nabla u + \overrightarrow{b_2}\nabla \overline{u}$$

with $\Delta_a = \partial_j a_{jk}\partial_k$, $\Delta_b = \partial_j b_{jk}\partial_k$.

We will solve it in two steps. In order to fix the ideas assume that $a \in \mathbb{R}$, $b \in \mathbb{C}$ are constant. Then let us look at the constant coefficient PDE:

$$\partial_t u = ia\Delta u + ib\Delta \overline{u}$$

Writing it as a system we get

$$\partial_t \begin{pmatrix} u \\ \bar{u} \end{pmatrix} = i \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} \Delta \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$$

which can be easily diagonalized because

$$\det \begin{pmatrix} a-\lambda & b \\ -\bar{b} & -a-\lambda \end{pmatrix} = 0 \qquad ; \qquad \lambda^2 - a^2 + |b|^2 = 0$$

$$\partial_t \begin{pmatrix} u \\ \bar{u} \end{pmatrix} = i\mathcal{O} \begin{pmatrix} \sqrt{a^2-|b|^2} & 0 \\ 0 & -\sqrt{a^2-|b|^2} \end{pmatrix} \Delta \mathcal{O}^{-1} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$$

and then $a^2 - |b|^2 > 0$ is a necessary condition.

This can be easily generalized to the variable coefficent situation using classical pseudo-differential equations.

Hence defining the new unknown

$$\mathcal{O}^{-1}u = v$$

we are reduced to look at the problem

$$\partial_t v = i\Delta_a v + \overrightarrow{b_1}\nabla v + \overrightarrow{b_2}\nabla \bar{v}.$$

In order to get a positive commutator the $\nabla \bar{u}$ term does not have the right simmetry.

Let us consider therefore the simplified problem

$$\partial_t v = i\Delta v + \overrightarrow{b_2}\nabla \bar{v}.$$

In this case we have to consider the system

$$\partial_t \begin{pmatrix} v \\ \overline{v} \end{pmatrix} = i \begin{pmatrix} \Delta & \overrightarrow{b_2}\nabla \\ \overrightarrow{\overline{b}_2}\nabla & -\Delta \end{pmatrix} \begin{pmatrix} v \\ \overline{v} \end{pmatrix}$$

and the “eigenvalues” are given by

$$0 = \det \begin{pmatrix} \Delta - \lambda & \overrightarrow{b_2}\nabla \\ \overrightarrow{\overline{b}_2}\nabla & -\Delta - \lambda \end{pmatrix} = \lambda^2 - \Delta^2 - \left|\overrightarrow{b_2}\nabla\right|^2 .$$

Hence

$$\lambda = \pm\Delta\sqrt{1 - \Delta^{-2}\left|\overrightarrow{b_2}\nabla\right|^2}$$

which is easily handled as a classical pseudo-differential operator.

As a conclusion the algebra of classical pseudo-differential operators allows us after some work to obtain both

- L^2 bound
- local smoothing estimate (positive commutator)

Non-elliptic case

It is much more delicate. Assume for simplicity $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$H = \Delta_x - \Delta_y$$

Then a quite simple algebraic argument implies that the only possible second order perturbation in $\overline{u}$ will be of the same type

$$\partial_t u = i\left(\Delta_x u - \Delta_y u\right) + b\left(\Delta_x - \Delta_y\right)\overline{u} \quad |b| < 1$$

With respect to first order perturbations:

$$\text{(3)} \qquad \partial_t u = i\left(\Delta_x - \Delta_y\right)u + \overrightarrow{b_2} \cdot \nabla \overline{u}$$

The diagonalization procedure given above does not work because

$$\left(\Delta_x - \Delta_y\right)^{-2} \left|\overrightarrow{b_2} \cdot \nabla\right|^2$$

it is of “order zero” which is not enough.

Then we have to follow a different approach based on building the integrating factor to almost kill the first order terms. We define a new unknown

$$v = Ku$$

such that if u solves (3) then v solves

$$\partial_t v = i\,(\Delta_x - \Delta_y)\,v + \epsilon\ (\text{first order}) + \epsilon^{-1}\ (\text{zero order}).$$

In order to find K we need a larger class than the one of classical pseudo-differential operators. Then a fundamental part of our work is to develop the corresponding calculus. I will skip the details. However it is important to say that the construction of K heavily relies on the qualitative and quantitative properties of the solutions to the bicharacteristic equations associated to the hamiltonian.

The theorem

Given a "good" u_0 there exists a time T>0 and a unique solution to the IVP

$$\text{(NLS)} \begin{cases} \partial_t u &= i\partial_j a_{jk}\partial_k u + F(u, \nabla u, \bar{u}, \nabla\bar{u}) \\ u(x,0) &= u_0(x) \end{cases}$$

with $x = (x_1, x_2) \in \mathbb{R}^{n+m}$, $0 < t < T$ F non–linear.

- $a_{jk} = a_{jk}(x, t, u, \nabla u, \bar{u}, \nabla\bar{u})$, $A = (a_{jk})_{jk}$ real, symmetric and invertible

$$A \sim \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix} \qquad |x| \sim \infty.$$

- The solutions of the hamiltonian flow associated to $(a_{jk})_{jk}$ are non trapping.
- There exists N such that

$$\left(1 + |x|^2\right)^{N/2} |\partial^\alpha u_0| \in L^2 \quad \text{for } |\alpha| \leq N.$$

Some open problems

The method I have showed above uses techniques developed in Harmonic Analysis. In particular the pseudo-differential calculus plays a fundamental role. In this section I would like to mention some open problems that have appeared in a natural way in the process of solving equation (NLS).

The maximal function

In order to be able to use the local smoothing estimate it is necessary to understand the following maximal fucntion introduced by L. Carleson in the late 70's.

$$u^*(x) = \sup_{|t|\leq 1} \left| \int_{\mathbb{R}} e^{it\xi^2 + ix\xi} \widehat{u}_0(\xi) d\xi \right|$$

Question 1.- Is the following estimate true?

$$\|u^*(x)\|_{L^2} \leq c \|u_0\|_{H^{1/2}}$$

It could be useful for proving the stability of the Landau-Lifshitz solution.

The symbols

In order to construct the integrating factor the following pseudo-differential operator appears

$$a(x,D)f = \int_{\mathbb{R}^2} e^{ix\xi} a(x,\xi)\widehat{f}(\xi)d\xi$$

$$a(x,\xi) = e^{ix\cdot\frac{\xi^\perp}{|\xi|}}\theta(|\xi|) \qquad \xi^\perp = (\mp\xi_2,\xi_1) \quad \xi = (\xi_1,\xi_2),$$

with θ a function which is zero in a neighborhood of the origin.

Question 2.- Is *a*(*x*,D) bounded in L^p?

The 1d cubic NLS

Consider the IVP

$$\begin{cases} i\partial_t u &= \partial_{xx} u \pm |u|^2 u \\ u(x,0) &= \text{a finite measure} \end{cases}$$

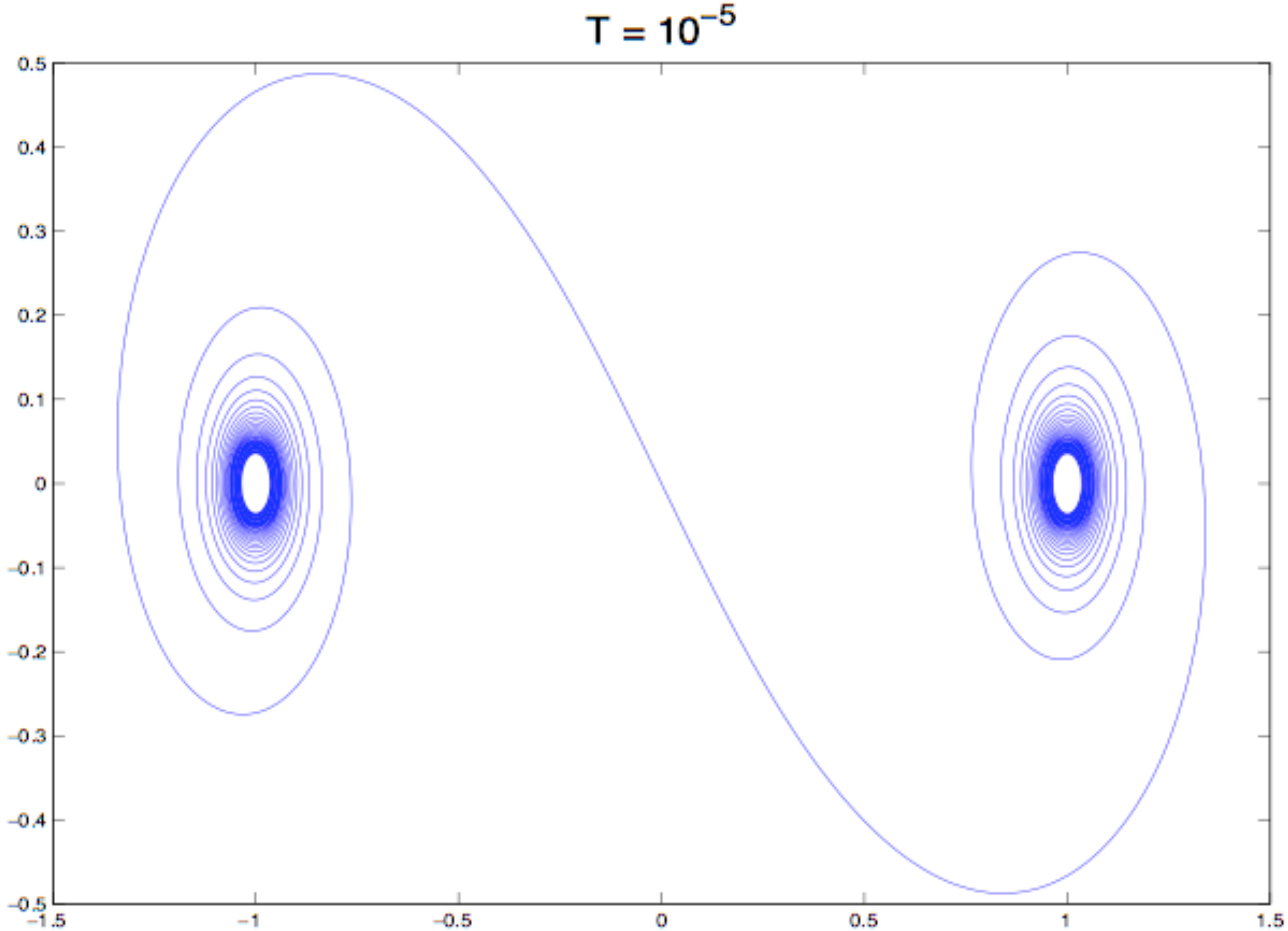
Question 3.- **Is it locally (globally) well-posed for**

- $x \in \mathbb{R}$?

- $x \in \mathbb{T}$?

Vargas-Vega, Grünrock, Christ.

Connection with Euler-Cornu spiral.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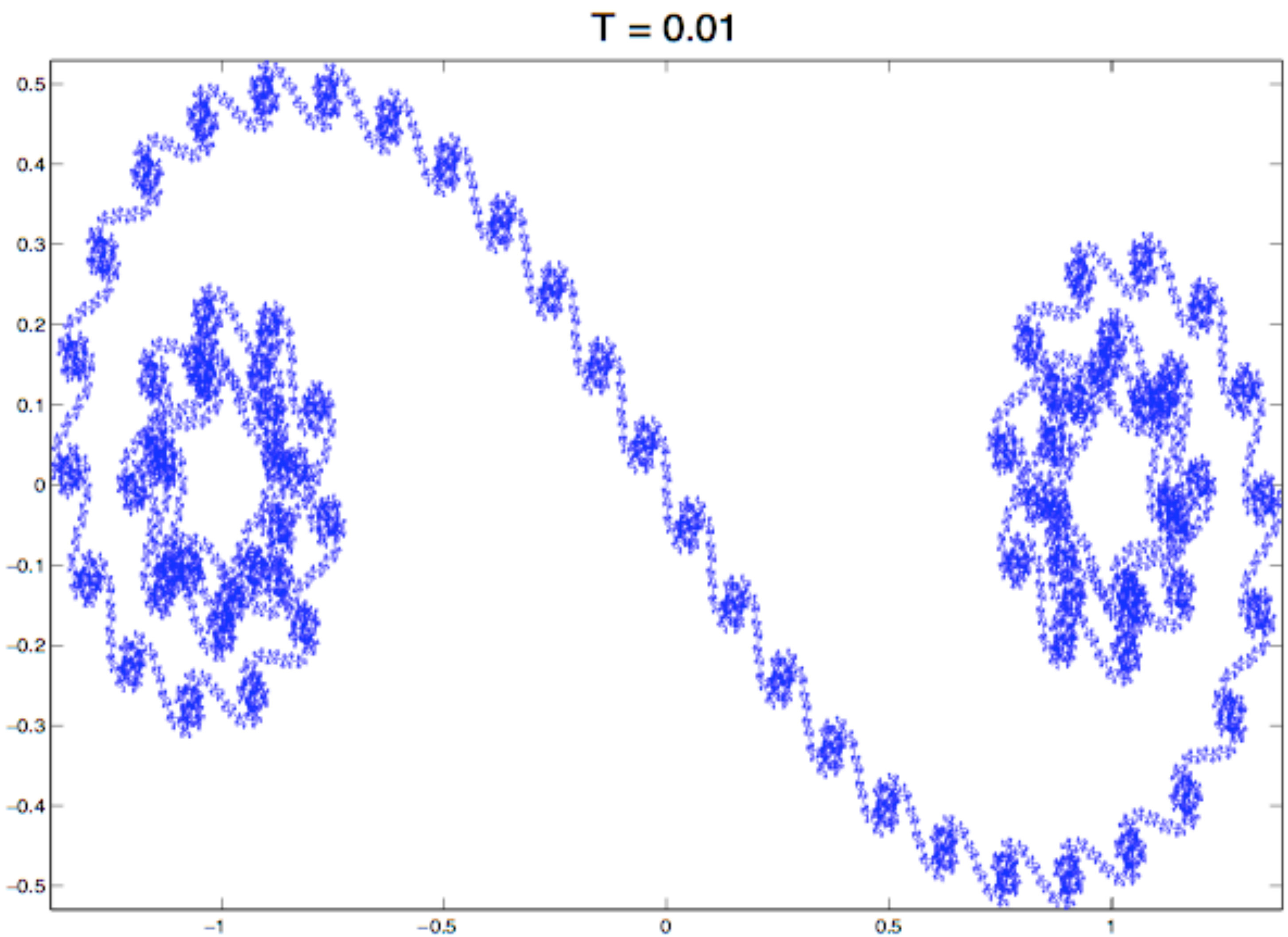

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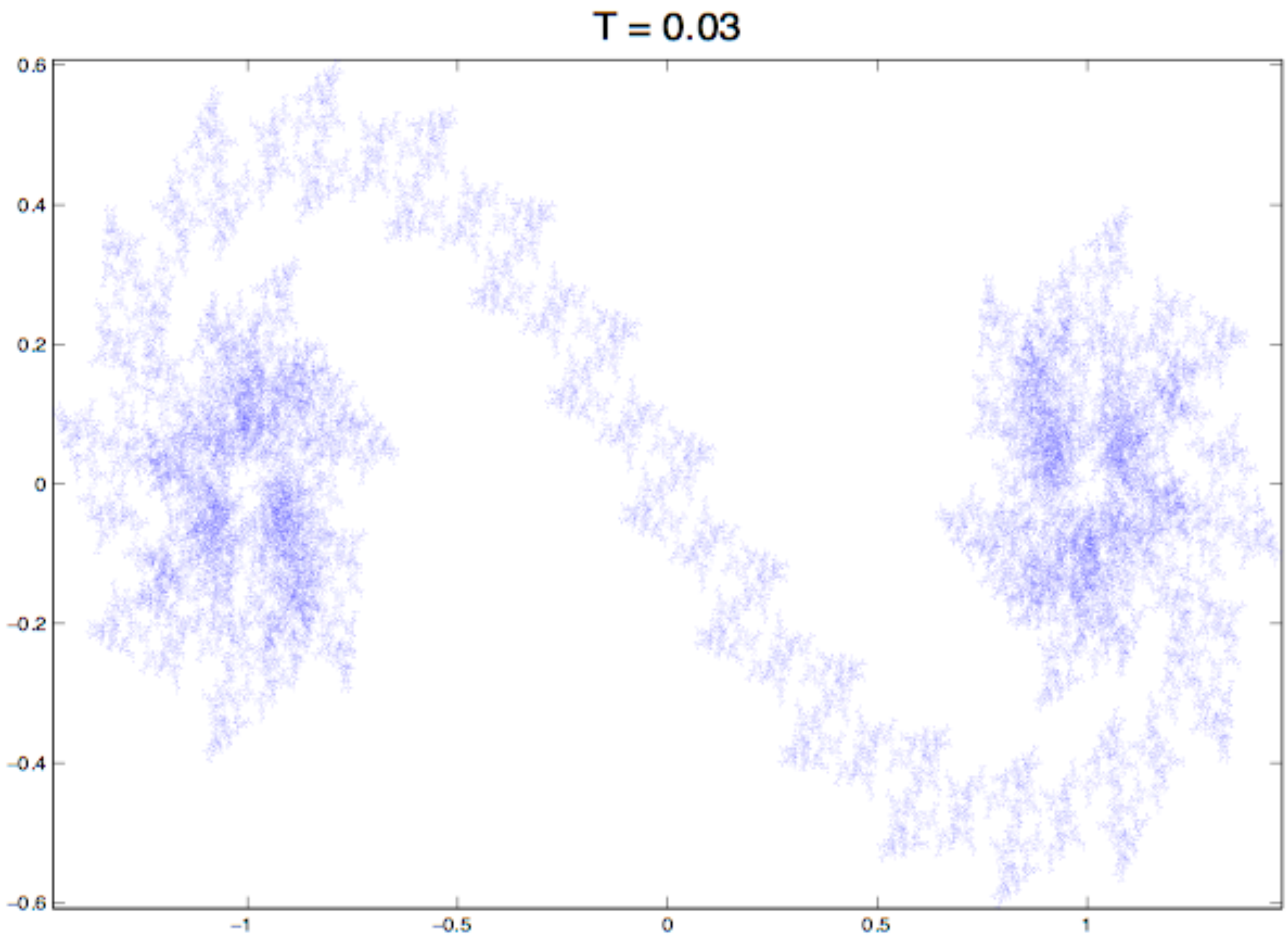
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