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Mini-Workshop: Convergence of Adaptive Algorithms

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Introduction by the Organisers

The Oberwolfach mini-workshop "Convergence of Adaptive Algorithms" originated from a previous Oberwolfach meeting 16/2004 on the topic of "Self-adaptive Methods for Partial Differential Equations" which took place in Spring 2004. One motivation for the mini-workshop was the resolution of the key issues of 'error reduction' in adaptive finite element schemes and the necessity, or otherwise, for 'coarsening strategies' in adaptive algorithms. While the former topic might be regarded as more theoretical, the latter has important practical repercussions in the sense that essentially every practical numerical example would indicate that coarsening is unnecessary. However, the existing proofs of optimal complexity would seem to suggest that coarsening is essential if one is to control discretisation error at an optimal computational cost.

Set against this background, the mini-workshop comprised of 18 leading experts on the convergence of adaptive finite element methods representing 8 different countries and three continents, who identified and discussed the following specific open questions:

(1) For which class of problems and adaptive finite element methods can convergence and error reduction properties be guaranteed?
(2) In what sense and for which classes of algorithm and mesh refinement schemes are adaptive algorithms optimal?
(3) Is coarsening necessary to guarantee the optimality of an adaptive algorithm?
(4) Can the proofs of convergence for adaptive algorithms be carried over from the bulk criterion to other more widespread criteria often used in practice, such as the maximum criterion?

During the mini-workshop 11 talks were given concerning adaptive finite element methods and covering a range of new extensions to the classical convergence analysis were presented. The talks directly addressed the important issues including the role of coarsening, marking rules, hp-adaptive refinement strategies, discrete weighted residual (DWR) adaptive methods in addition to the convergence of non-conforming and mixed methods. The participants also presented very recent work on applications to new classes of equations, e.g. for rough and non-conforming obstacles, for the Laplace–Beltrami operator and the Stokes equations.

The presentations were complemented by several more wide-ranging discussion sessions on open questions and future directions in the field. In particular, it was widely felt that in the case of the class of adaptive algorithms for which there is a proof of optimality, more numerical experiments are necessary to achieve a deeper understanding of the insights and issues highlighted by the abstract analysis. Moreover, numerical experiments were seen to be important in providing quantitative information on the generic constants that appear in the abstract error bounds, where it appears infeasible to derive realistic estimates of the constants that arise in the existing theory. In a similar vein, it would also be of considerable interest to quantify the saving in computational effort through the use of different adaptive schemes and in comparison to uniform refinement. Furthermore, the theory may be used to identify specific examples where coarsening steps are really needed to attain an optimal algorithm. More generally, the identification of a suite benchmark tests and comparisons with other adaptive strategies, for which current theory is lacking, was also suggested.

The importance of understanding the relationship between the numerical solution and the best approximation in the pre-asymptotic range as one can construct problems for which the cost of computations in the asymptotic range is prohibitively high.

Duality-based adaptive strategies compute a weighting of the relevance of the data in the course of the calculation. Starting with this aspect, it was also discussed how the convergence analysis of adaptive algorithms can be related to a data analysis of the problem. Participants proposed that the analysis of duality-based strategies provides an indication that after sufficiently many adaptive refinement steps it may simply be the case that the best strategy to continue the computation with uniform refinement. The presence of singularities in the solution may play a subtle role here.

Part of the session was dedicated to adaptive refinement strategies in three space dimensions, including the question of convergence of adaptive methods in this setting.

Another topic hotly discussed were outstanding hp-approximation issues. Participants agreed that automatic decision mechanisms when h- and when p-refinement is preferable but that there is a definite need for further fundamental improvements. The issue of the development and analysis of reliable and efficient error estimators is less developed for the p- and hp-version of the finite element method than for the h-version. Similarly, convergence proofs for hp-adaptive finite element methods need to be addressed in future in more detail.

MSC Classification: 65N12, 65N15, 65N30, 65N50

Mini-Workshop: Convergence of Adaptive Algorithms

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Abstracts

Some Thoughts about Convergence of the DWR Method

ROLAND BECKER AND ROLF RANNACHER

1. INTRODUCTION

The convergence theory for adaptive finite elements has recently received tremendous improvements, which are reported on in this mini-workshop. Convergence of the adaptively generated sequence of discrete solutions towards the solution of an elliptic partial differential equation in the energy norm is proved [5, 6], and even the optimal complexity can be shown in certain cases [4]. Unfortunately, these results cannot directly be used to prove that similar convergence properties also hold for the DWR method [1] for "goal-oriented" mesh adaptation. In this talk we investigate some topics related to convergence of this method.

Let $\Omega \subset \mathbb{R}^d, d = 2,3$, be a bounded domain with polygonal boundary $\partial\Omega$. For given $f \in L^2(\Omega)$, we consider the usual model problem

$$-\Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1}$$

Let $V_h \subset H_0^1(\Omega)$ be a conforming finite element space constructed from a partition $\mathscr{T}_h$ of Ω, and $u_h \in V_h$ the Ritz projection of the solution $u \in H_0^1(\Omega)$, defined by

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{2}$$

Here, $(\cdot,\cdot)$ denotes the L^2 scalar product on Ω. The basic idea of goal-oriented a posteriori error estimation is to bound the error with respect to a given functional $J \in H^{-1}(\Omega)$. In many interesting cases the functional is more singular and has to be regularized. We then introduce the adjoint solution $z \in H_0^1(\Omega)$ by

$$(\nabla v, \nabla z) = J(v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

With this we find that, with an appropriate interpolation operator I_h,

$$\begin{aligned} J(u) - J(u_h) &= (\nabla(u-u_h), \nabla z) = (f, z - I_h z) + (\nabla u_h, \nabla(z - I_h z)) \\ &= \sum_{K \in \mathscr{T}_h} \left\{ (f + \Delta u_h, z - I_h z)_K - \tfrac{1}{2}([\partial_n u_h], z - I_h z)_{\partial K} \right\}, \end{aligned} \tag{4}$$

and the standard mesh adaptation procedures can be employed. However, here the unknown function z has in general to be approximated, and a *lower* bound for the estimator with respect to the error is generally impossible. Such a lower bound exists for estimation with respect to the energy norm and is a key property for convergence proofs.

2. SOME TECHNICAL PROBLEMS

The convergence analysis of the DWR method depends on two critical ingredients, which are presented next. They can be formulated as independent questions concerning the finite-element Ritz projection.

2.1. **Stability of second-order differences.** The first hypothesis concerns the stability of the finite element solution. Let K denote the cells of the mesh $\mathscr{T}_h$, and E an edge of the cell boundary ∂K. Then, we conjecture that for $u \in C^2(\overline{\Omega})$

$$\sum_{E \subset \partial K} h_E^{-3} \|[\partial_n u_h]\|_E^2 \leq C(u), \quad K \in \mathscr{T}_h,\ h > 0. \tag{5}$$

The left-hand side can be viewed as a second-order difference quotient which we denote by D_h^2 and the inequality (5) resembles a local version of the continuous a priori estimate $\|\nabla^2 u\| \leq C\|f\|$. We further conjecture that (5) even holds true for u with certain singularities.

It is easy to prove that (5) holds on quasi-uniform meshes which are characterized by the additional property that the ratio of maximal cell-width to minimal cell-width is bounded, i.e., $h_{max}/h_{min} \leq C$. The proof relies on a quasi-optimal L^∞-error estimate and the main idea is shortly given:

$$\begin{aligned} |D_h^2 u_{h|K}| &\leq h_K^{-1} \|D_h^2 u_h\|_K \\ &\leq h_K^{-1} \|D_h^2 (u_h - I_h u)\|_K + h_K^{-1} \|D_h^2 I_h u\|_K \\ &\leq c h_K^{-2} \|\nabla e\|_{\tilde{K}} + c h_K^{-2} \|\nabla (u - I_h u)\|_{\tilde{K}} + h_K^{-1} \|D_h^2 I_h u\|_K \\ &\leq c h_K^{-2} \|\nabla e\|_{\tilde{K}} + c h_K^{-1} \|\nabla^2 u\|_{\tilde{K}} \leq c \|\nabla^2 u\|_\infty \end{aligned}$$

where $\tilde{K}$ denotes a cell-patch neighborhood of K. Unfortunately, this argument only works on *quasi-uniform* meshes, since the *local* error estimate (see [7])

$$\|\nabla e\|_{\infty;K} \leq h_K c(u)$$

does not hold in this strong form on meshes with $h_{\max}/h_{\min} \to \infty$.

2.2. **Accuracy of averaging on locally refined meshes.** Our next assumption is again well-established on quasi-uniform meshes. It is needed to theoretically justify the approximation of the interpolation error of the adjoint solution, either by computation with higher accuracy (finer mesh or high-order polynomials) or by local post-processing of the discrete solution computed on the same mesh.

If we suppose that the mesh $\mathscr{T}_h$ is strictly *uniform* with mesh-width h, then, it is known that in the nodal points, the error $z - z_h$ allows an asymptotic expansion in powers of h which can be expressed in the form (see [3])

$$I_h z - z_h = I_h(z - z_h) = h^2 I_h w + h^3 \tau_h, \tag{6}$$

with some h-independent function $w \in H_0^1(\Omega)$ and a remainder $\|\tau_h\| \leq c\|\nabla^3 z\|$.

Our hypothesis is that a similar error expansion as (6) holds on locally refined meshes, where the function w depends on the mesh but this dependency can be localized. To be more precise we assume that the domain can be decomposed into two mesh domains Ω_h and $\Omega \setminus \Omega_h$, such that

$$I_h z - z_h = h^2 I_h w + h^3 \tau_h, \quad x \in \Omega_h, \qquad |\Omega \setminus \Omega_h| \leq Ch. \tag{7}$$

In order to see how assumption (7) might be used for estimating the error in approximation of the dual solution z we give a heuristic argument.

Let $\widetilde{I}_h$ denote a post-processing operator, for example patch-wise quadratic interpolation. The simplest error estimator of the DWR method uses approximation of z by $\widetilde{I}_h z_h$ where z_h is the dual solution on the same mesh. The error term introduced by this can be written as (setting $e := u - u_h$)

$$(\nabla e, \nabla(z - \widetilde{I}_h z_h)) = (\nabla e, \nabla(z - \widetilde{I}_h z)) + (\nabla e, \nabla \widetilde{I}_h (z - z_h)).$$

We only need to estimate the last term, supposing enough regularity of z using the expansion (7). The leading error term can now be estimated by

$$\begin{aligned} (\nabla e, \nabla \widetilde{I}_h (I_h z - z_h)) &= h^2 (\nabla e, \nabla(\widetilde{I}_h w - I_h w))_{\Omega_h} + \text{h.o.t} \\ &\leq Ch^3 \|\nabla^2 w\| + \text{h.o.t.} \end{aligned}$$

Hence, (7) seems to be the key to prove that the proposed approximation is actually of higher order.

3. A CONVERGENT ALGORITHM

In order to guarantee the convergence of the DWR method, we introduce at each step of the mesh refinement iteration an additional finer mesh which is used to control the accuracy of the dual solution. We call these *gendarme meshes*. Reasoning for the corresponding spaces, we have the following scheme:

$$\begin{array}{ccccccc} \dots & \widetilde{V}_{k-1} & & \widetilde{V}_k & & \widetilde{V}_{k+1} & \dots \\ & \cup & \rightarrow & \cup & \rightarrow & & \\ \dots & V_{k-1} & & V_k & & V_{k+1} & \dots, \end{array}$$

At each step of the iteration the meshes are refined by one of the two following rules. The first rule is to refine the base mesh V_k according to the estimator obtained by approximation of the dual solution on the gendarme mesh:

$$\eta_k \approx \sum_{K \in \mathscr{T}_k} \|R(u_h)\|_K \|\widetilde{z}_k - I_k \widetilde{z}_k\|_K.$$

The same refinement procedure is performed for the gendarme mesh. The second rule is to refine the gendarme mesh by an energy error estimator for the adjoint equation. In order to decide which of the two rules is used, we introduce a sequence of tolerances (ε_l), $\varepsilon_l \to 0$. With this we define a sequence of subindices k_l which are defined such that k_l is the first index for which the tolerance ε_l is satisfied, $\eta_{k_l} \leq \varepsilon_l < \eta_{k_l - 1}$. The description of the algorithm is completed by asking for a global refinement of the gendarme mesh if $k = k_l$ for some l, and refinement with the weighted estimator otherwise. By construction, the meshes are always nested, but the basic meshes do not need to contain a sequence of global refinement. Since this is however true for the gendarme mesh, we easily obtain the following result.

Proposition 1. The "gendarme algorithm" ensures convergence in the sense that

$$\lim_{k \to \infty} J(u_k) = J(u).$$

4. Complexity estimate

For analyzing the algorithmical complexity of the DWR method, we may try to adopt the arguments presented in [2, 8] for energy-norm error control. Meshes are generated by hierarchical refinement, which leads to a tree of *admissible* meshes. Let $\mathscr{V}_N := \{V \text{ admissible}, \dim(V) = N\}$. Suppose, we have an error measure $\phi : \cup_{N\in\mathbb{N}} \mathscr{V}_N \to \mathbb{R}$, for example constructed from the error representation (4), which is quasi-monotone,

$$V \subset \tilde{V} \quad \Rightarrow \quad \phi(V) \le c_\phi\, \phi(\tilde{V}). \tag{8}$$

For given $\varepsilon > 0$, we define $\mathscr{V}_\varepsilon := \{V \text{ admissible}, \ \phi(V) \le \varepsilon\}$, and assume that

$$\sup_{\varepsilon>0} \varepsilon^{-1/s} \inf_{V\in\mathscr{V}_\varepsilon} \dim(V) < +\infty, \tag{9}$$

for some $s \ge 0$. We make the following crucial assumption on the refinement algorithm. The spaces V_k are constructed such that, with a constant $\rho < 1$, there holds, with $\widehat{\mathscr{V}}_k := \{V \supset V_{k-1} : \ \phi(V) \le \rho^k \phi(V_1)\}$,

$$N_k = \dim(V_k) \le \dim(V), \quad V \in \widehat{\mathscr{V}}_k, \tag{10}$$

Proposition 2. Let the meshes be constructed such that (10) holds and further assume that (9) is satisfied. Let $\varepsilon > 0$ be given. Suppose that $\mathscr{T}_n$ is the first mesh for which $\rho^n \le \varepsilon$. Then there exists a constant c independent of ε and n such that

$$N_n - N_1 \le c\varepsilon^{-1/s}. \tag{11}$$

Of course, this gives an optimal complexity estimate only if condition (10) can be satisfied with optimal complexity, which is a hard problem still to be solved.

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Adaptive Finite Element Methods with Optimal Complexity

PETER BINEV
(joint work with Wolfgang Dahmen and Ronald DeVore)

Adaptive methods are frequently used to numerically compute solutions to elliptic equations. While these methods have been shown to be very successful computationally, the theory describing the advantages of such methods over their non-adaptive counterparts is still not complete. Recently, it was proven (see [14], [18]) the convergence of such methods. These proofs of convergence still do not show any guaranteed advantage of these adaptive methods since there is no analysis of their *rate of convergence* in terms of the number of degrees of freedom or the number of computations. The only known algorithm with a proven rate of convergence was the one for the univariate case [3].

In our recent paper [2] we propose an Adaptive Finite Element Method (AFEM) and prove convergence rates for this method using as a model example the Poisson problem

$$-\Delta u = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and $\partial\Omega$ is its boundary. As approximations of the solution u we consider piecewise linear elements using a very specific adaptive refinement strategy (called newest vertex bisection) well-known in the FEM literature. We show that if the solution u can be approximated (using complete knowledge of u) in the energy norm by a piecewise linear function on the triangulation with n triangles to accuracy $O(n^{-s})$, $n \to \infty$, then the adaptive method will do the same using *only* knowledge of u gained through the adaptive iteration.

Our algorithm is not much different from existing adaptive methods based on bulk chasing of a posteriori error estimators. The one main difference is the utilization of a coarsening strategy. The role of coarsening in the algorithm is to ensure that at any iteration the approximated solution has near optimal representation in terms of degrees of freedom. We should mention that coarsening also played an important role in the analysis of adaptive wavelet methods (see [4],[5]).

In our analysis we rely on the theory of nonlinear approximation by piecewise polynomials. Since adaptive methods are a form of nonlinear approximation, this theory will on the one hand help us to provide a benchmark for measuring the success of adaptive methods, and on the other hand, provide an effective implementation for the coarsening.

As it was emphasized in [2], we consider our algorithm mainly as a contribution to the theory of adaptive methods. However, the ideas suggested in [2] can be useful in practice. The goal of this presentation is to give an overview of the coarsening strategy and its possible implications in the development of practical adaptive algorithms.

An important feature of the coarsening strategy is that it is applicable practically to any problem for which there exists an error reduction algorithm. In order to avoid complications of the presentation we shall sometimes refer to (1). In this way, the essentials of our arguments will be clear and we can also call on several known results concerning a-posteriori error estimates that can be found in the literature. In particular, we shall make use of the error reduction property given in [18]. However, by no means is the theory

restricted to this particular problem. On the contrary, as it will be made clear in the text, we expect the coarsening strategy to give higher payoff for complicated problems.

An adaptive procedure can be related to a decision tree. Thus it is natural to assume that the space of finite element functions $\mathscr{S}$ is associated to an infinite master tree $\mathscr{T}$, namely, that every function $v \in \mathscr{S}$ corresponds to a tree $T(v) \subset \mathscr{T}$. The trees that correspond to a function from $\mathscr{S}$ are called *admissible* and their set is denoted by $\mathscr{T}_a$. To indicate that the final leaves of a tree $T \subset \mathscr{T}$ correspond to cells of some partition of the domain Ω, we shall call them *cells*. Then the adaptive process can be described as follows. We start with some initial tree $T_0 \in \mathscr{T}_a$ and mark a set M_1 of certain cells for subdivision. After doing these subdivisions we arrive at the tree T_1'. This tree is not necessarily admissible and so we shall make some additional subdivisions which will *complete* T_1' to an admissible partition $T_1 \in \mathscr{T}_a$. We then repeat this process of marking and completing to obtain sets M_k and trees T_k, $k = 1, ..., n$. It will be important for us to see that the completion process does not seriously inflate $\#(T_n)$, the number cells in T_n. Namely, that there is a constant $C_0 > 0$ depending only on T_0 such that

$$\#(T_n) \leq \#(T_0) + C_0(\#(M_1) + \cdots + \#(M_n)) \,. \tag{2}$$

In [2] we proved (2) for the newest vertex bisection procedure.

Given the functions v from the finite element space $\mathscr{S}$, we denote by Σ_N the set of all of them for which $\#(T(v)) \leq N$. The best approximation of a function u is defined by

$$\sigma_N(u) := \inf_{v \in \Sigma_N} \|u - v\| \,.$$

It is unreasonable to expect that any numerical procedure will result in finding the best approximant to u. Usually, the goal is to find a procedure which exhibits the same rate of convergence for the error as $\sigma_N(u)$. Here we have higher standards requiring that for every N our procedure finds a *near best* approximation u_N to u with $\#(T(u_N)) \leq N$ and the property

$$\|u - u_N\| \leq C_1 \sigma_{c_2 N}(u) \tag{3}$$

with some absolute constants C_1 and c_2. In [1] we consider a *tree approximation* procedure based upon special error functionals placed on the nodes of $\mathscr{T}$. In the case the norm of the error $\|u - u_N\|$ is equivalent to the discrete norm of these error functionals over the leaves of $T(u_N)$, we have shown that (3) is valid with constants C_1 and c_2 relatively close to 1. Moreover, the number of computations needed to find u_N is $\mathscr{O}(N)$. In [2] we apply these results to find a near best approximation in H^{-1}-norm to the right hand side f of (1), as well as to find a near best approximation in the energy norm to the approximate Galerkin solution in the coarsening step.

The decisions in standard AFEM are often based upon *local error estimators* which sum $\Phi(T)$ gives a reliable estimate of the square of the error in the energy norm for a given tree T. This estimate is used as stopping criteria in different procedures. The knowledge of $\Phi(T)$ is also important in the choice of the numerical precision of the algorithms in AFEM.

A basic ingredient of AFEM is the *error reduction procedure*. Given approximate solution u_T with a tree T, it finds a refinement T^+ of T and an approximate solution u_{T^+} which error is at least C_3 times smaller than the one for u_T. In addition, we have

that $\#(T^+) \leq C_4\#(T)$ and the number of calculations needed does not exceed $C_5\#(T)$. This procedure can be composed as several iterations of bulk chasing (see [14], [18]), or could be just several consecutive uniform refinements of the current partition tree T. We denote the result of the latter by $\mathscr{R}(T)$. It is important to note that for fixed C_3 we can design different algorithms that have different constants C_4 and C_5. Keeping the optimality constant C_4 low will increase the efficiency constant C_5 and vice versa.

Finally, the *coarsening step* takes the current approximate solution u_{T^+} and finds a sparse near best approximation $u_{T^*} \in \mathscr{S}$ to it using the tree algorithm with stopping criterium based upon the estimate of the error $\Phi(T^+)$. This could eventually increase the error by at most $C_3/2$ times but will keep the solution near optimal in terms of the complexity of T^*. In case $\Phi(T^+)$ is equivalent to the square of the error in the energy norm, the approximation u_{T^*} to u is near best in terms of (3). Theoretically, the algorithm continues with the error reduction procedure setting $T := T^*$. However, in practice we may use a certain refinement of T^* instead.

Algorithmically, the biggest advantage of the coarsening strategy is the possibility to use error reduction with a small constant C_5 and by this increasing the efficiency of the algorithm. In comparison, the algorithms without coarsening should keep the constant C_4 low (in a hope that the optimal convergence rates will be preserved) which could result in an enormous number of iterations with a very small bulk.

The utilization of coarsening strategy could take different forms. The coarsening step could be used as an inexpensive test for the optimality of the solution. In case $\#(T^+)$ is not much higher than $\#(T^*)$, we can continue with $T := T^+$ instead of $T := T^*$. It should be also clear that using coarsening strategy does not always mean that the partition should be coarsen. For example, we can immediately set $T^+ := \mathscr{R}(T^*)$ and calculate the local error estimators only to use $\Phi(T^+)$ as a threshold in the tree algorithm. In this case the adaptivity of the procedure comes from near optimality properties of the tree approximation. Blending this approach with bulk chasing could be very beneficial for solving complicated problems which require heavy calculations to find the local error estimators and/or do not have lower estimates to the error of the approximate solution. Some ideas from the tree algorithms can also be used in the design of the error reduction procedure in practice, although the theory for this is still under development.

In conclusion, the coarsening strategy not only provides an AFEM with best performance rates and a near best approximation of the solution, it also can give new opportunities of improving the existing practical algorithms especially for problems with complicated solutions.

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Survey on the Convergence of Adaptive Finite Element Methods

CARSTEN CARSTENSEN

State of the art simulations in computational mechanics aim reliability and efficiency via adaptive finite element methods (AFEMs) with a posteriori error control. The a priori convergence of finite element methods is justified by the density property of the sequence of finite element spaces which essentially assumes a quasi-uniform mesh-refining. The advantage is guaranteed convergence for a large class of data and solutions; the disadvantage is a global mesh refinement everywhere accompanied by large computational costs.

AFEMs automatically refine exclusively wherever the refinement indication suggests to do so and so violate the density property on purpose. Then, the a priori convergence of AFEMs is not guaranteed automatically and, in fact, crucially depends on algorithmic details. The advantage of AFEMs is a more effective mesh accompanied by smaller computational costs in many practical examples; the disadvantage is that the desirable error reduction property is *not always* guaranteed a priori. Efficient error estimators can justify a numerical approximation a posteriori and so achieve reliability. But it is *not* clear from the start that the adaptive mesh-refinement will generate an accurate solution at all.

This paper discusses particular versions of an AFEMs and their analyses for error reduction, energy reduction, and convergence results for linear and nonlinear problems.

1. INTRODUCTION

This paper discusses limitations and generalisations of the recent convergence theory of adaptive finite element methods (AFEMs) so far mainly established for the Laplacian and the p-Laplacian in [4, 14, 18, 24]. Therein and below, AFEMs consist of recursive loops of the form

$$\text{SOLVE} \;\to\; \text{ESTIMATE} \;\to\; \text{MARK} \;\to\; \text{REFINE}. \tag{1}$$

There exists a vast literature on a posteriori error control for step ESTIMATE and we refer to the books [1, 3, 15, 23] and the reference included therein plus some select references [2, 6, 11, 17, 20, 21] for elastoplasticity.

A typical reliable error estimator, such as the explicit error estimator, results in local contributions η_M associated with an edge, face, or element M in the current mesh and their sum $\eta^2 := \sum_M \eta_M^2$ over all such objects M. Frequently in the literature, a maximum criterion marks a subset $\mathscr{M}$ according to

$$M \in \mathscr{M} \text{ if and only if } \eta_M \geq \Theta \max \eta,$$

where $\max \eta$ denotes the maximum of all of the η_M and $0 \leq \Theta < 1$ is a parameter. Even though the titles of corresponding articles and books suggest adaptive algorithms, those

mesh-refinement strategies are considered therein without any mathematical analysis. The numerical experiments reported in those references appear to be extremely successful. This success, however, is *not* understood in theory and hence *not* guaranteed in the forthcoming refinement loops.

This papers follows [14, 4, 18] and adopts the bulk-criterion in the step MARK which defines a set $\mathscr{M}$, the marked objects, by

$$\sum_{M\in\mathscr{M}} \eta_M^2 \geq \Theta\,\eta^2$$

with a parameter $0<\Theta\leq 1$. Section 2 introduces the adaptive algorithm and the mesh-refinement strategy for an edge-oriented explicit error estimator. Section 3 discusses the error reduction property and some counter example. Section 4 studies the main arguments for energy reduction. Further details, proofs, and software will appear elsewhere [9, 10, 12].

2. Adaptive Mesh-Refining

A typical adaptive algorithm is sketched below where, on each level $\ell=0,1,2,3,\ldots$, the discrete stress σ_ℓ is piecewise constant with respect to the triangulation $\mathscr{T}_\ell$ with the set $\mathscr{E}_\ell$ of edges and faces in 2D and 3D, respectively. Then, for each edge or face $E\in\mathscr{E}_\ell$ of diameter $h_E:=\operatorname{diam}(E)$ and with unit normal ν_E, the contribution

$$\eta_E^{(\ell)} := h_E^{1/2}\Big(\int_E |[\sigma_\ell]\nu_E|^2\,ds\Big)^{1/2} \tag{2}$$

accounts for the jump $[\sigma_\ell]\nu_E$ of the discrete stresses across the interior edge E in the normal direction.

Adaptive Algorithm (AFEM)
Input: Coarse shape-regular triangulation $\mathscr{T}_0$ of Ω into triangles with set of edges $\mathscr{E}_0$; $0<\Theta<1$.
For $\ell=0,1,2,3\ldots$ **do (a)—(e):**
(a) Solve the discrete problem with respect to the actual mesh $\mathscr{T}_\ell$ and corresponding FE spaces. Let u_ℓ denote the FE displacement and let σ_ℓ denote the discrete stress field.
(b) Compute $\eta_E^{(\ell)}$ for all edges or faces $E\in\mathscr{E}_\ell$ and $\eta_\ell:=(\sum_{E\in\mathscr{E}_\ell}(\eta_E^{(\ell)})^2)^{1/2}$ as stress-error estimator.
(c) Generate a set $\mathscr{M}_\ell$ of edges or faces in $\mathscr{E}_\ell$ such that

$$\Theta\,\eta_\ell^2 \leq \sum_{E\in\mathscr{M}_\ell}(\eta_E^{(\ell)})^2. \tag{3}$$

(d) Control oscillations OSC_ℓ and (possibly) add further edges to $\mathscr{M}_\ell$ to decrease $\mathrm{OSC}_{\ell+1}\leq\Theta\,\mathrm{OSC}_\ell$.
(e) Run closure algorithm to avoid handing nodes; refine all triangles T with some edge or face E in $\mathscr{M}_\ell$ with bisc5(T) and all other elements with red-green-blue or newest-vertex bisection refinement after Figure 1. Let $\mathscr{T}_{\ell+1}$ denote the resulting shape-regular triangulation.

Output: Sequence of discrete stress fields $\sigma_0, \sigma_1, \sigma_2, \dots$ in $L^2(\Omega; \mathbb{R}^{d\times d}_{sym})$.

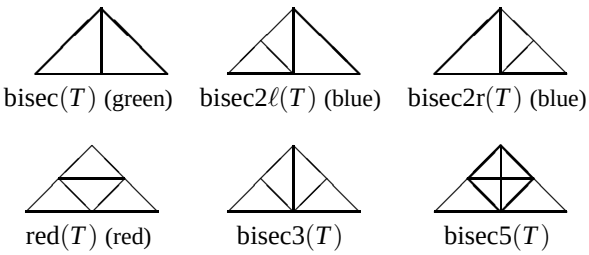

FIGURE 1. Possible refinements of a triangle for red-green-blue and newest-vertex bisection refinement [4, 8, 12].

This algorithm yields a strongly convergent stress field in linear and nonlinear elasticity as well as in elastoplasticity [10].

3. Error Reduction Property

The error reduction property is frequently also called saturation assumption when it is used as a hypothesis. However, this property has to be guaranteed by the mesh-design. To fix ideas, suppose that σ_ℓ is the finite element stress approximation to the exact stress field σ in level ℓ as in Algorithm 2. Then, consider the error reduction property in the form

$$|||\sigma - \sigma_{\ell+1}|||^2 \le \rho\, |||\sigma - \sigma_\ell|||^2 + C\,\mathrm{OSC}_\ell^2 \qquad \text{for all } \ell = 0,1,2,\dots. \tag{4}$$

Here, $|||\cdot|||$ denotes the energy norm and $\rho < 1$ is the reduction factor. The last term $C\,\mathrm{OSC}_\ell^2$ accounts for oscillations of the data, i.e. in f when we suppose an equilibrium of the (strong) form

$$f + \operatorname{div}\sigma = 0 \text{ in } \Omega$$

and suppose Dirichlet boundary conditions on the entire boundary $\Gamma_D := \Gamma := \partial\Omega$ of the domain Ω such that no traction conditions appear (which would arise further data oscillations [10]). Then,

$$\mathrm{OSC}_\ell(f) := \Big(\sum_{z\in\mathscr{K}_\ell} h_z^2 \int_{\omega_z} |f(x) - f_{\omega_z}|^2\, dx \Big)^{1/2}, \tag{5}$$

where $f_{\omega_z} := \int_{\omega_z} f(x)\,dx/|\omega_z|$ abbreviates the integral mean of f over the patch

$$\omega_z := \{x \in \Omega : 0 < \varphi(x)\} \text{ and so } \overline{\omega_z} = \cup\mathscr{T}(z) \quad \text{where } \mathscr{T}(z) = \{T \in \mathscr{T}_\ell : z \in T\}.$$

This ω_z is the interior of the union of the set $\mathscr{T}(z)$ of neighbouring elements of a free node $z \in \mathscr{K}_\ell := \mathscr{N}_\ell \setminus \Gamma_D$ and has volume $|\omega_z|$ and size $h_z := \operatorname{diam}(\omega_z)$.

The first observation on error reduction (4) is that the oscillations cannot be omitted. In fact, given a sequence of (even successive e.g. uniform refinements and associated)

discrete spaces $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ there exists a problem with some exact solution u and with discrete solutions u_ℓ and

$$|||u - u_\ell||| = \mathrm{dist}_{|||\cdot|||}(u; V_\ell) \to 0 \quad \text{as slow as we want.}$$

This is a general fact from approximation theory in separable Hilbert spaces. The point is that the regularity of the exact solution could be as bad as we want and so the convergence could be as poor as possible and this contradicts a linear convergence implied by (4) for $C = 0$. The good news is that the additional term is given in terms of the data and not completely in terms of the unknown regularity of the unknown exact solution.

The second observation is that there exist good and bad refinements. A simple Poisson problem with constant right-hand side $f \equiv 1$ and homogeneous Dirichlet boundary conditions on the unit square for P_1 finite element methods on the meshes $\mathscr{T}_0$ and $\mathscr{T}_1$ of Figure 2 allows no error reduction in this refinement step.

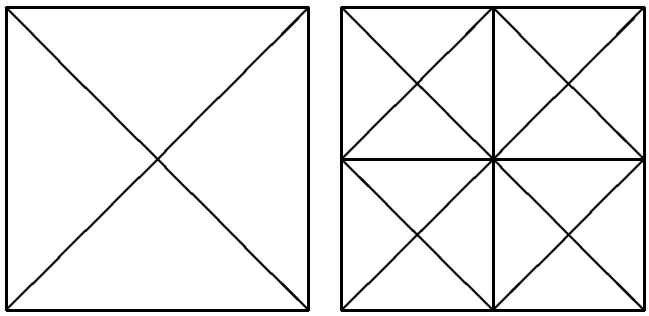

FIGURE 2. Counter example for error reduction: Two meshes with finite element solutions $u_0 = u_1$ and so with $\rho = 1$ and $\mathrm{OSC}_\ell(f) = 0$ in (4).

4. ENERGY REDUCTION PROPERTY

For nonlinear elasticity such as Hencky elastoplasticity with hardening let $E(v)$ denote the elastic energy and $\delta_\ell := E(u_\ell) - E(u)$ the difference of the discrete energy $E(u_\ell)$ to the minimal energy $E(u)$. Assume that the energy E is uniformly convex and its derivative DE is Lipschitz continuous. A typical example is the nonlinear Hencky material or nonlinear Hooke's law of [19, Sect. 3.3] and [25, Sect. 62.8].

The energy reduction property reads, with some $0 < \rho < 1$,

$$\delta_{\ell+1} \le \rho\, \delta_\ell + C\, \mathrm{OSC}_\ell^2 \qquad \text{for all } \ell = 0, 1, 2, \ldots \tag{6}$$

The energy reduction is equivalent to the error reduction in linear problems (e.g. for Poisson or Lamé equations). Observe in the general case that the energy reduction property

implies an error decrease and the R-order of convergence (up to higher-order terms) in the form

$$|||\sigma-\sigma_\ell|||^2 \le \rho^\ell\,\delta_0+\sum_{k=0}^{\ell-1} C\rho^{\ell-1-k}\,\mathrm{OSC}_k^2 \quad \text{for } \ell=1,2,3,\dots \tag{7}$$

The remaining part of this section is devoted to arguments sufficient for energy reduction (6). Under the present assumptions on the energy functional, one may define a computable residual $R_\ell(v) := -DE(u_\ell;v)$ with a norm

$$\|R_\ell\|_{V^*} = \sup_{v\in V\setminus\{0\}} R_\ell(v)/\|v\|_V.$$

A first key argument is equivalence of $\|R_\ell\|_{V^*}$ to the error or the energy differences [9, 10], e.g.

$$\delta_\ell+\alpha|||u-u_\ell|||^2 \le R_\ell(u-u_\ell). \tag{8}$$

A similar consequence of uniform convexity reads

$$\alpha|||u_{\ell+1}-u_\ell|||^2 \le \delta_\ell-\delta_{\ell+1}. \tag{9}$$

A second key argument is refined reliability of the error estimator [13, 10] which reads in terms of the residual as

$$\|R_\ell\|_{V^*} \le C_1\,\eta_\ell + C_2\,\mathrm{OSC}_\ell(f). \tag{10}$$

A third key argument is the residual coverage [9] which leads to a discrete efficiency in the form

$$\eta_\ell \le C_3|||u_{\ell+1}-u_\ell||| + C_4\,\mathrm{OSC}_\ell\,. \tag{11}$$

The proof of (11) follows as in the linear case from the design of a local trial function $b_E \in V_{\ell+1}$ with

$$(\eta_E^{(\ell)})^2 = \int_\Omega \sigma_\ell : Db_E\,dx$$

followed by the equilibrium condition (with respect to the refined mesh and induced finite element space)

$$\int_\Omega \sigma_{\ell+1} : Db_E\,dx = \int_\Omega f\cdot b_E\,dx$$

plus the extra mean property

$$\int_\Omega b_E\,dx = 0.$$

For this extra property, one needs an inner-node property of the mesh-refining. That is, for each marked edge, either one of the neighbouring elements requires a bisec5-refinement or one complete patch is refined in an appropriate way. The counter example of Figure 2 clearly implies that one needs at least one red-refinement [or a perturbation of the configuration in that the newest-vertex must not be the mid-point for all the triangles] in a patch [9].

The combination of the foregoing three identities, namely (with a proper mean f_E of f)

$$(\eta_E^{(\ell)})^2 = \int_\Omega (\sigma_\ell-\sigma_{\ell+1}) : Db_E\,dx + \int_\Omega (f-f_E)\cdot b_E\,dx,$$

plus the bulk criterion eventually lead to (10) [9, 10].

To finish the proof of the energy reduction (6), one employs (8) and (10) to obtain

$$\delta_\ell + \alpha |||u-u_\ell|||^2 \le (C_1\,\eta_\ell + C_2\,\mathrm{OSC}_\ell(f))\|u-u_\ell\|_V \le \alpha\,|||u-u_\ell|||^2 + C_5\,\eta_\ell^2 + C_6\,\mathrm{OSC}_\ell(f)^2.$$

The immediate estimate of δ_ℓ combines first with (11) and second with (9) to

$$\delta_\ell \le C_7\,|||u_{\ell+1}-u_\ell|||^2 + C_8\,\mathrm{OSC}_\ell(f)^2 \le C_7/\alpha(\delta_\ell - \delta_{\ell+1}) + C_8\,\mathrm{OSC}_\ell(f)^2.$$

This implies (6) with $\rho = (1-\alpha/C_7)$ and $C = \alpha C_8/C_7$.

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Automatic hp-Adaptivity. A Progress Report

LESZEK DEMKOWICZ
(joint work with Jason Kurtz, David Pardo)

With appropriately selected element size h and order of approximation p, hp-Adaptive Finite Element Methods should deliver exponential convergence, both for regular and singular solutions, [13]. The first fully automatic hp-adaptive strategy for elliptic problems was proposed and verified for 1D and 2D elliptic problems in [8]. The method was further developed and generalized to 2D Maxwell equations in [5], and for 3D elliptic problems in [12]. For a comprehensive presentation of the whole technology and details of the method, we refer to [4].

The lecture presented a further development of the method focusing on two directions:

- a new implementation of the method for 3D elliptic problems, driven by a class of acoustics scattering problems,
- a fully automatic, goal-oriented hp-adaptive method, with applications to simulations of EM logging.

Automatic hp-Adaptivity for 3D Elliptic Problems. Referring to [9] for details, we shall review quickly the main idea of the method and discuss the accomplishments of the new implementation. The main idea of the method is based on the notion of the *Projection Based Interpolation*, see [6, 3, 7, 2, 4]. Given a hexahedral finite element K, possibly refined into two, four, or eight element-sons, and a function $u \in H^{3/2+\varepsilon}(K)$, we define the corresponding projection-based interpolant u^p as a sum of vertex, edge, face, and interior contributions,

$$\Pi u = u^p = u_1 + \underbrace{\sum_e u^p_{2,e}}_{u^p_2} + \underbrace{\sum_f u^p_{3,f}}_{u^p_3} + u_4 \,.$$

Here u_1 is the standard trilinear vertex interpolant of u. The edge contribution u^p_2 is obtained by summing up contributions $u^p_{2,e}$ from individual edges obtained by projecting

the difference $u-u_1$ onto the "edge bubbles",

$$\|u^p_{2,e}-(u-u_1)\|_{L^2(e)} \to \min .$$

By the edge bubbles we mean the traces of the element shape functions on the particular edge, vanishing at the edge endpoints. For an unrefined element and edge of order p_e, these will be simply polynomials of order p_e, vanishing at the end points. For a refined edge, we mean piece-wise polynomials. The edge contribution is extended to the rest of the element using the element shape functions. Similarly, function u^p_3 is obtained by summing up contributions $u^p_{3,f}$ obtained by projecting difference $u-u_1-u^p_2$ onto face bubbles,

$$|u^p_{3,f}-(u-u_1-u^p_2)|_{H^{1/2}(f)} \to \min .$$

Finally, the element interior contribution is obtained by projecting the difference $u-u_1-u^p_2-u^p_3$ onto the element bubbles,

$$|u^p_4-(u-u_1-u^p_2-u^p_3)|_{H^1(K)} \to \min .$$

It has been shown (for the latest versions of the theory, see [7, 2]) that the projection-based interpolation delivers optimal p and h convergence rates. The energy-driven automatic hp-adaptivity produces a sequence of coarse/fine hp meshes where the fine mesh is obtained from the coarse one by a global hp-refinement, i.e. each hexa in the coarse mesh is broken into eight element-sons, and the order is raised uniformly by one. Both meshes may be very non-uniform, including possible strong anisotropies (both in h and p). The two meshes paradigm is essentially different from standard adaptive methods working only with a single mesh and an error indicator (estimator). The problem is solved on the fine mesh. The fine mesh solution $u=u_{h/2,p+1}$ is then interpolated on the coarse mesh and a sequence of meshes obtained from the coarse mesh with various h and p refinements, to determine the mesh that maximizes the rate with which the interpolation error decreases,

$$\frac{|u-\Pi_{hp}u|_{H^1(K)}-|u-\Pi_{hp_{opt}}u|_{H^1(K)}}{N_{hp}-N_{hp_{opt}}} \to \max .$$

Here $\Pi_{hp}u$ denotes the interpolant on the coarse mesh, and $\Pi_{hp_{opt}}u$ denotes the interpolant on a optimally refined coarse mesh to be determined.

The algorithm explores the logic of the projection-based interpolation by determining first optimal refinements of the coarse element edges, then faces and, finally, the coarse element interiors. The result of each of the steps determines initial conditions for the next (discrete) optimization problem. A *competition* between various types of refinement determines not only whether an edge, face or element is to be h- or p-refinement but also a proper kind of (possibly) anisotropic refinement (recall that a face can be h-refined in three different ways, and an element can be h-refined in seven different ways. Enabling the competitions on face and element levels is probably the most significant departure from the earlier implementations that staged the competition at the edge levels only, and explored the (possibly anisotropic) structure of the error to choose between the anisotropic and isotropic h-refinements.

We list shortly other significant advancements of the new implementation and differences with the previous work.

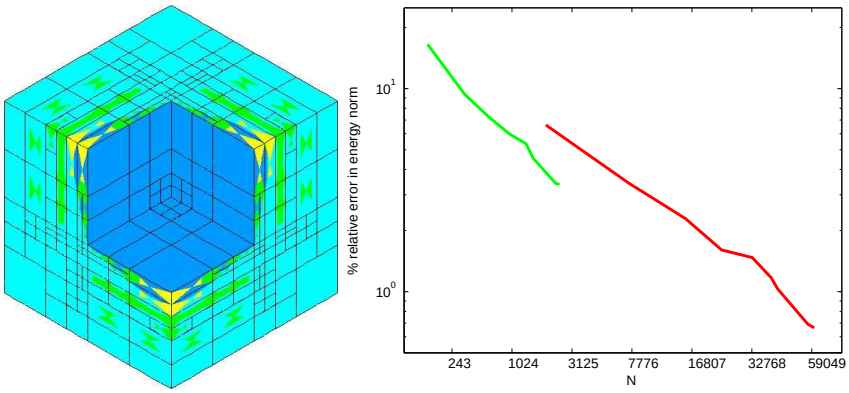

FIGURE 1. Fichera's problem. Optimal coarse grid and convergence history for the coarse (green) and fine (red) meshes

- The new code has been written as a stand alone package.
- Consistently with [5], the method ignores constrained nodes in the projections. This restores a full locality of the interpolation and commutativity of de Rham diagram.
- Two enabling technologies: order $O(p^7)$ integration (with the $O(p^2)$ speed-up *fully realized in practice*), and a telescopic solver for a sequence of dynamically determined nested hp meshes, make the competition on all levels feasible.
- A special version of 1-irregular meshes algorithm (based on "virtual refinements") enforces the optimal h-refinements with the final mesh *independent* of order in which the refinements have been performed, and elements refined *only in one shot* into two, four or eight elements.
- Upon communicating the unwanted h-refinements to the mesh optimization package, an optimal distribution of orders p is returned.
- Due to the algorithmical improvements, the size of the package has been reduced to 15k lines.
- The complexity of the algorithm is of the same order as the complexity of a multifrontal linear equations solver. In performed numerical experiments the time spent to determine optimal refinements has always been smaller that the time needed by the fine grid solver (MUMPS [10]).

Fig. 1 presents an optimal (coarse) hp mesh for the Fichera problem, and the corresponding exponential convergence of the coarse and fine grid errors. The results were obtained on a four-year-old Dell laptop with 512Mb memory and Intel 3 processor in less than 20 minutes.

Automatic Goal-oriented hp-Adaptivity. Simulations of EM waves in the borehole environment presents a class of challenging problems that are not solvable not only with classical discretization methods but also with h-adaptive schemes. In presence of casing,

the modeling involves the solution of an elliptic or Maxwell problem (only axisymmetric simulations are reported here) with a material contrast up to 10-12 orders of magnitude, and a dynamic range (ratio of signal at the radiating antenna to the signal at the receiving antenna) of up to 8-10 orders of magnitude. The problem is out of range for the energy driven hp algorithm discussed above, but also for a goal-driven h-adaptivity paradigm [1]. Only a combination of the goal-driven adaptivity with the automatic hp-adaptivity has made the solution of the problem possible.

Let

$$\begin{cases} u \in V \\ b(u,v) = l(v), \quad \forall v \in V \,, \end{cases}$$

be the usual (abstract) variational boundary-value formulation corresponding to an elliptic or Maxwell problem. The main idea of the goal-driven adaptivity is to introduce the dual problem,

$$\begin{cases} v \in V \\ b(u,v) = g(u), \quad \forall u \in V \,, \end{cases}$$

where $g(u)$ represents a *goal functional*. Standard derivation based on Galerkin orthogonality property leads then to a representation of the error in goal,

$$|g(u-u_{hp})| = \inf_{w_{hp}\in V_{hp}} |b(u-u_{hp}, v-w_{hp})| \leq \inf_{w_{hp}\in V_{hp}} \sum_K |b_K(u-u_{hp}, v-w_{hp})| \,.$$

Here u_{hp} is the discrete solution for an hp mesh, V_{hp} denotes the corresponding finite element space, and b_K stands for element K contribution to the global bilinear form. The representation holds also for exact solutions u, v replaced with *fine grid solutions* $u = u_{h/2,p+1}$ and $v = v_{h/2,p+1}$. Replacing the coarse grid solution with the coarse grid interpolant, we obtain,

$$\begin{aligned} |g(u-u_{hp})| &= \inf_{w_{hp}\in V_{hp}} |b(u-u_{hp}, v-w_{hp})| \\ &\leq |b(u-u_{hp}, v-\Pi_{hp}v)| \\ &= |b(u-\Pi_{hp}u, v-\Pi_{hp}v) + \underbrace{b(\Pi_{hp}u-u_{hp}, v-\Pi_{hp}v)}_{\text{neglected}}| \,. \end{aligned}$$

We use the error representation *not* for an error estimate but for the mesh optimization. We redefine then our mesh optimization problem as,

$$\frac{\sum_K |b_K(u-\Pi_{hp}u, v-\Pi_{hp}v)| - \sum_K |b_K(u-\Pi_{hp_{opt}}u, v-\Pi_{hp_{opt}}v)|}{N_{hp}-N_{hp_{opt}}} \to \max \,.$$

The goal-oriented hp algorithm is then a generalization of the energy driven algorithm. Referring to [11] for details, we point to only to essential differences of the new implementation when compared with the original version of the algorithm presented in [14].

- In presence of strong material contrast, the use of energy norms rather than generic Sobolev norms is very essential. The same comment applies to indefinite wave propagation problems where the norm has to include in particular the wave number.

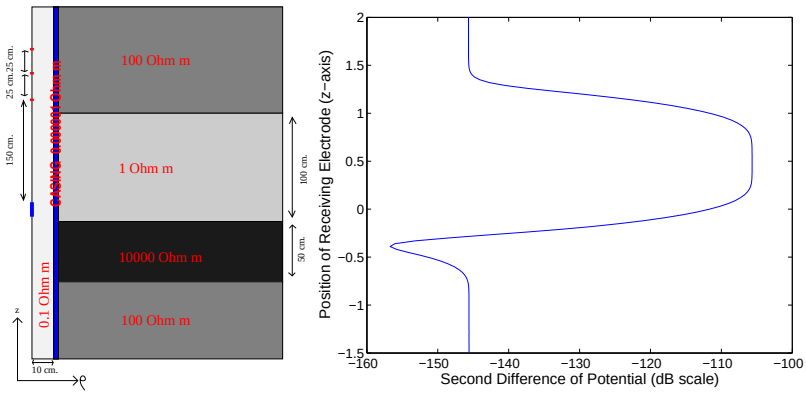

FIGURE 2. Trough Casing Resistivity Login Problem (DC). Material data and a final log obtained using goal-oriented hp-adaptivity. The log represents a second difference of the potential corresponding to three receiving antennas as a function of the position of the tool

- The "edge norms" have to be derived from the element energy norms using an approximation to minimum energy extensions.

Fig. 2 represents a fully reproduced log corresponding to 80 positions of an inductive EM tool with the relative error varying in the range $0.01 - 0.1$ percent and the problem size not exceeding 14k unknowns. The result was obtained in 20 minutes on a laptop. We mention that, in case of Maxwell problems, the problem can be solved using both H^1- and $H(curl)$-conforming elements, with the two formulations delivering results that are identical in first 8 digits in the quantity of interest !

Conclusions. The hp-adaptive finite elements not only can deliver solutions with an accuracy not accessible for other versions of finite elements [1] in finite time and on small platforms, but enable solutions of challenging problems with strong material contrasts and large dynamic range.

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[1] 0.1 percent error is typical

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Convergence of an Adaptive *hp* Finite Element Method

WILLY DÖRFLER, VINCENT HEUVELINE

Aim. We consider finite element methods with varying meshsize as well as varying polynomial degree. Such methods have been proven to show exponentially fast convergence in some classes of partial differential equations if an adequate distribution of *h*– and *p*–refinement is chosen [10, Ch. 4.5]. In order to find *hp*–refinement strategies that show up automatically with optimal complexity, it is a first step to establish convergent adaptive algorithms.

To be specific, we will refer to the model problem

$$\int_\Omega u'v' = \int_\Omega \{ fv - gv' \} \qquad \text{for all } v \in H^1_0(\Omega) \tag{1}$$

for $\Omega := (0,1)$. We develop a strategy that automatically constructs a solution adapted approximation space by combining local *h*– and *p*–refinement and can be proven to be convergent at a linear rate.

For this we need an a posteriori error estimate that gives rigorous bounds to the error in the energy norm from above and below, both uniformly in h and p. Such a result is so far open in the case of more than one space dimension [7].

Theorem 1 (A posteriori error estimate). [10, Ch. 3.5] Let $V_N \subset H^1_0(\Omega)$ be the finite element space over the discretisation $\mathscr{K}$ of $\Omega = (0,1)$, $\overline{\Omega} = \bigcup_{K\in\mathscr{K}} K = \bigcup_{k=1}^n [x_{k-1}, x_k]$,

with polynomial degree p_K on K. Let $u \in H_0^1(\Omega)$ a solution of (1) and $u_N \in V_N$ be a solution of the discrete problem

$$\int_\Omega u_N' v_N' = \int_\Omega \{ f v_N - g v_N' \} \qquad \text{for all } v_N \in V_N. \tag{2}$$

Then, we have the a posteriori bound

$$\|(u-u_N)'\|_{L^2(\Omega)}2 \le \sum_{K\in\mathscr{K}} \{\eta_K^2 + \delta_K^2\} \tag{3}$$

for the error in the energy norm. Here,

$$\eta_K^2 := \frac{1}{p_K(p_K+1)} \|\sqrt{\omega_K}(u_N'' + f_{q_K} + g_{q_K}')\|_{L^2(K)}2, \tag{4}$$

$$\delta_K^2 := \frac{1}{4}\frac{h_K^2}{p_K^2}\|f - f_{q_K}\|_{L^2(K)}2 + \|g - g_{q_K}\|_{L^2(K)}2 \tag{5}$$

with $\omega_K(x) := (x_k - x)(x - x_{k-1})$ for all $x \in K = [x_{k-1}, x_k]$ and $f_{q_K}, g_{q_K} \in \mathbb{P}_{q_K}$, for some $q_K \in \mathbb{N}$ with $q_K \ge p_K$, being arbitrary approximations to $f\lfloor_K$ and $g\lfloor_K$, respectively. Furthermore, we have the lower bound

$$2c_0\,\eta_K \le \|(u-u_N)'\|_{L^2(K)} + \frac{1}{2}\frac{h_K}{p_K}\|f - f_{q_K}\|_{L^2(K)} + \|g - g_{q_K}\|_{L^2(K)}. \tag{6}$$

The constant c_0 is a number in $(0,1)$ that depends on $\max_{K\in\mathscr{K}}\{q_K/p_K\}$.

The adaptive algorithm.

0. Initialization. We usually start with a (coarse) uniform decomposition $\mathscr{K}^{(0)}$ and polynomial degree 1.

1. Error estimation. On the given grid $\mathscr{K}^{(j)}$, $j \ge 0$, we solve for the discrete solution $u^{(j)} \in V^{(j)}$, the finite element space on $\mathscr{K}^{(j)}$. It is assumed here, that $u^{(j)}$ is the exact solution of the resulting linear system. Now compute and store the values $[\eta_K]_K$ and stop the loop if $\left(\sum_{K\in\mathscr{K}}\{\eta_K^2 + \delta_K^2\}\right)^{1/2}$ is below a prescribed tolerance. Compute and store the values $\beta_K^{(\ell)}$ for $\ell = 1, \ldots, r$ that are described below.

2. Marking elements. Let the numbers η_K and $\beta_K^{(\ell)}$ be given for $\ell = 1, \ldots, r$ and $K \in \mathscr{K}$. We seek $\mathscr{A} \subset \mathscr{K}$ and $[\ell_K]_{K\in\mathscr{K}}$ to be the solution of the following minimization problem

$$\sum_{K\in\mathscr{K}} \frac{w_K^{(\ell_K)}}{\beta_K^{(\ell_K)}} \longrightarrow \min,$$

under the constraint

$$\sum_{K\in\mathscr{A}} (\beta_K^{(\ell_K)}\eta_K)^2 \ge \theta^2 \sum_{K\in\mathscr{K}} \eta_K^2$$

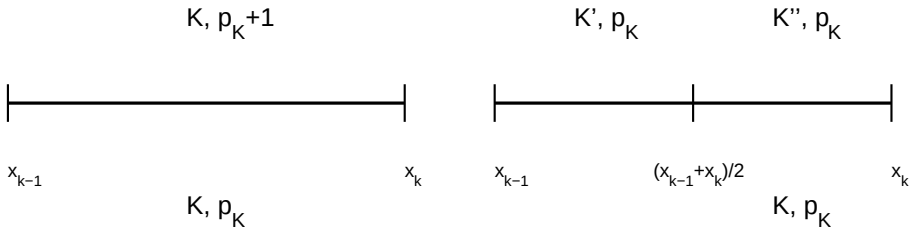

FIGURE 1. Refinement pattern on $K = [x_{k-1}, x_k]$. Left: Increase polynomial degree by 1 ($\ell = 1$). Right: Bisection of K into K', K'' with $p_{K'} = p_{K''} = p_K$ ($\ell = 2$).

for some $\theta \in (0,1)$. The weighting factors $w_K^{(\ell_K)}$ are here set to the number of degrees of freedom that the local finite element space would have after having followed the refinement pattern ℓ. In practice we proceed as follows: first we define ℓ_K by

$$\frac{w_K^{(\ell_K)}}{\beta_K^{(\ell_K)}} = \min_{\ell=1,\dots,r} \left\{ \frac{w_K^{(\ell)}}{\beta_K^{(\ell)}} \right\}$$

and then we construct a minimal possible $\mathscr{A}$ fulfilling the constraint in the usual way [3]. In case the constraint cannot be fulfilled, we set $\mathscr{A} := \mathscr{K}$ and let $\ell_K := 2$ for all $K \in \mathscr{K}$.

3. Element refinement. Refine $V^{(j)}$ on K according to the refinement pattern ℓ_K.

If necessary (e.g., when using iterative solvers), $u^{(j)}$ has to be interpolated onto the new space $V^{(j+1)}$. After having established the new decomposition, one sets up the new linear system and continues with step 1.

Refinement pattern and computation of $\beta_K^{(\ell)}$. We consider for example the following two possible refinements of the finite element space on the interval K (see Figure 1): we keep K but increase the polynomial degree by 1 ($\ell := 1$) or we bisect K into halves while maintaining the polynomial degree in both the new intervals ($\ell := 2$). Other refinements, like graded bisection with fixed and optimized grading may also be added to get a list of pattern indexed by $\ell = 1, \dots, r$ with $r \geq 2$. Applying the refinement pattern ℓ on K will lead to a locally refined new finite element space $\widetilde{V}_{N;K}^{\ell}$ of functions compactly supported in K. We now define a number $\beta_K^{(\ell)}$ through the optimization problem

$$\beta_K^{(\ell)} \frac{1}{p_K} \| \sqrt{\omega_K}\, \mathrm{res}_K \|_{L^2(K)} = \sup_{\widetilde{w}_N \in \widetilde{V}_{N;K}^{\ell}} \left\{ \frac{\int_K \mathrm{res}_K \widetilde{w}_N}{\| \widetilde{w}_N' \|_{L^2(K)}} \right\}. \tag{7}$$

If $\widetilde{z}_N^{(\ell)} \in \widetilde{V}_{N;K}^{\ell}$ is the solution of

$$\int_K \widetilde{z}_N^{(\ell)\prime} \widetilde{w}_N' = \int_K \mathrm{res}_K \widetilde{w}_N \qquad \text{for all } \widetilde{w}_N \in \widetilde{V}_{N;K}^{\ell}, \tag{8}$$

then the right hand side of (7) is given by $\| \widetilde{z}_N^{(\ell)\prime} \|_{L^2(K)}$.

Theorem 2 (Convergence of an adaptive method). Assume in addition to Theorem 1 that f_{q_K} is the L^2–projection to $f_{\lfloor K}$ onto $\mathbb{P}_{q_K}$ for some $q_K \geq p_K$ and that $\sum_{K\in\mathscr{K}} \delta_K^2 \leq \mu^2 \sum_{K\in\mathscr{K}} \eta_K^2$ holds for some sufficiently small $\mu > 0$. Construct the refined space $V_{\widetilde{N}}$ from V_N using the strategies either (I) or (II) presented before. If $u_{\widetilde{N}}$ is the solution in the finite element space $V_{\widetilde{N}}$, we have decrease of the energy error

$$\|(u-u_{\widetilde{N}})'\|_{L^2(\Omega)} \leq \rho \, \|(u-u_N)'\|_{L^2(\Omega)}$$

for some $\rho < 1$. ρ and μ depend both on c_0, θ, β_0 in case (I), and c_0, θ in case (II).

Bibliographical notes. Convergence and optimal complexity proofs for the h–method have been published in [3] [6] [2] [12]. Other automatic hp refinement strategies have been proposed in [4] [8] [9] [1] [11] [7] [5].

Open questions.

- Extend proof to higher space dimensions.
- Optimal complexity for an automatic hp-method.

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Convergence Analysis of Adaptive Mixed and Nonconforming Finite Element Methods

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(joint work with Carsten Carstensen)

We are concerned with a convergence analysis of adaptive mixed and nonconforming finite element methods for second order elliptic boundary value problems. We note that in case of standard conforming Lagrangian type finite element approximations, such methods have been considered in [4, 5]. The methods presented in this contribution provide a guaranteed error reduction and thus imply convergence of the adaptive loop which consists of the essential steps 'SOLVE', 'ESTIMATE', 'MARK', and 'REFINE'. Here, 'SOLVE' means the efficient solution of the finite element discretized problems with respect to a given coarse shape-regular triangulation $\mathscr{T}_H(\Omega)$ of the computational domain Ω. The following step 'ESTIMATE' is devoted to the a posteriori error estimation of the global discretization error. A greedy algorithm is the basic tool of the step 'MARK' to indicate selected elements for refinement. The final step 'REFINE' deals with the technical realization of the refinement process resulting in a refined triangulation $\mathscr{T}_h(\Omega)$.
The analysis is carried out for a model problem, namely the 2D Poisson equation in a bounded polygonal domain Ω under homogeneous Dirichlet boundary conditions. Discretization by the lowest order Raviart-Thomas elements with respect to the triangulation $\mathscr{T}_H(\Omega)$ amounts to the computation of $(p_H^M, u_H^M) \in RT_0(\Omega;\mathscr{T}_H(\Omega)) \times P_0(\Omega;\mathscr{T}_H(\Omega))$ such that

$$\begin{aligned}(p_H^M, q_H)_{L^2(\Omega)} + (u_H^M, \nabla\cdot q_H)_{L^2(\Omega)} &= 0 \quad \text{for all } q_H \in RT_0(\Omega;\mathscr{T}_H(\Omega)) \,,\\ (\nabla\cdot p_H, v_H)_{L^2(\Omega)} &= -(f, v_H)_{L^2(\Omega)} \quad \text{for all } v_H \in P_0(\Omega;\mathscr{T}_H(\Omega)) \,,\end{aligned}$$

where $RT_0(\Omega;\mathscr{T}_H(\Omega)$ stands for the associated Raviart-Thomas space and $P_0(\Omega;\mathscr{T}_H(\Omega)$ refers to the linear space of elementwise constants.
The residual-type a posteriori error estimator η_H consists of edge residuals

$$\eta_H := \Big(\sum_{E\in\mathscr{E}_H(\Omega)} \eta_E^2\Big)^{1/2} \quad \text{with } \eta_E^2 := h_E \|\tau_E\cdot[p_H^M]_E\|^2_{L^2(E)} \,.$$

Here, $\mathscr{E}_H(\Omega)$ is the set of interior edges and $\tau_E\cdot[p_H^M]$ denotes the jump of the tangential component of the discrete flux across an interior edge E. The convergence analysis further invokes the data term $\|Hf_H\|_{0,\Omega}$ and the data oscillation osc_H as given by

$$\|Hf_H\|_{0,\Omega} := \Big(\sum_{T\in\mathscr{T}_H} h_T^2 \Big|\int_T f(x)dx\Big|^2\Big)^{1/2} \,,\; osc_H := \Big(\sum_{E\in\mathscr{E}_H} h_E^2 \|f - f_{\omega_E}\|^2_{0,\omega_E}\Big)^{1/2} \,,$$

where $\omega_E := T_1\cup T_2$ is the patch consisting of the two triangles $T_1, T_2 \in \mathscr{T}_H(\Omega)$ sharing E as a common edge and f_{ω_E} is the integral mean of f with respect to the patch ω_E.
In the step 'MARK', we select a set $\mathscr{M}_E$ of edges $E \in \mathscr{E}_H(\Omega)$ such that for some universal constant $0 < \theta < 1$

$$\theta\eta_H^2 \le \sum_{E\in\mathscr{M}_E} \eta_E^2 \,.$$

We can show the following error reduction property which implies R-linear convergence of the mixed flux approximations:

Theorem 1. Let p be the flux and assume that p_h^M and p_H^M are the mixed finite element approximations to p with respect to $\mathscr{T}_h(\Omega)$ and $\mathscr{T}_H(\Omega)$. Then, there exist positive constants $\rho < 1$ and C depending only on θ and on the shape regularity of $\mathscr{T}_h(\Omega)$ and $\mathscr{T}_H(\Omega)$ such that

$$\|p - p_h^M\|_{0,\Omega}^2 \;\leq\; \rho \|p - p_H^M\|_{0,\Omega}^2 \;+\; C\big(\|H f_H\|_{0,\Omega} \;+\; osc_H\big) osc_H \;.$$

On the other hand, discretizing the model problem by the lowest order Crouzeix-Raviart finite elements and denoting by $CR_0^1(\Omega;\mathscr{T}_H(\Omega)$ the associated nonconforming finite element space, we are looking for $u_H^N \in CR_{1,0}(\Omega;\mathscr{T}_H(\Omega)$ such that

$$\sum_{T\in\mathscr{T}_H} (\nabla_H u_H^N, \nabla_H v_H)_{0,T} = (f, v_H)_{0,\Omega} \quad \text{for all } \; v_H \in CR_{1,0}(\Omega;\mathscr{T}_H(\Omega) \;.$$

A novel residual-type a posteriori error estimator η_H is derived in terms of edge residuals involving the jump of the tangential derivatives across interior edges

$$\eta_H \;:=\; \big(\sum_{E\in\mathscr{E}_H(\Omega)} \eta_E^2\big)^{1/2} \quad \text{with} \quad \eta_E^2 \;:=\; h_E \|[\partial u_H^N/\partial s]\|_{0,E}^2 \;.$$

Moreover, the convergence analysis requires the consideration of the data term

$$\mu_H \;:=\; \big(\sum_{T\in\mathscr{T}_H(\Omega)} |T| \|f\|_{0,T}^2\big)^{1/2}.$$

and the data oscillation osc_H. Again, in 'MARK' we select $\mathscr{M}_E \subset \mathscr{E}_H(\Omega)$ such that for some $0 < \theta_1 < 1$

$$\theta_1 \sum_{E\in\mathscr{E}_H(\Omega)} h_E \|[\partial u_H^N/\partial s]\|_{0,E}^2 \leq \sum_{E\in\mathscr{M}_E} h_E \|[\partial u_H^N/\partial s]\|_{0,E}^2 \;.$$

We further assume that the refined regular triangulation $\mathscr{T}_h(\Omega)$ from 'REFINE' satisfies

$$\mu_h^2 \;\leq\; \rho_2 \mu_H^2 \quad , \quad osc_h^2 \;\leq\; \rho_3 osc_H^2 \;,$$

for some $0 < \rho_\nu < 1, 2 \leq \nu \leq 3$, which can always be achieved by including the data terms and data oscillations in the selection step 'MARK'.
Under these assumptions, the following error reduction property can be verified:

Theorem 2. Let $p = \nabla u$ and denote by $u_h^N \in CR_{1,0}(\Omega;\mathscr{T}_h(\Omega))$ and $u_H^N \in CR_{1,0}(\Omega;\mathscr{T}_H(\Omega))$ the nonconforming finite element approximations to u and by $p_H^N = \nabla_H u_H^N$ and $p_h^N = \nabla_h u_h^N$ the associated discrete fluxes. Then, there exist positive constants $\rho_1 < 1$, and C_1, C_2 depending only on θ_1 and on the shape regularity of the triangulations such that

$$\begin{pmatrix} \|p - p_h^N\|_{0,\Omega}^2 \\ \mu_h^2 \\ osc_h^2 \end{pmatrix} \leq \begin{pmatrix} \rho_1 & C_1 & C_2 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix} \begin{pmatrix} \|p - p_H^N\|_{0,\Omega}^2 \\ \mu_H^2 \\ osc_H^2 \end{pmatrix} .$$

The essential steps in the proofs of Theorem 1 and Theorem 2 are the reliability of the estimator, a discrete local efficiency, and quasi-orthogonality properties. Also, we strongly

utilize the following fundamental relationship between the discrete mixed and nonconforming fluxes

$$p_H^M(x) \;=\; p_H^N(x) \;-\; \frac{1}{2} f_T (x - x_T) \quad , \quad x \in T \;,\; T \in \mathscr{T}_H(\Omega) \;,$$

where f_T is the integral mean of f on T and x_T refers to the center of gravity.
In contrast to the convergence analysis of standard Lagrangian finite element discretizations, there are no special assumptions with regard to the refinement process. In particular, we do not need an internal node property. The convergence proofs do not require any regularity of the solution nor do they make use of duality arguments.

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On Adaptivity in hp-FEM

MARKUS MELENK

(joint work with Tino Eibner, Barbara Wohlmuth)

In the hp-version of the finite element method (FEM) convergence can be achieved by refining the mesh or by increasing the approximation order or by a combination of both. In fact, suitable combinations of both techniques can lead, for a large class of problems, to very fast, exponential, convergence. The adaptive algorithm presented aim at realizing this exponential convergence.

We consider the elliptic model problem

$$-\Delta u = f \text{on } \Omega \subset \mathbb{R}^2, \qquad u|_{\partial\Omega} = 0.$$

The hp-FEM space used is $S^p(\mathscr{T}) := \{u \in H_0^1(\Omega) \,|\, u|_K \in \mathscr{P}_{p_K}\}$, where $\mathscr{T}$ is a shape-regular triangulation of the polygon Ω and a polynomial degree $p_K \in \mathbb{N}$ is associated with each element $K \in \mathscr{T}$. The FE-solution $u_{FE} \in S^p(\mathscr{T})$ is then given by the projection of u onto $S^p(\mathscr{T})$ in the energy norm $\|v\|_E^2 := \int_\Omega |\nabla v|^2$.

1. Residual based error estimation

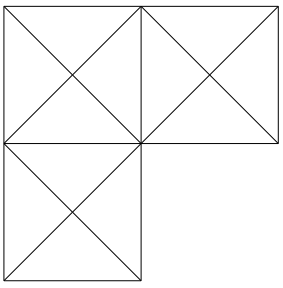
In a first step, the adaptive algorithm identifies elements with large errors. This is done with the aid of the error indicators

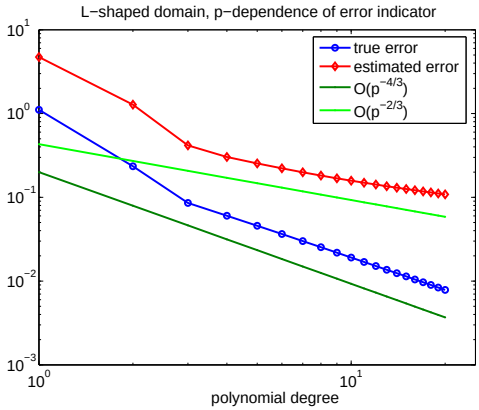
$$\eta_K^2 := \frac{h_K^2}{p_K^2}\|f+\Delta u_{FE}\|^2_{L^2(K)} + \frac{h_K}{p_K}\sum_{e:e\subset\Omega \text{ is edge of } K}\|[\partial_n u_{FE}]\|^2_{L^2(e)},$$

where K denotes an element of the mesh and $[\partial_n u_{FE}]$ denotes the jump of the normal derivative of the FE-solution u_{FE} across the edge e. These error indicators η_K have the following properties, [2]:

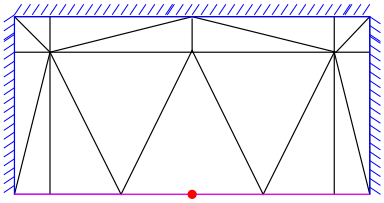
$$\|u-u_{FE}\|_E^2 \le C\sum_{K\in\mathscr{T}}\eta_K^2, \qquad \sum_{K\in\mathscr{T}}\eta_K^2 \le Cp^{2+\varepsilon}\{\|u-u_{FE}\|_E^2+\mathrm{osc}(f)\},$$

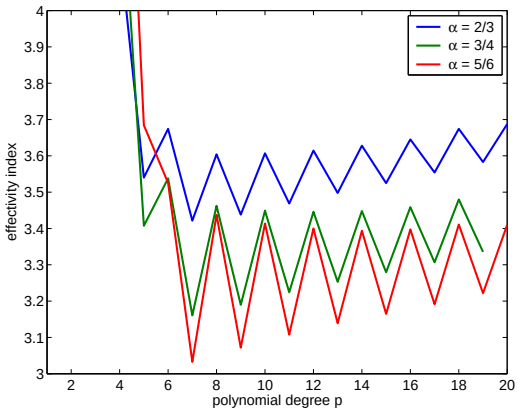
where $\mathrm{osc}(f)$ is typically of higher order and $p=\max_{K\in\mathscr{T}}p_K$; the constant $\varepsilon>0$ is arbitrary. The presence of the factor $p^{2+\varepsilon}$ points to a reliability-efficiency gap. This gap is not entirely an artefact of the method of proof as the following numerical example illustrates.
Example: We consider the p-version FEM (i.e., the mesh is fixed as depicted below and the polynomial degree $p_K=p$ for all $K\in\mathscr{T}$) on an L-shaped domain with exact solution (given in polar coordinates where the origin is located at the reentrant corner) $u=r^{2/3}\sin(2/3\varphi)\chi$, where χ is a smooth cut-off function. We note that the singularity of u is located at a mesh point.





Example: In order to get more insight into the p-dependence of the error estimator, we performed calculations where the exact solution (in polar coordinates) has the form $u(r,\varphi)=r^\alpha\sin(\alpha\varphi)\chi$ for a smooth cut-off function χ and different choices of the parameter α. Neumann boundary conditions are prescribed on the bottom part of the computational domain (shown below) and the error indicators η_K for the elements K touching the Neumann part of the boundary are appropriately adjusted. Again the mesh is fixed (as shown below) and the polynomial degree is uniformly raised. We note that the singularity (marked by a dot in the geometry below) is not at a mesh point. The effectivity index $\frac{\sqrt{\sum_{K\in\mathscr{T}}\eta_K^2}}{\|u-u_{FE}\|_E}$ is plotted versus the polynomial degree p in the following graph. We note that the dependence on the polynomial degree p is significantly reduced as compared with the preceding example.





2. Adaptive algorithms

We present an algorithm that is based on locally testing for analyticity; details and references to related work can be found in [1]. For an interval $I = (-1,1)$, it is well-known that a function $u \in L^2(I)$ is analytic on $\overline{I}$ if and only if the coefficients u_n of the expansion $u = \sum_{n\in\mathbb{N}_0} u_n P_n^{(0,0)}$, where the functions $P_n^{(0,0)}$ are the classical Legendre polynomials, decay exponentially in n. Tensor product arguments extend this result to domains with product structure (squares, hexahedra, etc.). The following result generalizes this observation to triangles and tetrahedra:

Theorem: Let $T = \{(\xi_1,\xi_2)|\, -1 < \xi_1 < 1, -1 < \xi_2 < \xi_1\}$ be the reference triangle and define the $L^2(T)$-orthogonal polynomials

$$\psi_{i,j}(\xi_1,\xi_2) := P_i^{(0,0)}(2\frac{1+\xi_1}{1-\xi_2} - 1)\left(\frac{1-\xi_2}{2}\right)^i P_j^{(2i+1,0)}(\xi_2),$$

where the polynomials $P_i^{(\alpha,\beta)}$ are the standard Jacobi polynomials. Then any $u \in L^2(T)$ can be expanded as $u = \sum_{i,j\in\mathbb{N}_0} u_{i,j}\psi_{i,j}$, and we have the following characterization of analyticity: u is analytic on $\overline{T}$ if and only if there exist $C, b > 0$ such that

$$|u_{i,j}| \le Ce^{-b(i+j)} \qquad \forall i,j \in \mathbb{N}_0.$$

An analogous statement holds for tetrahedra as well, [1].

With this characterization in hand, one can formulate an hp-adaptive algorithm whose inner loop is as follows:

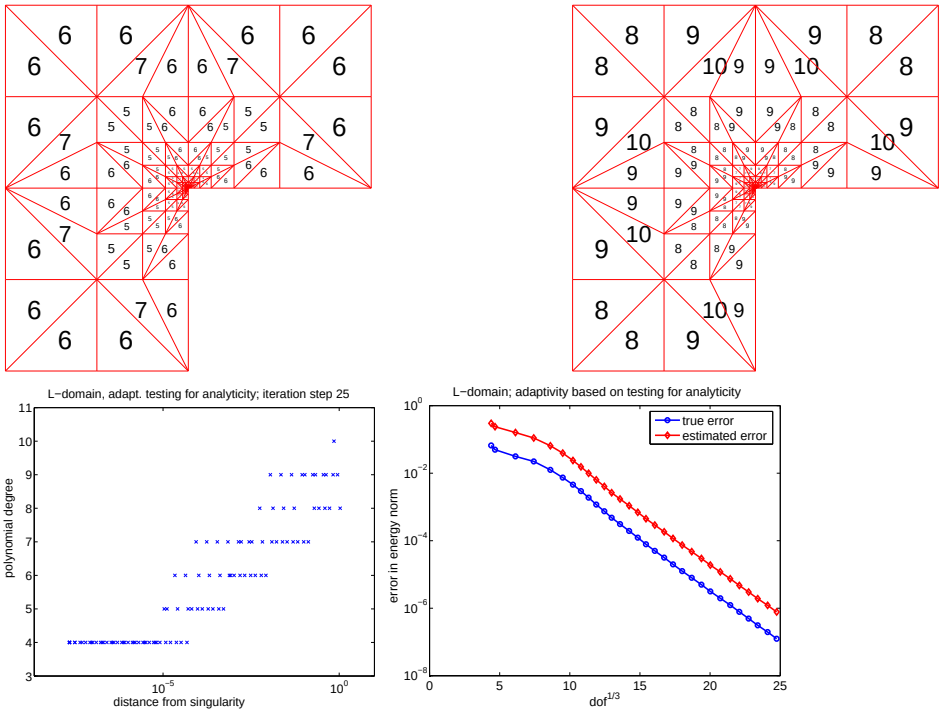
(1) based on the error indicators η_K identify the elements with large error (in the numerical example below: $\eta_K^2 \ge \sigma\overline{\eta}^2$, where $\overline{\eta}^2 = \frac{1}{\#\mathcal{T}}\sum_{K\in\mathcal{T}}\eta_K^2$ and $\sigma = 0.75$)
(2) for each element K that has a large error do:
  (a) expand $\widehat{u}_K := u_{FE}|_K \circ F_K$ (here, $F_K : T \to K$ is the element map) as $\widehat{u}_K = \sum_{i+j\le p_K} u_{i,j}\psi_{i,j}$
  (b) determine C_K, b_K by fitting (in a least squares sense) the coefficients $u_{i,j}$ to the law $u_{i,j} = C_K e^{-b_K(i+j)}$

(c) If $b_K \geq b$ (in the numerical example: $b = 0.9$), then increase the polynomial degree p_K of element K by 1. If $b_K < b$, then split the element K into 4 elements ("red" refinement) and perform the appropriate mesh closure (remove hanging nodes).

The following numerical example illustrates the performance of the algorithm.

Example:

We consider the L-shaped domain as depicted below with an exact solution u of the form $u(r,\varphi) = r^{2/3}\sin(2/3\varphi)\chi$, where the cut-off function is smooth and the origin is located at the reentrant corner. In order to ensure that a sufficient number of coefficients in the expansion is available for all elements, the initial polynomial degree is $p_K = 3$ for all elements. We depict the meshes and polynomial degree distributions for steps 15 and 25 of the adaptive algorithm. Additionally, we show for step 25 of the algorithm the polynomial degree distribution along the line connecting the origin with the point $(1/2,1)$; furthermore, we plot the error versus $N^{1/3}$, where N is the problem size. We observe that the algorithm yields a convergence behavior of the form $\|u-u_{FE}\|_E \leq Ce^{-bN^{1/3}}$.



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A Convergent Adaptive Algorithm for the Laplace-Beltrami Operator

PEDRO MORIN
(joint work with Khamron Mekchay, Ricardo H. Nochetto)

We consider a surface $\Gamma \subset \mathbb{R}^d$ as a graph of a function $z(\mathbf{x})$ defined on a bounded polygonal region $\Omega \subset \mathbb{R}^{d-1}$, $d \geq 2$, namely,

$$\Gamma := \left\{ (\mathbf{x}, z(\mathbf{x})) \in \mathbb{R}^d \mid \mathbf{x} \in \Omega \subset \mathbb{R}^{d-1} \right\},$$

where $z : \Omega \to \mathbb{R}$ is a C^1 function. In general we may also include the case when z is $C^{0,1}$ where discontinuities of ∇z align with polygonal lines on Ω.

We consider a Dirichlet boundary value problem for Poisson's equation on Γ

$$-\Delta_\Gamma u = f \text{ on } \Gamma, \tag{1}$$

$$u = 0 \text{ on } \partial\Gamma, \tag{2}$$

where $f \in L^2(\Gamma)$. Here, Δ_Γ denotes the Laplace-Beltrami operator on the surface Γ. In weak form this problem reads

$$u \in H^1_0(\Gamma): \quad \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_\Gamma f\varphi \quad \forall\, \varphi \in H^1_0(\Gamma), \tag{3}$$

where $\nabla_\Gamma u \in \mathbb{R}^d$ denotes the tangential gradient of u computed on Γ.

To find a discrete approximation we consider a polyhedral approximation Γ_k of Γ, a finite element space $\mathbb{V}_k$, and define

$$u_k \in \mathbb{V}_k: \quad \int_\Gamma \nabla_\Gamma u_k \cdot \nabla_\Gamma \varphi_k = \int_\Gamma F_k \varphi_k \quad \forall\, \varphi_k \in \mathbb{V}_k. \tag{4}$$

For $T \in \mathscr{T}_k$, we define the *energy error indicator* $\eta_k(T)$ by

$$\eta_k^{2(T)} := h_T^2 \, \|\mathscr{R}_T(u_k)\|^2_{L^2(T)} + \sum_{S \in \mathscr{S}_k^o : S \subset \partial T} h_S \, \|\mathscr{J}_S(u_k)\|^2_{L^2(S)},$$

where

$$\mathscr{R}_T(u_k) := (\Delta_{\Gamma_k} u_k + F_k)|_T, \tag{5}$$

$$\mathscr{J}_S(u_k) := (\nabla_{\Gamma_{u_k}})^+ \cdot n_S^+ + (\nabla_{\Gamma_{u_k}})^- \cdot n_S^-, \tag{6}$$

and the *energy error estimator* $\eta_k := (\sum_{T \in \mathscr{T}_k} \eta_k^2(T))^{1/2}$.

Similarly, for $T \in \mathscr{T}_k$ we define *the geometric error indicator* $\zeta_k(T)$ by

$$\zeta_k(T) := \|\nu - \nu_k\|_{L^\infty(T)} \, \|\nabla_\Gamma u_k\|_{L^2(\tilde{T})},$$

and the *geometric error estimator* $\zeta_k := (\sum_{T \in \mathscr{T}_k} \zeta_k^2(T))^{1/2}$. Here ν and ν_k denote the normals to Γ, Γ_k respectively. We also define the *geometric oscillation*

$$\lambda_k(T) := \|\nu - \nu_k\|_{L^\infty(T)} \quad \text{and} \quad \lambda_k := \max_{T \in \mathscr{T}_k} \lambda_k(T).$$

With this definition we prove the following **Upper Bound**

There exist constants C_1 and C_2 depending only on shape regularity and the surface Γ such that

$$\|\nabla_\Gamma(u-u_k)\|^2_{L^2(\Gamma)} \le C_1\eta_k^2 + C_2\zeta_k^2. \tag{7}$$

We also prove the following **lower bound**

There exist constants c_3, c_4 and c_5 depending on shape regularity and the surface Γ such that for $T \in \mathscr{T}_k$

$$\eta_k^2(T) \le c_3\|\nabla_\Gamma(u-u_k)\|^2_{L^2(\omega_k(T))} + c_4\mathsf{osc}_k^2(\omega_k(T)) + c_5\zeta_k^2(\omega_k(T)), \tag{8}$$

where $\omega_k(T)$ region consists all elements in $\mathscr{T}_k$ that share at least a side with T.

For $T \in \mathscr{T}_k$ the oscillation term is defined by

$$\mathsf{osc}_k^2(T) := h_T^2\left\|\mathscr{R}_T(u_k) - \overline{\mathscr{R}}_T\right\|^2_{L^2(T)} + h_T\sum_{S\subset\partial T}\left\|\mathscr{J}_S(u_k) - \overline{\mathscr{J}}_S\right\|^2_{L^2(S)},$$

where $\overline{\mathscr{R}}_T$ and $\overline{\mathscr{J}}_S$ are L^2-projections of $\mathscr{R}_T(u_k)$ and $\mathscr{J}_S(u_k)$ onto $\mathbb{P}_m(T)$ and $\mathbb{P}_m(S)$, respectively, the spaces of polynomial functions of degree $\le m$ on T, and S, respectively.

For $\omega_k(T) \subset \Gamma_k$ we define $\mathsf{osc}_k^2(\omega_k(T)) := \sum_{T\subset\omega_k(T)}\mathsf{osc}_k^2(T)$ and denote $\mathsf{osc}_k := \mathsf{osc}_k(\Gamma_k)$; and analogously we define $\zeta_k^2(\omega_k(T))$.

The final ingredient for convergence is given by a quasi-orthogonality relation:

The exist constants $C_6, C_7 > 0$ and a number $k_* \ge 0$ such that $\Lambda_0 := (\frac{1}{2} - \rho_2 C_6\lambda_{k_*}^2) \in [\frac{1}{4}, \frac{1}{2})$, and for any $k \ge k_*$

$$\begin{aligned}\|\nabla_\Gamma(u-u_{k+1})\|^2_{L^2(\Gamma)} &\\ \le \|\nabla_\Gamma(u-u_k)\|^2_{L^2(\Gamma)} - \Lambda_0\|\nabla_\Gamma(u_k-u_{k+1})\|^2_{L^2(\Gamma)} &+ C_7\zeta_k^2,\end{aligned} \tag{9}$$

provided λ_k is decreasing.

This is a *conditional* quasi-orthogonality relation between $\nabla_\Gamma(u-u_k)$ and $\nabla_\Gamma(u_k-u_{k+1})$, which is valid only if λ_k is small enough. But this does not matter. Our algorithm ensures a monotone reduction of λ_k which in turn will lead to convergence. To be more specific, we define the following:

`Marking Strategy:` Given parameters $0 < \theta_e, \theta_g, \theta_o < 1$, construct a subset $\widehat{\mathscr{T}_k}$ of $\mathscr{T}_k$ such that the followings hold:

$$(\mathsf{M1}): \quad \sum_{T\in\widehat{\mathscr{T}_k}}\eta_h^2(T) \ge \theta_e^2\eta_h^2, \tag{10}$$

$$(\mathsf{M2}): \quad \sum_{T\in\widehat{\mathscr{T}_k}}\zeta_h^2(T) \ge \theta_g^2\zeta_h^2, \tag{11}$$

$$(\mathsf{M3}): \quad \sum_{T\in\widehat{\mathscr{T}_k}}\mathsf{osc}_h^2(T) \ge \theta_o^2\mathsf{osc}_h^2. \tag{12}$$

`Interior Node Property:` Refine each marked element $T \in \widehat{\mathscr{T}_k}$ to obtain a new mesh $\mathscr{T}_{k+1}$ compatible with $\mathscr{T}_k$ such that

T and the adjacent elements $T' \in \mathscr{T}_k$ of T, as well as their common sides, contain a node of the finer mesh $\mathscr{T}_{k+1}$ in their interior.

`Reduction of geometric oscillation:` Given a reduction factor $\theta_\lambda < 1$, refine all $T \in \widehat{\mathscr{T}_k}$ such that for all $T' \in \mathscr{T}_{k+1}(T)$ we have

$$\lambda_{k+1}(T') \le \theta_\lambda \lambda_k(T),$$

where $\mathscr{T}_{k+1}(T) := \{T' \in \mathscr{T}_{k+1} \mid T' \text{ is obtained by refining } T\}$.

The procedure REFINE is performed in several steps as follow:

`Refining Strategy:` Given a sequence $\{a_k\} \searrow 0$, a marked set $\widehat{\mathscr{T}_k}$, geometric oscillations $\{\lambda_k(T)\}_{T\in\mathscr{T}_k}$, and a fixed reduction rate of element size $0 < \gamma_r < 1$;

(1) Refine all $T \in \widehat{\mathscr{T}_k}$ according to `Interior Node Property`;

(2) Refine more according to `Reduction of geometric oscillation`;

(3) Refine more if needed so that for any $T \in \mathscr{T}_k$

$$\forall T' \in \mathscr{T}_{k+1}(T): \qquad \lambda_{k+1}(T') \le \min\{a_k, \lambda_k(T)\};$$

(4) Refine more if needed so that for any $T' \in \mathscr{T}_{k+1}(T)$, $T \in \mathscr{T}_k$,

$$\frac{|T'_k|}{|T'|} \le \gamma_T \left(\frac{|T|}{|T'|}\right)^{\frac{2}{d-1}} \quad \text{where} \quad \gamma_T := \begin{cases} \gamma_r & \text{if } T \in \widehat{\mathscr{T}_k} \\ 1 & \text{if } T \notin \widehat{\mathscr{T}_k} \end{cases}, \tag{13}$$

and $T'_k \subset T$ is the projection of T' back to T.

This algorithm ensures that $\lambda_k \to 0$ in a monotonic way, which implies that eventually the orthogonality relationship (9) will eventually start to hold and will continue to hold thereof. Consequently, following similar ideas to those in [1], we can prove the following *error reduction* formula:

There exists a number $k_0 \ge 0$ and constants $\xi < 1$, γ, $C > 0$ such that for any $k \ge k_0$ the sequence generated by a repeated application of the refinement procedure above satisfies

$$(\|\nabla_\Gamma(u - u_{k+1})\|^2 + \gamma\zeta_{k+1}^2) \le \xi(\|\nabla_\Gamma(u - u_k)\|^2 + \gamma\zeta_k^2) + C\mathrm{osc}_k^2. \tag{14}$$

Since this holds for all $k \ge k_0$ we have *convergence* of the adaptive loop.

A full version of this result, together with all the details and the missing proofs will be published in a forthcoming article.

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Convergence of the Equidistribution Strategy

KUNIBERT G. SIEBERT
(joint work with Andreas Veeser)

We consider linear symmetric elliptic boundary value problems with homogeneous Dirichlet boundary values in two space dimensions. For the *adaptive approximation* of the exact solution u we use a discretization by *piecewise linear finite elements* over a sequence of conforming and shape-regular triangulations $\{\mathscr{T}_k\}_{k\in\mathbb{N}_0}$. Here, we start with an initial triangulation $\mathscr{T}_0$ and triangulation $\mathscr{T}_{k+1}$ is generated by (local) refinement of $\mathscr{T}_k$ using *bisectioning* of selected elements. This in turn leads to a sequence of *nested* finite element spaces $\{\mathring{\mathbb{V}}_k\}_{k\in\mathbb{N}_0}$. For estimating the true error we use the standard residual type error estimator where the error indicator on a single element splits into the *element* and *jump* residual [1, 9].

The convergence analysis of adaptive finite element discretizations in higher space dimensions was initiated by Dörfler in [3]. Involving a notion of data oscillation, Morin, Nochetto and Siebert could avoid the assumption of Dörfler, that given data on the initial grid is sufficiently resolved [5, 6, 7]. The ingredients for the convergence proof are

- Orthogonality of the error relating the actual error to the error on the next grid and error reduction due to the nesting of the finite element spaces.
- A refinement rule for single elements, which guarantees that for all marked elements in each of its sides and in its interior a new node in the next triangulation is created.
- A strict error reduction property up to data oscillation using the fixed fraction marking for the estimator proposed by Dörfler [3].
- A strict oscillation reduction up to error reduction using a fixed fraction marking for data oscillation by Morin et al. [5], generalized by [2] and [4].

However, adaptive algorithms used in practice, like the equidistribution strategy devised by Babuška and Rheinboldt e.g. [1], seem to converge without an interior node property, without a fixed fraction marking strategy, and without treating data oscillation. An adaptive algorithm using the equidistribution strategy is given by:

Given tolerance $\mathrm{TOL} > 0$ and safety parameter $\theta \in (0,1)$:

(1) Choose an initial mesh $\mathscr{T}_0$, set $k := 0$.
(2) Compute the discrete solution u_k on $\mathscr{T}_k$.
(3) Compute the local indicators $\mathscr{E}_k(T)$ and the total error estimate $\mathscr{E}_k(\mathscr{T}_k)$. If $\mathscr{E}_k(\mathscr{T}_k) \le \mathrm{TOL}$ stop.
(4) Define

$$\hat{\mathscr{T}}_k := \left\{ T \in \mathscr{T}_k \mid \mathscr{E}_k(T) \ge \theta\, \mathrm{TOL}\, |\mathscr{T}_k|^{-1/2} \right\}.$$

(5) Refine $\mathscr{T}_k$ into $\mathscr{T}_{k+1}$ by refining all elements in $\hat{\mathscr{T}}_k$ by two bisections.
(6) Set $k := k+1$ and goto (2).

In this talk, we prove the convergence of the above algorithm. To be more precise, we prove that any given tolerance $\mathsf{TOL} > 0$ is reached in a finite number of steps. Note, that no special marking for data oscillation is applied and the refinement rule does not require the creation of new nodes in the interior of marked elements. The convergence proof is based on the following results:

- The sequence of mesh-size functions $\{h_k\}_{k\in\mathcal{N}_0}$ converges to some limiting function h_∞ in $L_\infty(\Omega)$. Note, that in general $h_\infty \not\equiv 0$ holds.
- For any sequence of tolerances $\mathsf{TOL}_k > 0$ in the kth iteration of an adaptive procedure with $\lim_{k\to\infty} \mathsf{TOL}_k$ we prove that the limit of the element residual is 0. The proof is based on an idea used by Siebert and Veeser for controling the element residual in a convergent algorithm for the elliptic obstacle problem [8].
- Using the equidistribution strategy, the contribution of non-marked elements to the total estimate $\mathscr{E}_k(\mathscr{T}_k)$ satisfies $\mathscr{E}_k(\mathscr{T}_k \setminus \hat{\mathscr{T}}_k) < \theta\mathsf{TOL}$. Contributions of marked elements are controled via discrete local efficiency by the error reduction and the element residual [3] which both converge to 0.

The presentation is part of ongoing research and it seems that the presented ideas can also be used for proving convergence of other practical marking strategies, like the popular maximum strategy [1].

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Optimal Adaptive Finite Element Methods without Coarsening

ROB STEVENSON

Adaptive finite element methods for solving elliptic boundary value problems have the potential to produce a sequence of approximations to the solution that converges with a rate that is optimal in view of the polynomial order that is applied, also in the, common, situation that finite element approximations with respect to uniformly refined partitions

exhibit a reduced rate due to a lacking (Sobolev) regularity of the solution. The basic idea of an adaptive finite element method is, given some finite element approximation, to create a refined partition by subdividing those elements where local error estimators indicate that the error is large, and then, on this refined partition, to compute the next approximation, after which the process can be repeated. Although, because of their success in practice, during the last 25 years the use of these adaptive methods became more and more widely spread, apart from results in the one-dimensional case by Babuška and Vogelius ([1]), their convergence was not shown before the work by Dörfler ([6]), that was later extended by Morin, Nochetto and Siebert ([7]).

Although these results meant a break through in the theoretical understanding of adaptive methods, they do not tell anything about the rate of convergence, and so, in particular, they do not show that adaptive methods are more effective than, or even competitive with non-adaptive ones in the situation that the solution has a lacking regularity.

Recently, in [2], Binev, Dahmen and DeVore developed an adaptive finite element method which they showed to be of optimal computational complexity. Whenever for some $s > 0$, the solution is in the approximation class $\mathscr{A}^s$, meaning that there exists a sequence of partitions of the domain into n elements such that the best finite element approximation with respect to this partition has an error in energy norm of order n^{-s}, then the adaptive method produces a sequence of approximations that converge with the same rate, where, moreover, the cost of computing such an approximation is of the order of the number of elements in the underlying partition. A combination of the (near) characterization of $\mathscr{A}^s$ in terms of Besov spaces from [3], and Besov regularity theorems from [5, 4], indicate that under very mild conditions the value of s is indeed only restricted by the polynomial order. An additional condition was required on the right-hand side, the discussion of which we postpone to the end of this abstract.

The key to obtain the optimal computational complexity result was the addition of a so-called coarsening or derefinement routine to the method from [7], that has to be applied after each fixed number of iterations, as well as, in view of the cost, to replace the exact Galerkin solvers by inexact ones. Thanks to the linear convergence of the method from [7], and the fact that after this coarsening, the underlying partition can be shown to have, up to some constant factor, the smallest possible cardinality in relation to the current error, optimal computational complexity could be shown.

The result of [2] is of great theoretical importance, but the adaptive method may not be very practical. The implementation of the coarsening procedure is not trivial, whereas, moreover, numerical results indicate that coarsening is not needed for obtaining an optimal method. In this talk, we will present a proof of this fact (see [8]). We construct an adaptive finite element method, that, except that we solve the Galerkin systems inexactly, is very similar to the one from [7], and show that it has optimal computational complexity.

As in [2, 7], we restrict ourselves to the model case of the Poisson equation in two space dimensions, linear finite elements, and partitions that are created by newest vertex bisection. Our results, however, rely on three ingredients only, two dealing with residual based a posteriori error estimators, and one dealing with bounding the number of bisections needed to find the smallest conforming refinement of a partition. The two results on a

posteriori error estimators extend to more general second order elliptic differential operators, to more space dimensions, and to higher order finite elements. It can be expected that also the result about newest vertex bisection extends to more space dimensions, which, however, has to be investigated.

To solve a boundary value problem on a computer, it is indispensable to be able to approximate the right-hand side by some finite representation within a given tolerance. As (implicitly) in [7, 2], we use piecewise constant approximations, but, in particular for higher order elements, by a modification of the adaptive refinement routine, piecewise polynomial approximations of higher order can be applied as well. Our aforementioned result concerning optimal computational complexity is valid only under the additional assumption that if the solution $u \in \mathscr{A}^s$, then for any n we know how to approximate the right-hand side f by a piecewise constant function with respect to a partition of n elements such that the error in the dual norm is of order n^{-s}. For $s \in (0, \frac{1}{2}]$, which is the relevant range for piecewise linear elements, we conjecture that if $u \in \mathscr{A}^s$, then such approximations for the corresponding right-hand side exist, which, however, is something different than knowing how to construct them. For $f \in L_2(\Omega)$, however, the additional assumption is always satisfied, where for constructing the approximations of the right-hand side we may even rely on uniform refinements.

The adaptive methods from [7, 2] apply only to $f \in L_2(\Omega)$. Our additional assumption on the right-hand side is weaker than that of [2], but for $f \in H^{-1}(\Omega)$ not in $L_2(\Omega)$, it has to be verified for the right-hand side at hand.

At the end of the talk we discuss some work in progress. We present an extension of our results for solving the Stokes equations, and discuss a possible application for goal oriented adaptivity.

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Convergent Adaptive Finite Elements for Rough and Conforming Obstacles

ANDREAS VEESER
(joint work with Kunibert G. Siebert)

There has been recent progress in the analysis of adaptive finite element methods for linear elliptic problems, addressing also the issues of optimality, convergence rates, and complexity. For an overview of these results, we refer to the other contributions of this report and the references therein.

This contribution concerns adaptive finite elements and their convergence for the elliptic variant of the obstacle problem. The obstacle problem may be considered as a model case for variational inequalities – a problem class that is ubiquitous and includes, e.g., contact and phase transition problems. Important features of the obstacle problem are described by the following properties of its resolution operator: nonlinearity, nondifferentiability, and loos of information.

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a polyhedral Lipschitz domain and $f \in L_2(\Omega)$ a load term. The lower obstacle is given by a finite sequence of pairs $\{(K_i, \psi_i)\}_{i=1}^n$ such that

(1) each $K_i \subset \Omega$ is a nondegenerate m-simplex, $m \in \{d-1, d\}$,
(2) their interiors (with respect to the induced topology) are pairwise disjoint,
(3) each ψ_i is an affine function over K_i satisfying $\psi_i \leq 0$ on $\partial\Omega \cap K_i$.

Notice that this covers continuous, piecewise affine obstacles but allows also for (combinations of) discontinuous and 'singular' obstacles.

Let u be the typically unknown minimizer of the 'inhomogeneous Dirichlet energy'

$$I[v] := \int_\Omega \tfrac{1}{2}|\nabla v|^2 - fv$$

in the set

$$\mathscr{F} := \{v \in H_0^1(\Omega) \mid v \geq \psi_i \text{ on } K_i \text{ for } i = 1, \ldots, n\},$$

which is nonempty, convex, and closed thanks to the trace theorem. Such minimizer exists, is unique, and is characterized by the variational inequality

$$\forall v \in \mathscr{F} \quad \langle \nabla u, \nabla(v-u)\rangle \geq \langle f, v-u\rangle,$$

where $\langle\cdot,\cdot\rangle$ indicates the L_2-scalar product. We are interested in both the computational approximation of the minimum point u and the minimum value $I[u]$.

To this end, we design an adaptive algorithm with continuous linear finite elements. The algorithm is based upon an iteration of the following main steps:

$$\text{solve} \to \text{estimate} \to \text{mark} \to \text{refine},$$

i.e., solve for the minimum u_k of I in the current finite element subset $\mathscr{F}_k$ of $\mathscr{F}$ and estimate its error to test if it already meets a prescribed tolerance. If not, mark certain elements and refine them in order to obtain a new, enlarged discrete feasible set $\mathscr{F}_{k+1}$.

The realization of the steps 'estimate' and 'mark' involve a new a posteriori estimator $\mathscr{E}_k$ for the error in the energy minimum $I[u_k] - I[u]$. Although it is somehow related to the hierarchical estimator (see, e.g., [2]), it differs from other previous ones in various aspects,

e.g. error notion, accumulation of indicators, and range of covered obstacles. Our main result about the estimator $\mathscr{E}_k$ is as follows.

Theorem 1 (Upper bound)**.** The estimator $\mathscr{E}_k$ bounds the error in the energy minimum,

$$I[u_k] - I[u] \preccurlyeq \max\{\tfrac{1}{2}\mathscr{E}_k^2, \mathscr{E}_k\}.$$

The hidden constant depends on d, the shape regularity of the initial mesh, and, only if there are isolated contact nodes, in addition on the load term f and the lower obstacle.

The proof is based upon the inequality

$$\rho_k(-\mathscr{D}_k) \preccurlyeq \mathscr{E}_k \tag{1}$$

where the quantity

$$\rho_k(-\mathscr{D}_k) := \sup\{\langle -\mathscr{D}_k, \varphi\rangle \mid \varphi \in H_0^1(\Omega) \text{ s.th. } \|\nabla\varphi\| \leq 1,\, u_k + \varphi \in \mathscr{F}\}$$

is a modification of the $H^{-1}(\Omega)$-norm of the residual or the derivative $\mathscr{D}_k$ of I in the current approximate minimizer u_k, i.e.

$$\forall\, \varphi \in H_0^1(\Omega) \quad \langle \mathscr{D}_k, \varphi\rangle = \langle \nabla u_k, \nabla\varphi\rangle - \langle f, \varphi\rangle.$$

Notice that $\rho_k(-\mathscr{D}_k)$ combines the usual L_2-constraint for the gradient of the test function with a constraint of pointwise nature. The special form of the upper bound arises from the relationship $I[u_k] - I[u] \leq \max\{\frac{1}{2}\rho_k(-\mathscr{D}_k)^2, \rho_k(-\mathscr{D}_k)\}$, which is a generalization of the unconstrained case.

Important ingredients of the proof of (1) are the concept of full contact introduced in [1] and an adaptation of the projection operators on stars in [3].

The upper bound in Theorem 1 is accompanied by appropriate lower bounds such that, exploiting the technique in [5], one can derive the following result.

Theorem 2 (Convergence)**.** Suppose that the initial triangulation is subordinated to the lower obstacle.

Then the indicated algorithm converges in a finite number of steps or produces an infinite sequence of approximate minima $\{u_k\}_{k\in\mathbb{N}}$ such that

$$I[u_k] \to I[u] \quad \text{and} \quad u_k \to u \text{ in } H^1(\Omega) \quad (k \to \infty).$$

The algorithm has been implemented within the framework of the finite element toolbox ALBERTA [4]. Our numerical results corroborate and complement the theoretical results. In particular, they indicate that typically $I[u_k] - I[u] \approx \frac{1}{2}\mathscr{E}_k^2$ and that the convergence rate in terms of the number of unknowns coincide with the one of nonlinear approximation of u.

As an illustration, we present an example with a discontinuous and 'singular' obstacle, the exact solution of which has singularities that are related to reentrant corner singularities for the linear Poisson equation. Figure 1 depicts a corresponding finite element minimizer on the left and the decay of $\mathscr{E}_k$ versus the number of degrees of freedom in a log-log scale on the right. In this particular example, our theory ensures $\frac{1}{2}\|\nabla(u_k - u)\|^2 \leq I[u_k] - I[u] \approx \frac{1}{2}\mathscr{E}_k$ and so the shown estimator decay rate implies an almost maximum decay rate for linear finite elements in the H^1-norm, which here cannot be reached by uniform refinement.

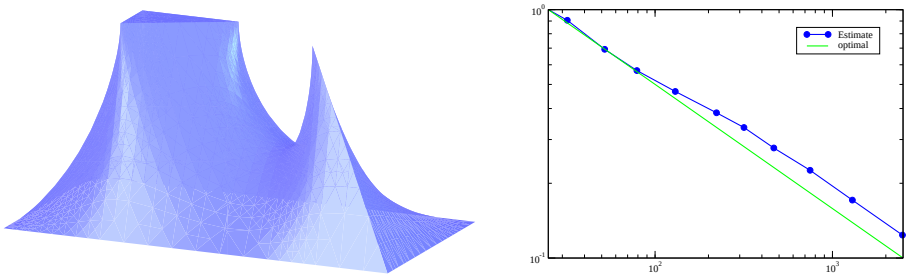

FIGURE 1. Example with a discontinuous and singular obstacle: a finite element minimizer (left) and estimator versus number of degrees of freedom with maximum decay rate for linear finite elements in log-log scale (right).

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