A TWO GRID SOLVER FOR SPD PROBLEMS

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg \min \| x^{(n+1)} - x \|_A = \frac{(A^{-1} r^{(n)}, S r^{(n)})_A}{(S r^{(n)}, S r^{(n)})_A}$$

Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$
1 iteration with $S = S_C = P A_C^{-1} R$

A TWO GRID SOLVER FOR SPD PROBLEMS

Error reduction and stopping criteria

Let $e^{(n)} = x^{(n)} - x$ the error at step n, $\tilde{e}^{(n)} = [I - S_C A]e^{(n)} = [I - P_C]e^{(n)}$. Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, S_F A \tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, (P_C + S_F A)\tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2}$$

Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} \leq \sup_e [1 - \frac{| (e, (P_C + S_F A)e)_A |^2}{\| e \|_A^2 \| S_F A e \|_A^2}] \leq C < 1$$

(Error Reduction)

For our stopping criteria, we want: Iterative Solver Error $\approx$ Discretization Error. That is:

$$\frac{\| e^{(n+1)} \|_A}{\| e^{(0)} \|_A} \leq 0{,}01$$

(Stopping Criteria)

A TWO GRID SOLVER FOR ELECTROMAGNETICS

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} A S] r^{(n)}$$
$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg \ \min \| x^{(n+1)} - x \|_B = \frac{(A^{-1} r^{(n)}, S r^{(n)})_B}{(S r^{(n)}, S r^{(n)})_B}$$ (NOT COMPUTABLE)

Then, we define our two grid solver for Electromagnetics as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$
1 iteration with $S = S_\nabla = \sum G_i^{-1}$ $+$
1 iteration with $S = S_C = P A_C^{-1} R$

A TWO GRID SOLVER FOR ELECTROMAGNETICS

A two grid solver for discretization of Maxwell's equations using hp-FE

Helmholtz decomposition:

$$H_D(\text{curl};\Omega) = (Ker(\text{curl})) \oplus (Ker(\text{curl}))^{\perp}$$

We define the following subspaces (T =grid, K =element, v =vertex, e =edge):

$\Omega^v_{k,i} = \text{int}(\bigcup\{\bar{K} \in T_k : v_{k,i} \in \partial K\})$; $\Omega^e_{k,i} = \text{int}(\bigcup\{\bar{K} \in T_k : e_{k,i} \in \partial K\})$ *Domain decomposition*

$M^v_{k,i} = \{u \in M_k : \text{supp}(u) \subset \Omega^v_{k,i}\}$; $M^e_{k,i} = \{u \in M_k : \text{supp(u)} \subset \Omega^e_{k,i}\}$ *Nedelec's elements decomposition*

$W^v_{k,i} = \{u \in W_k : \text{supp}(u) \subset \Omega^v_{k,i}\}$; $W^e_{k,i} = \{u \in W_k : \text{supp}(u) \subset \Omega^e_{k,i}\} = \emptyset$ *Polynomial spaces decomposition*

Hiptmair proposed the following decomposition of M_k:

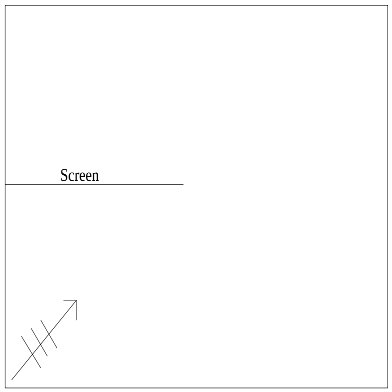
$$M_k = \sum_e M^e_{k,i} + \sum_v \nabla W^v_{k,i}$$

Arnold *et. al* proposed the following decomposition of M_k:

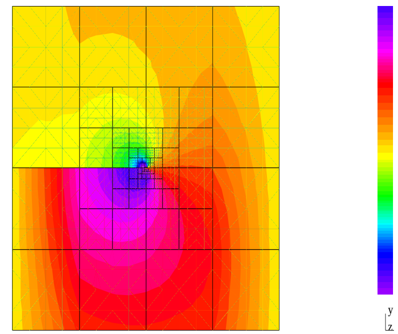
$$M_k = \sum_v M^v_{k,i}$$

PERFORMANCE OF THE TWO GRID SOLVER

Plane Wave incident into a screen (diffraction problem)

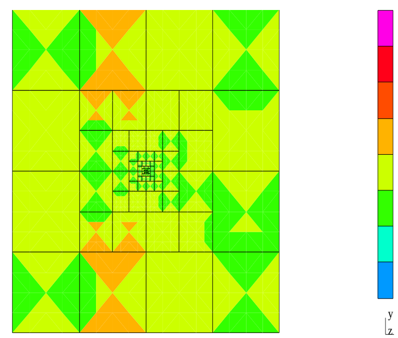


Geometry



Second component of electric field

Convergence history
(tolerance error= 0.1 %)

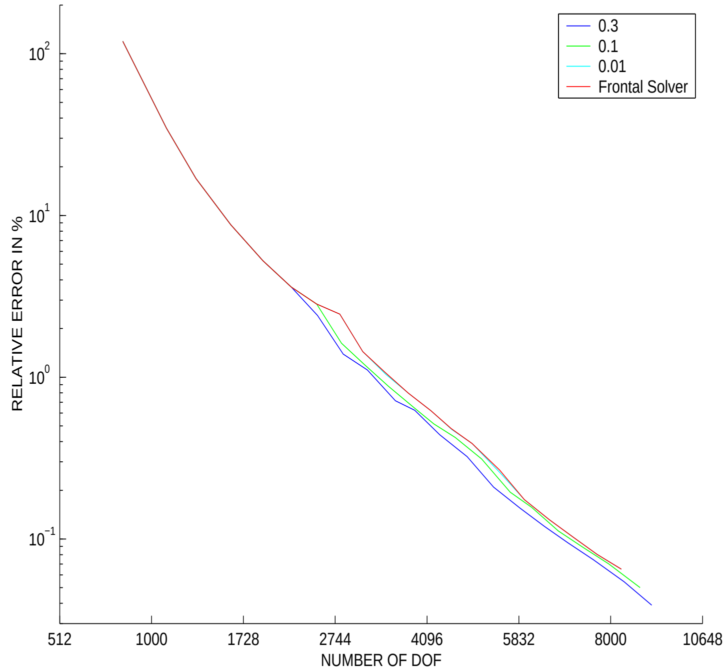


Final hp grid

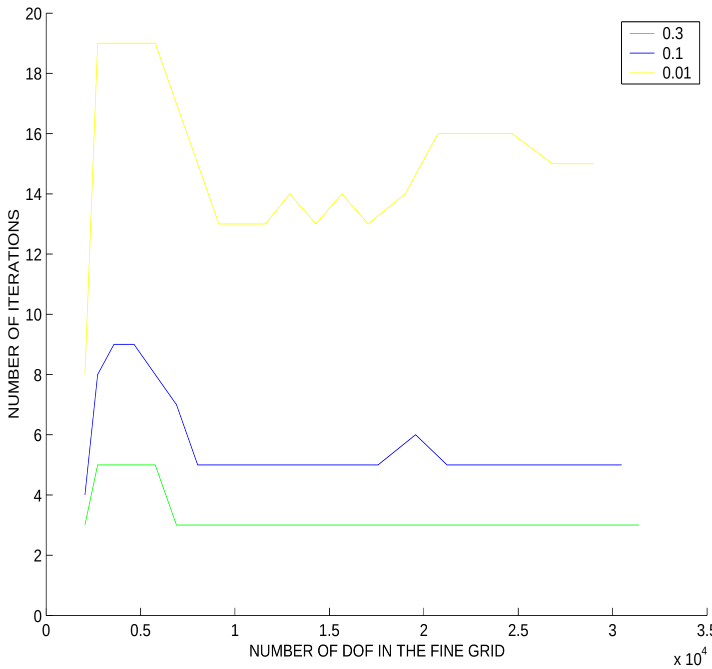
PERFORMANCE OF THE TWO GRID SOLVER

Guiding automatic hp-refinements

Diffraction problem. Guiding hp-refinements with a partially converged solution.



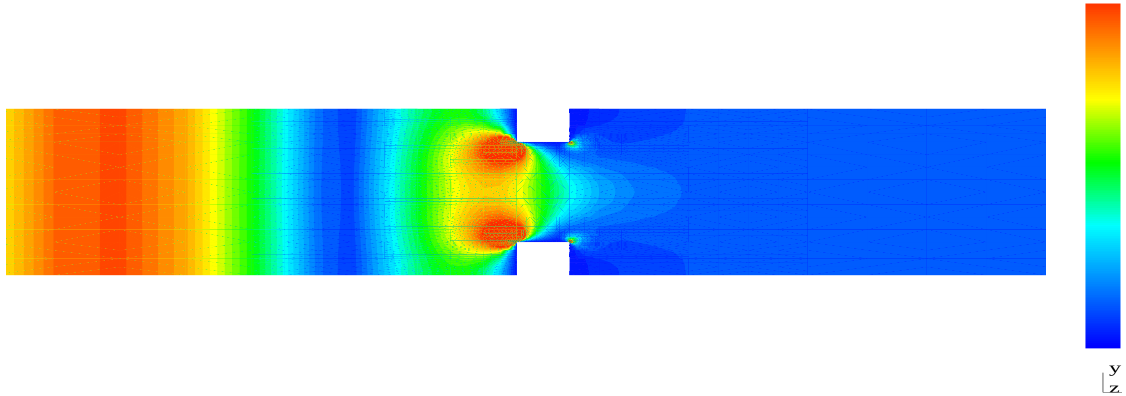
Discretization error



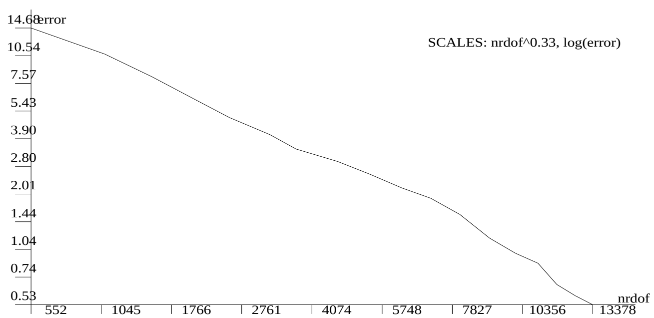
Number of iterations

PERFORMANCE OF THE TWO GRID SOLVER

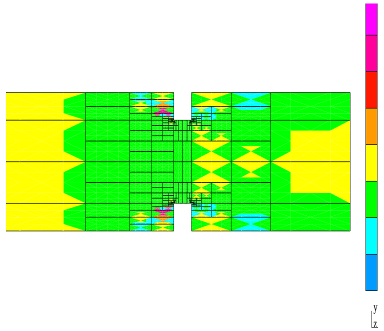
Waveguide example



Module of Second Component of Magnetic Field



Convergence history (tolerance error= 0.5 %)

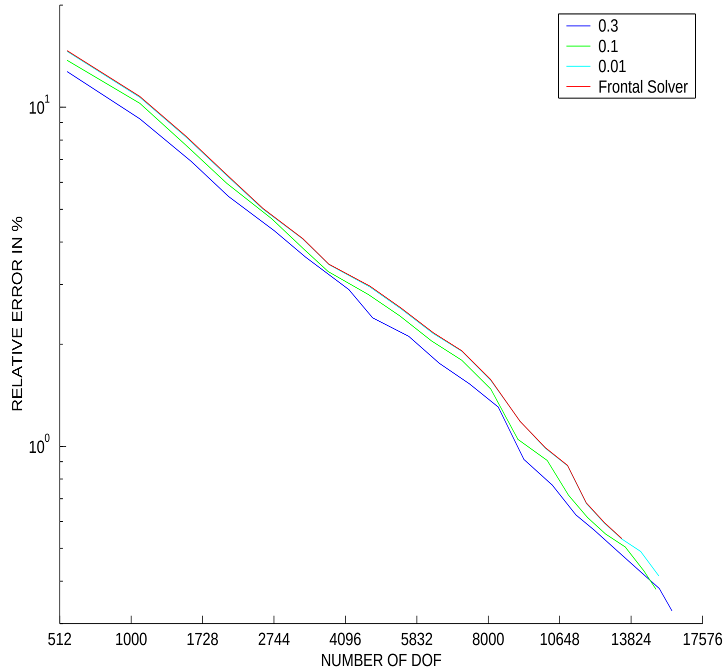


Final hp grid

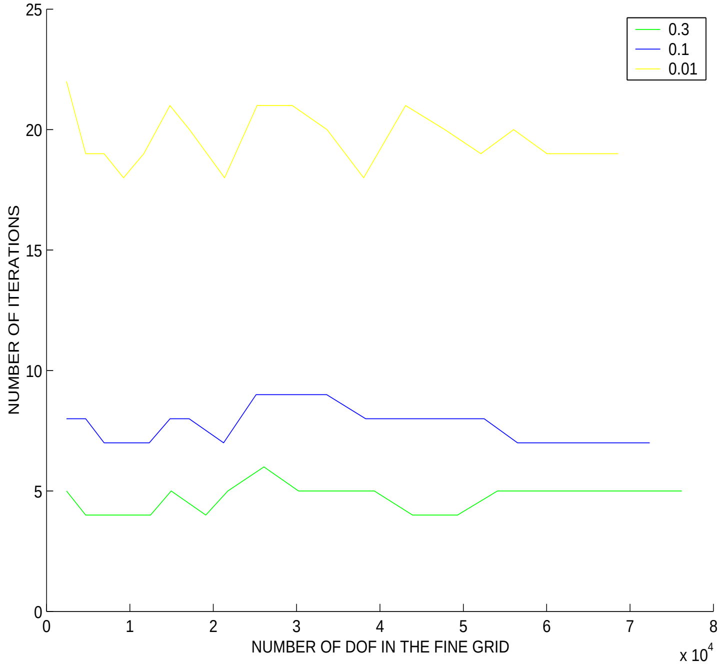
PERFORMANCE OF THE TWO GRID SOLVER

Guiding automatic hp-refinements

Waveguide problem. Guiding hp-refinements with a partially converged solution.



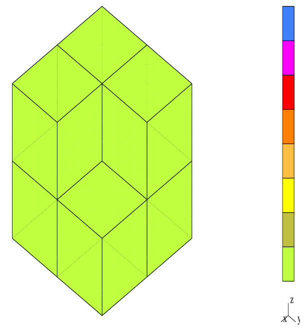
Discretization error



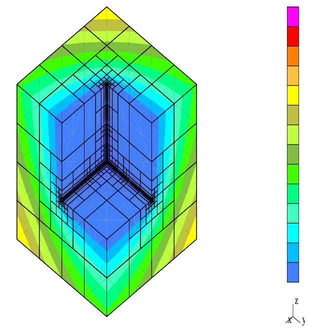
Number of iterations

PERFORMANCE OF THE TWO GRID SOLVER

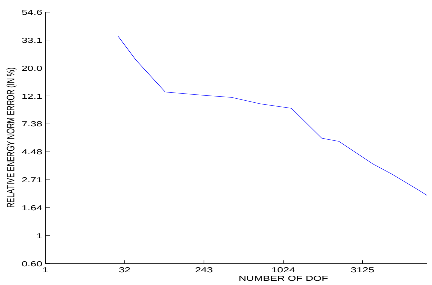
Fickera problem



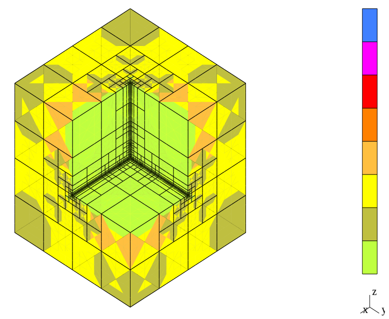
Equation: $-\Delta u = 0$
Boundary Conditions: Neumann, Dirichlet



Solution: unknown



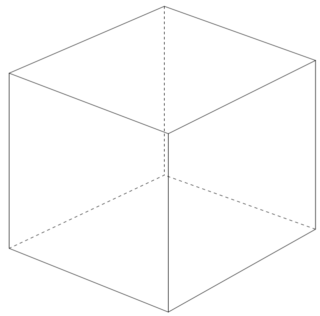
Convergence history
(tolerance error= 1 %)



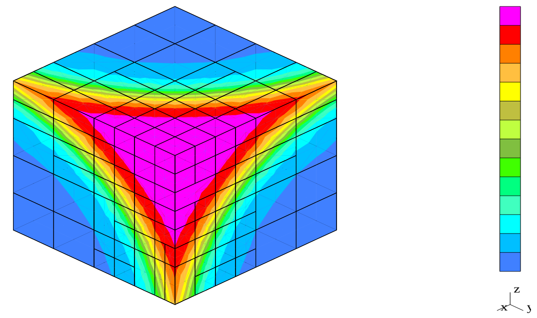
Final hp grid

PERFORMANCE OF THE TWO GRID SOLVER

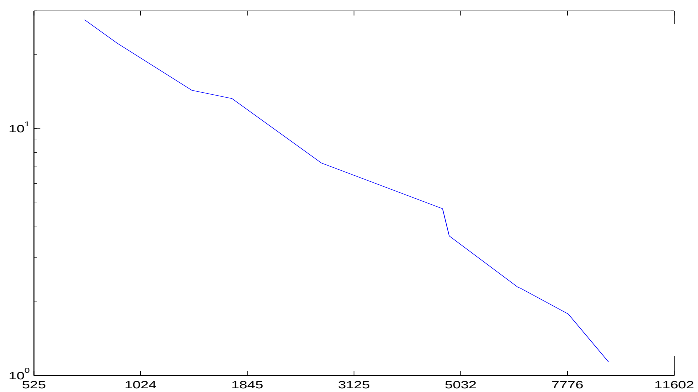
3D shock like solution example



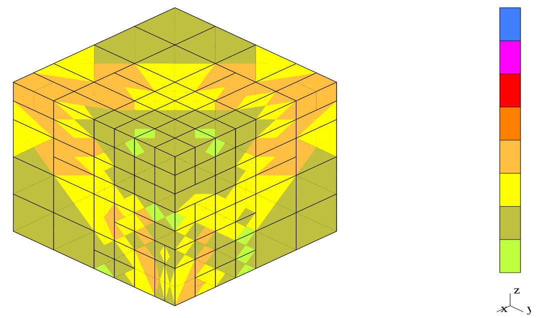
Equation: $-\Delta u = f$
Geometry: $[0,1]^3$



Solution: $u = atan(20 * \sqrt{r} - \sqrt{3}))$
$r = (x - ,25) * *2 + (y - ,25) * *2 + (z - ,25) * *2$
Boundary Conditions: Dirichlet



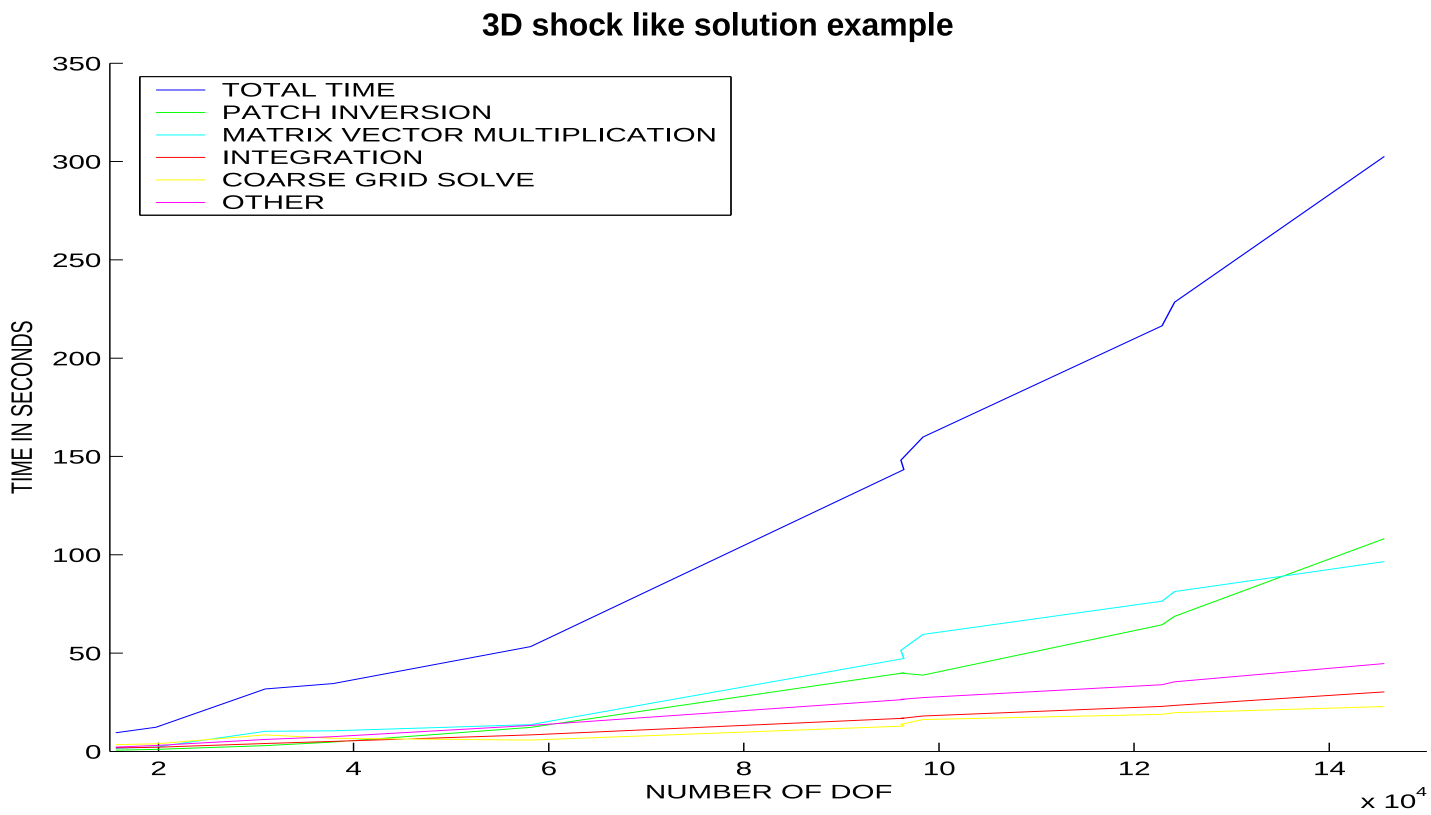
Convergence history
(tolerance error= 1 %)



Final hp grid

PERFORMANCE OF THE TWO GRID SOLVER

Performance of the two grid solver



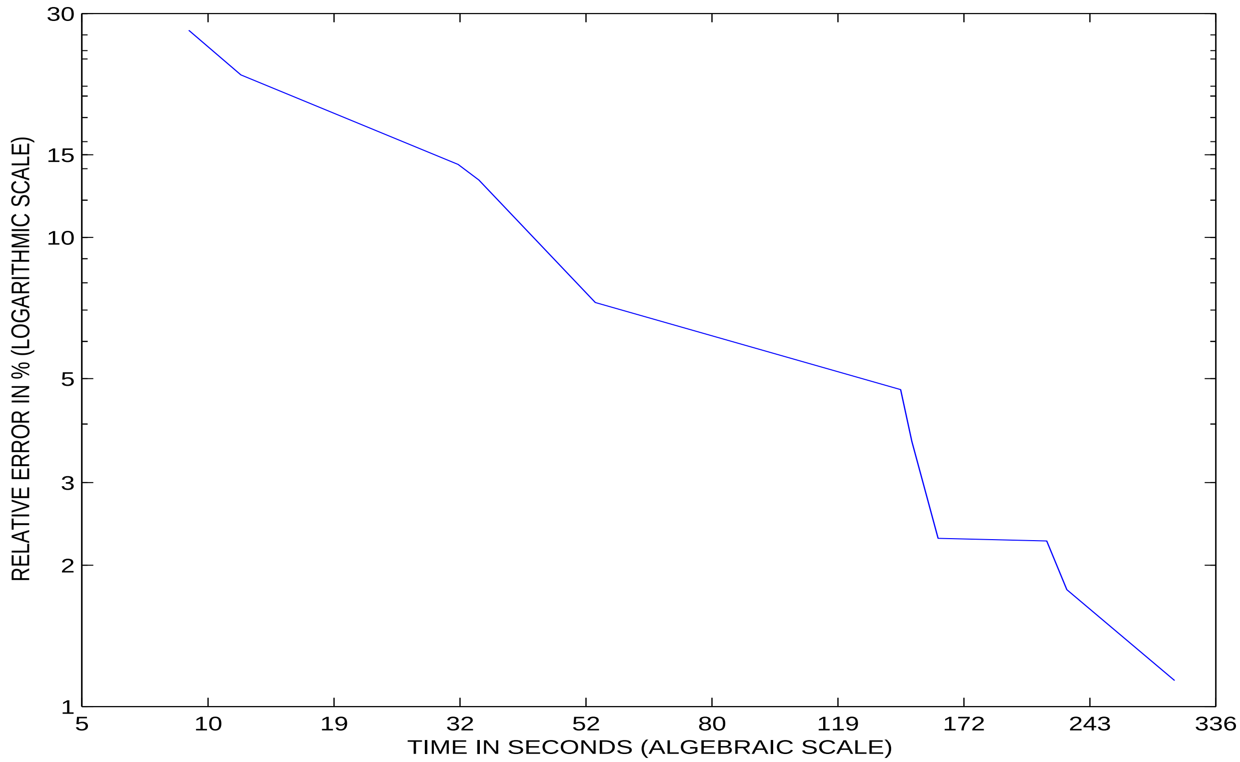
In core computations, AMD Athlon 1 Ghz processor.

PERFORMANCE OF THE TWO GRID SOLVER

Convergence history

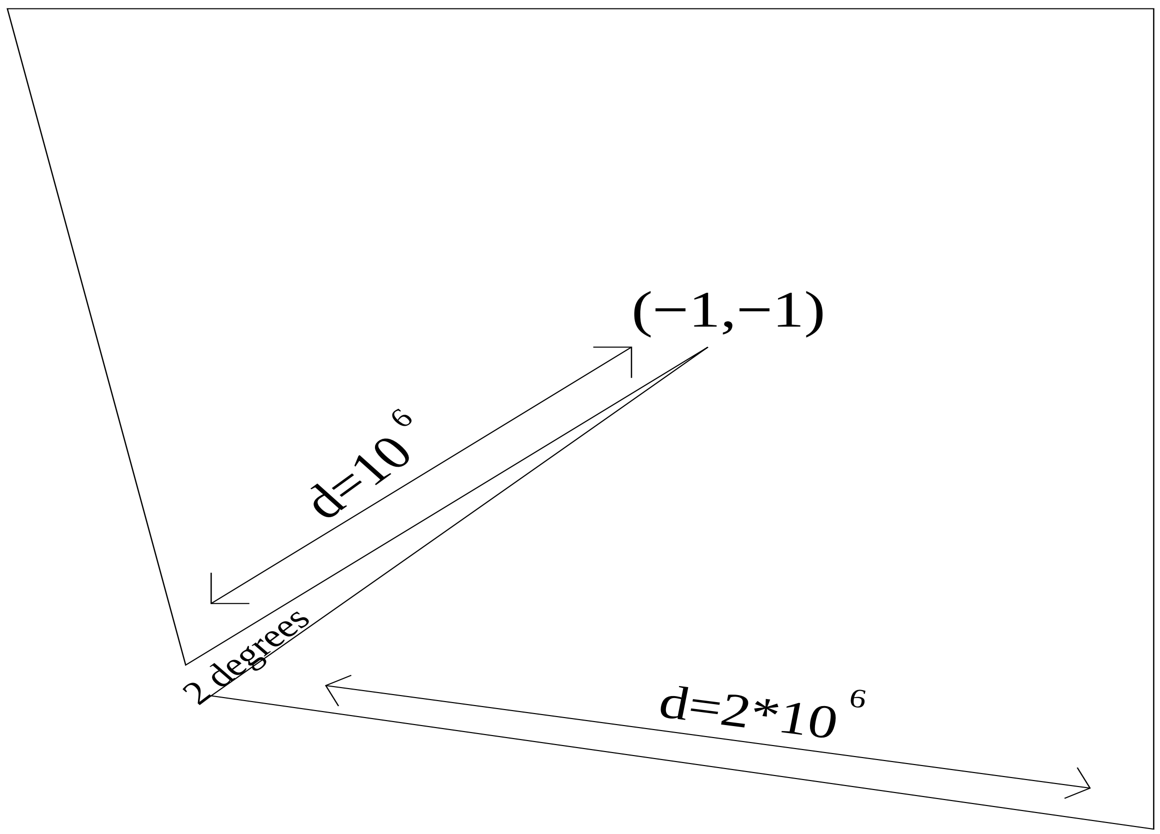
3D shock like solution example

Scales: ERROR VS TIME.



ELECTROMAGNETIC APPLICATIONS

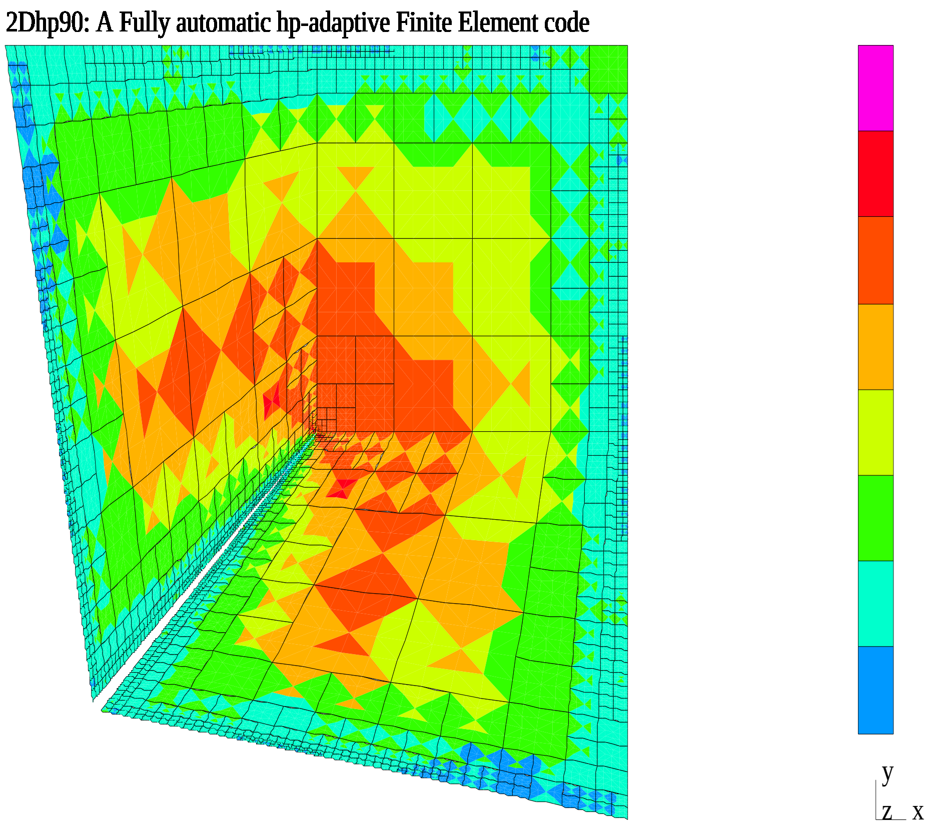
Edge diffraction example(Baker-Hughes): Electrostatics



Dirichlet Boundary Conditions
u(boundary)=−ln r, r=sqrt (x*x+y*y)

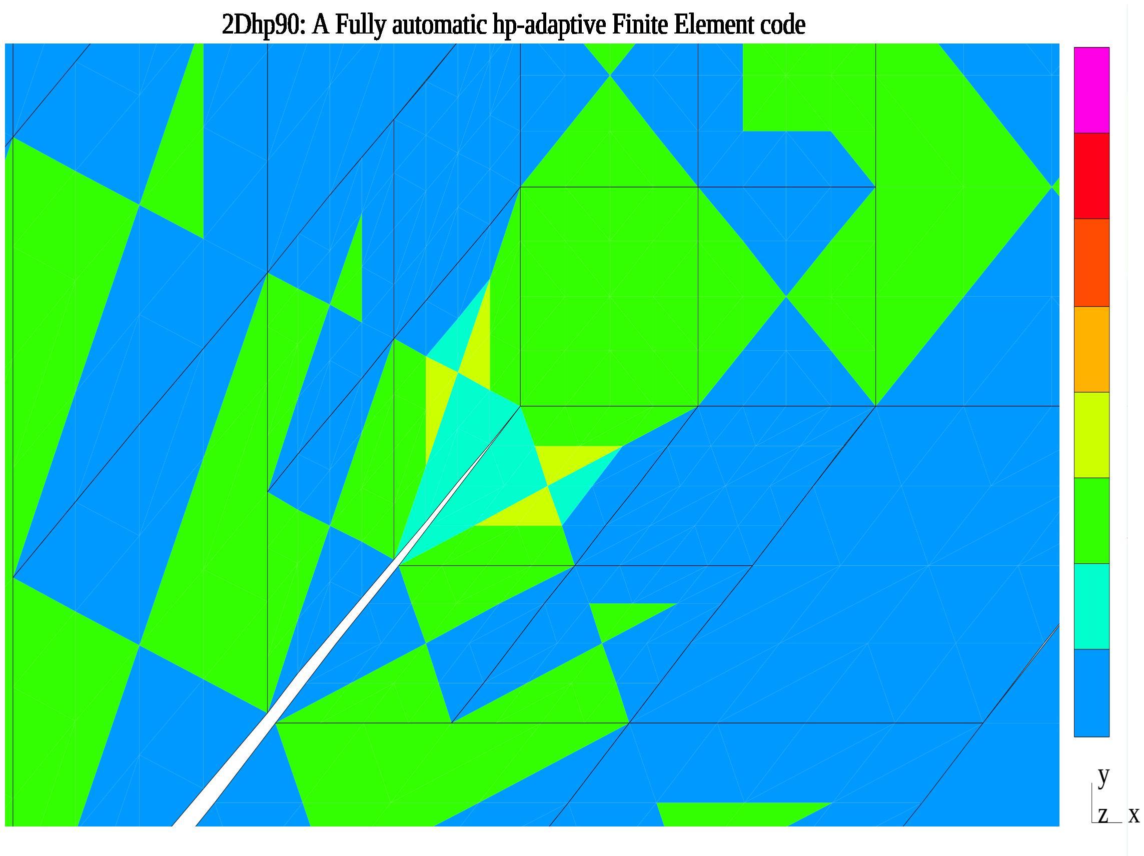
ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Final hp grid, Zoom = 1



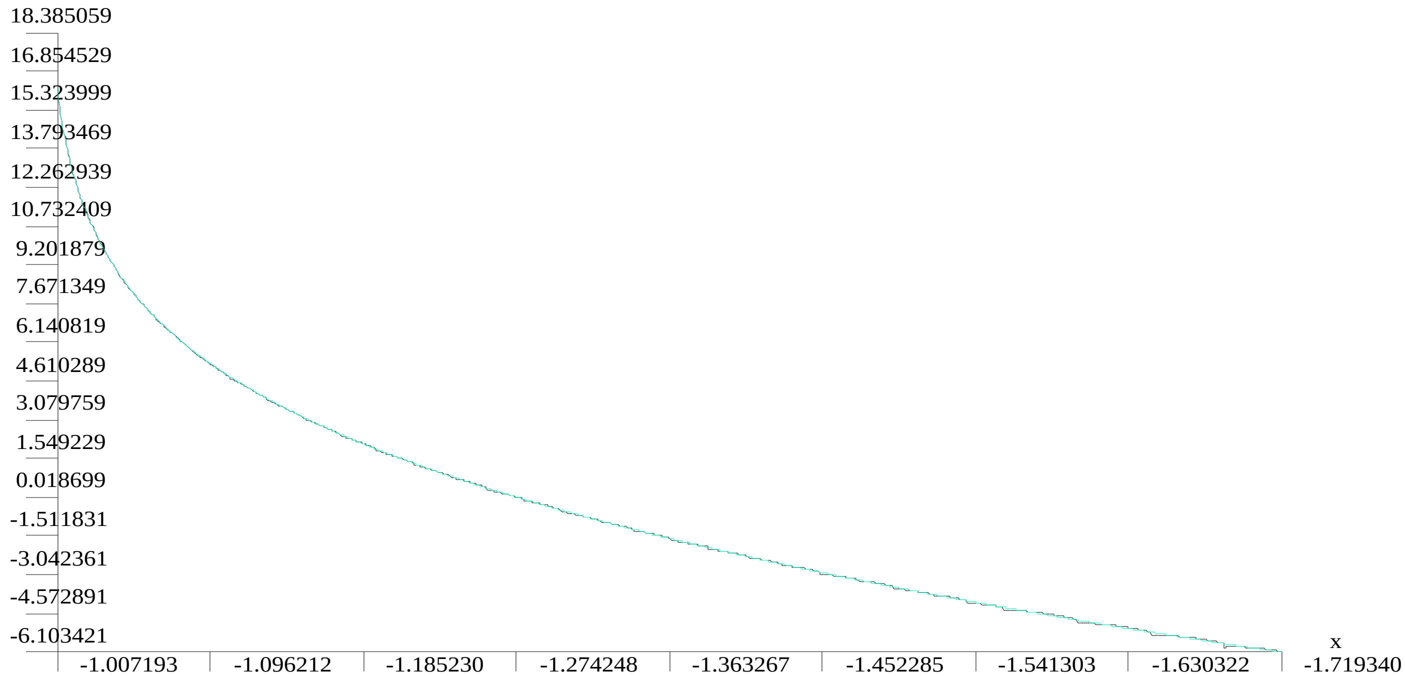
ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Final hp grid, Zoom = 10^{13}



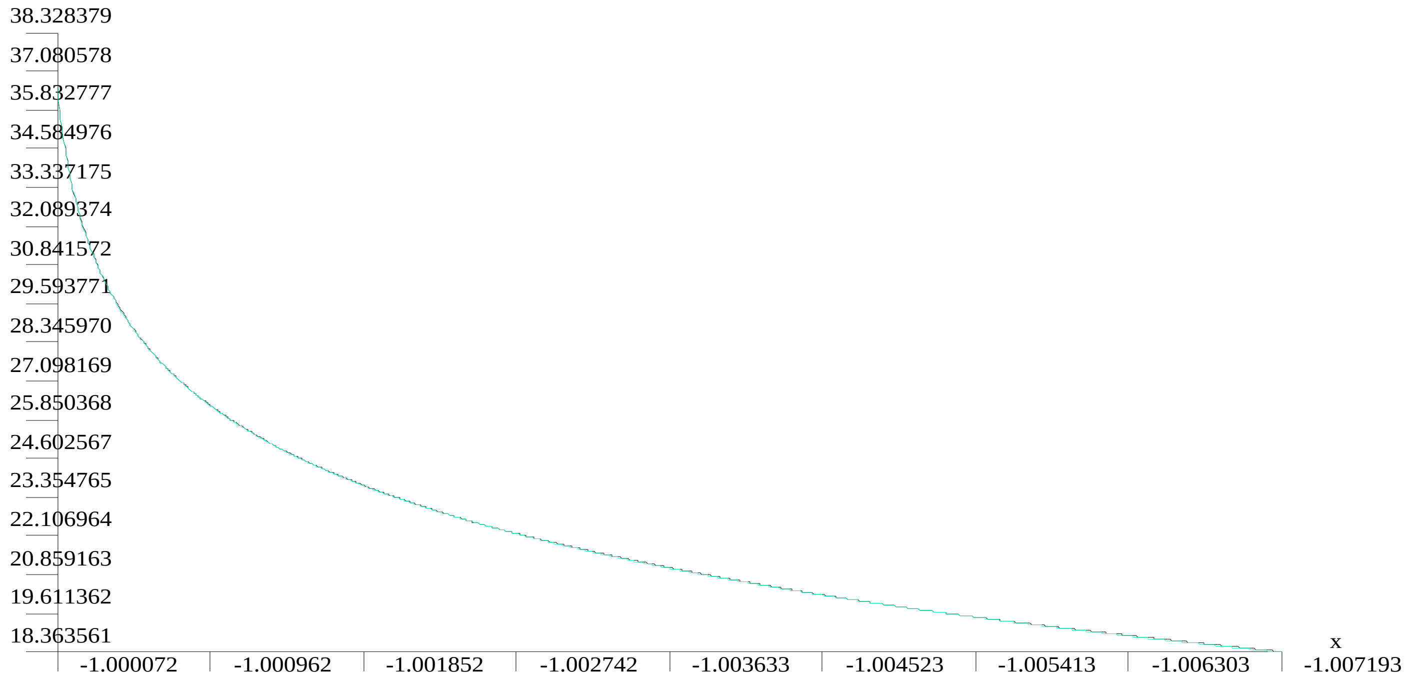
ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Comparison between exact and approximate solution at distances 0.01-1 from the singularity



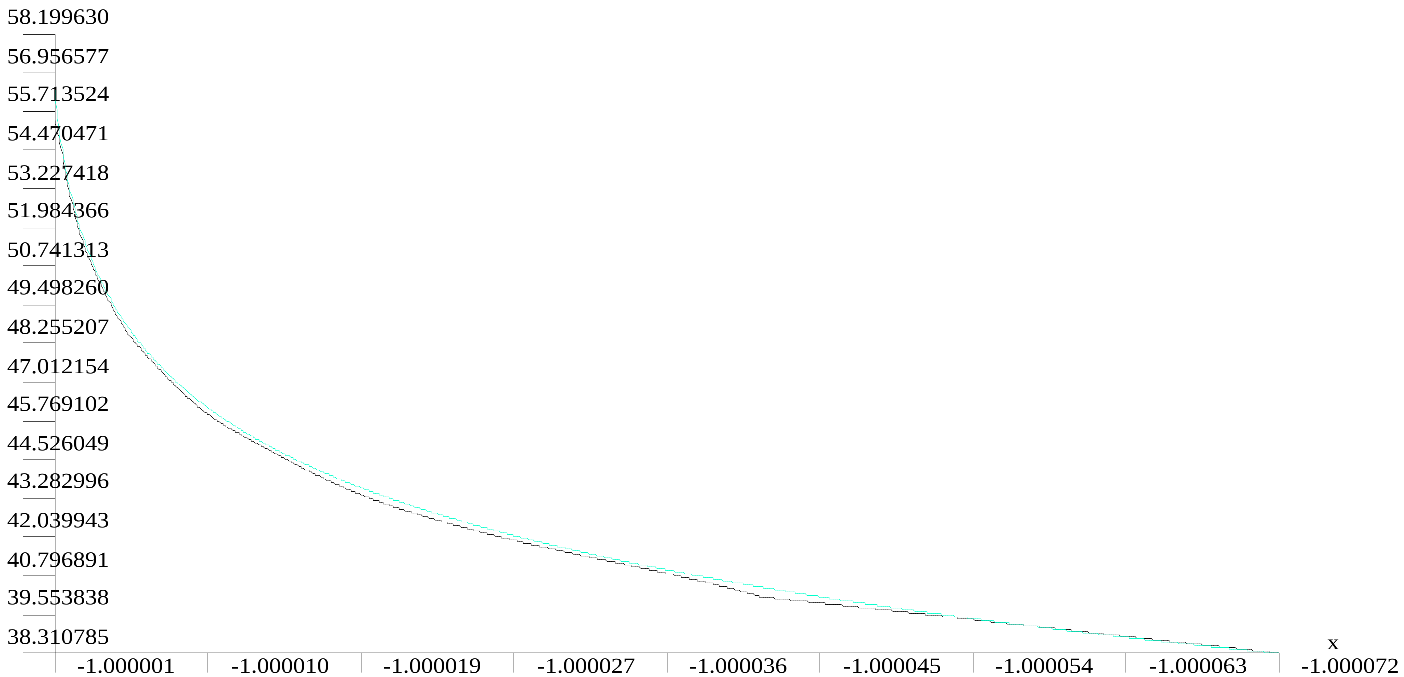
ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Comparison between exact and approximate solution at distances 0.0001-0.01 from the singularity



ELECTROMAGNETIC APPLICATIONS

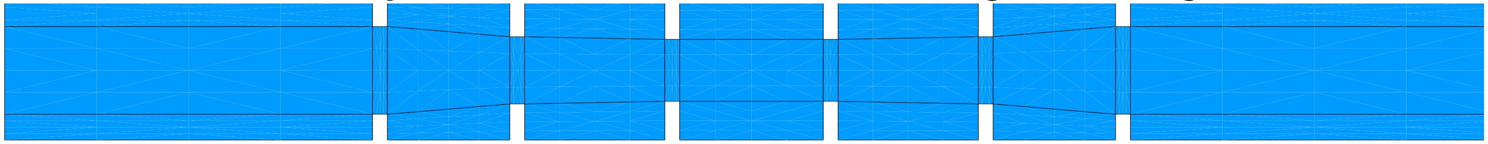
Edge diffraction example: Comparison between exact and approximate solution at distances 0.000001-0.0001 from the singularity

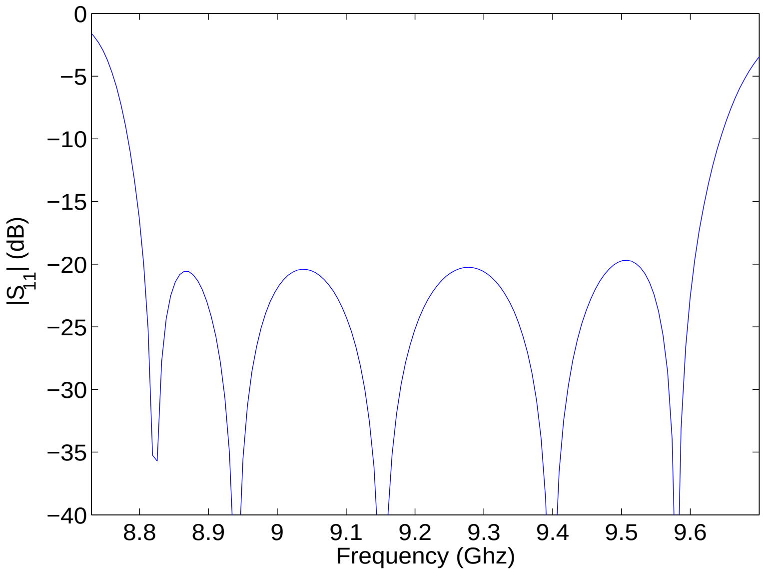


ELECTROMAGNETIC APPLICATIONS

Waveguide example with six iris

Geometry of a cross section of the rectangular waveguide





Return loss of the waveguide structure

H-plane six resonant iris filter.

Dominant mode (source): $TE_{10}-$mode.

Dimensions $\approx 20 \times 2 \times 1$ cm.

Operating Frequency $\approx 8{,}8 - 9{,}6$ Ghz

Cutoff frequency $\approx 6{,}56$ Ghz

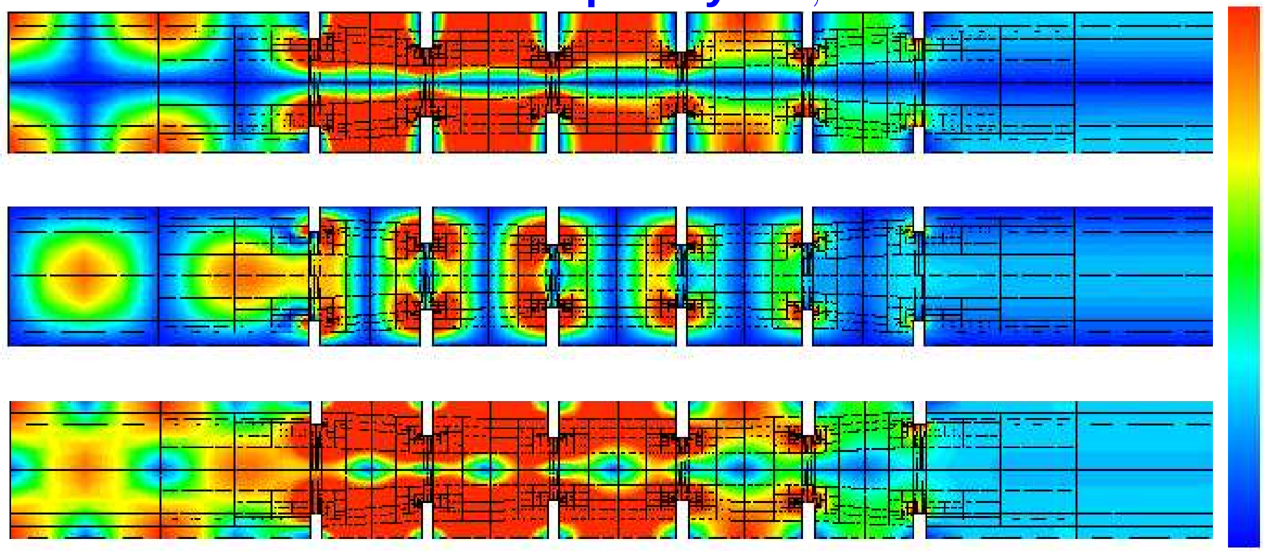
ELECTROMAGNETIC APPLICATIONS

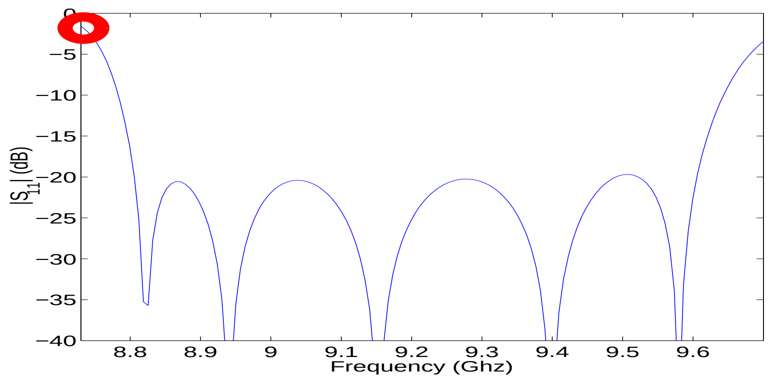
FEM solution for frequency $= 8{,}72$ Ghz

$|H_x|$

$|H_y|$

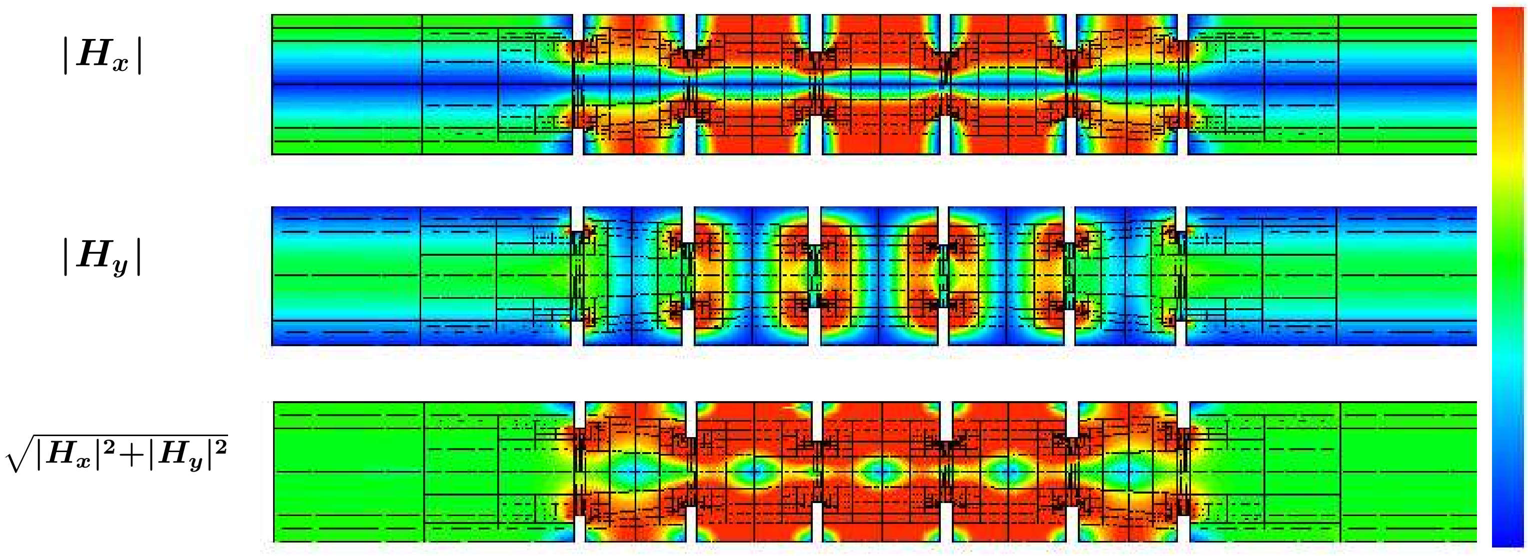
$\sqrt{|H_x|^2+|H_y|^2}$

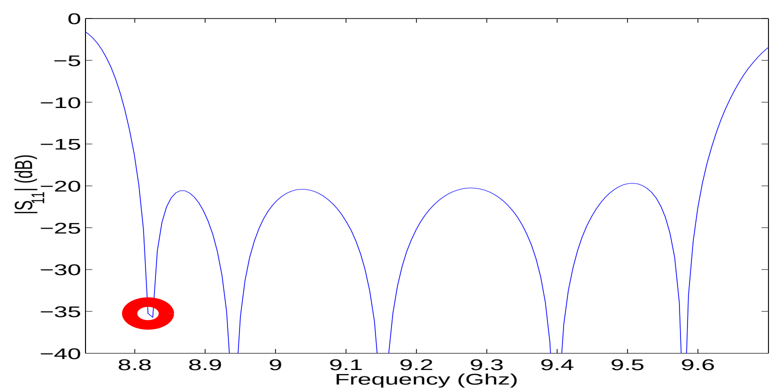




ELECTROMAGNETIC APPLICATIONS

FEM solution for frequency $= 8{,}82$ Ghz





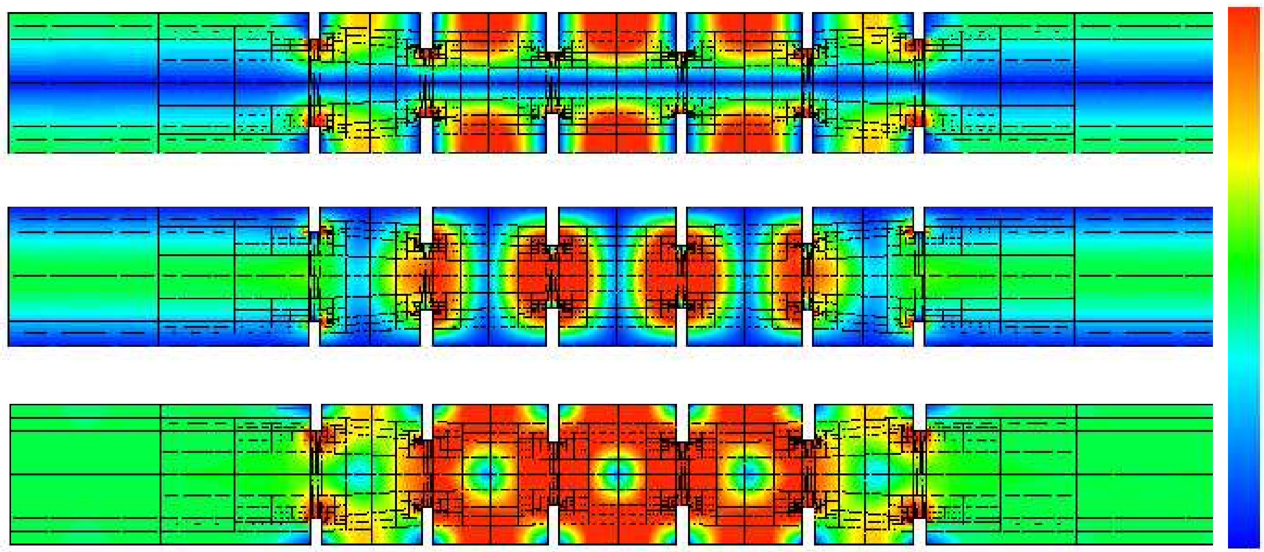
ELECTROMAGNETIC APPLICATIONS

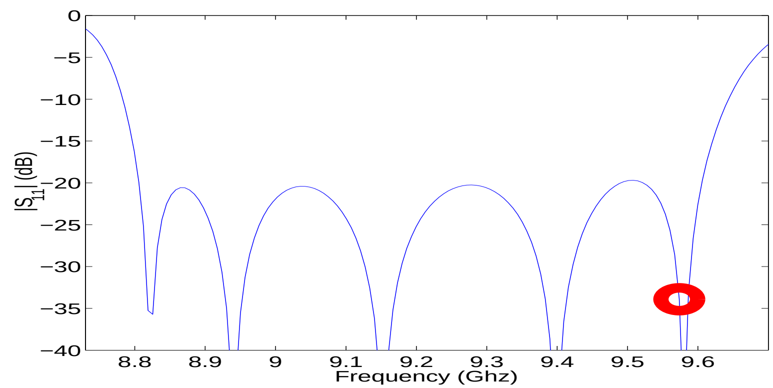
FEM solution for frequency $= 9{,}58$ Ghz

$|H_x|$

$|H_y|$

$\sqrt{|H_x|^2+|H_y|^2}$





ELECTROMAGNETIC APPLICATIONS

FEM solution for frequency $= 9{,}71$ Ghz