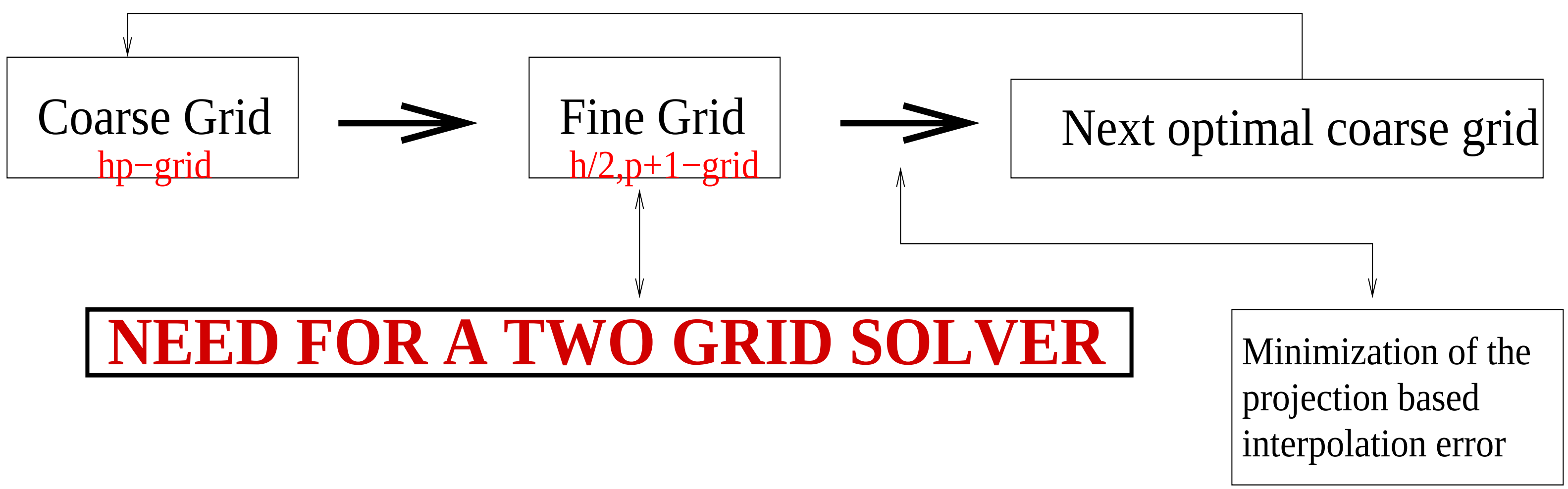
A TWO GRID SOLVER FOR ELECTROSTATICS

Automatic *hp*-adaptivity delivers exponential convergence and enables solution of challenging EM problems



A TWO GRID SOLVER FOR ELECTROSTATICS

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg\ \text{mín} \parallel x^{(n+1)} - x \parallel_A = \frac{(A^{-1} r^{(n)}, S r^{(n)})_A}{(S r^{(n)}, S r^{(n)})_A}$$

Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1} \quad +$
1 iteration with $S = S_C = P A_C^{-1} R$

A TWO GRID SOLVER FOR ELECTROSTATICS

Error reduction and stopping criteria

Let $e^{(n)} = x^{(n)} - x$ the error at step n, $\tilde{e}^{(n)} = [I - S_C A]e^{(n)} = [I - P_C]e^{(n)}$. Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} = 1 - \frac{|\ (\tilde{e}^{(n)}, S_F A \tilde{e}^{(n)})_A\ |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2} = 1 - \frac{|\ (\tilde{e}^{(n)}, (P_C + S_F A)\tilde{e}^{(n)})_A\ |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2}$$

Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} \leq \sup_e [1 - \frac{|\ (e, (P_C + S_F A)e)_A\ |^2}{\| e \|_A^2 \| S_F A e \|_A^2}] \leq C < 1 \qquad \text{(Error Reduction)}$$

For our stopping criteria, we want: Iterative Solver Error $\approx$ Discretization Error. That is:

$$\frac{\| e^{(n+1)} \|_A}{\| e^{(0)} \|_A} \leq 0{,}01 \qquad \text{(Stopping Criteria)}$$

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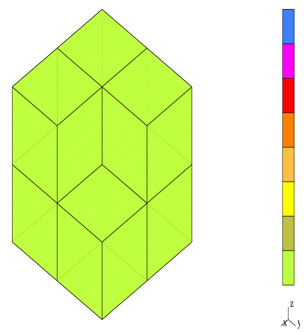
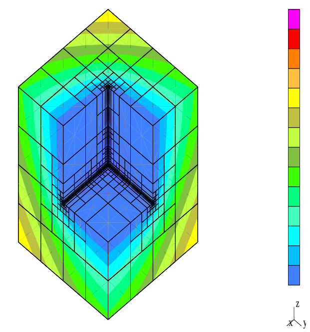
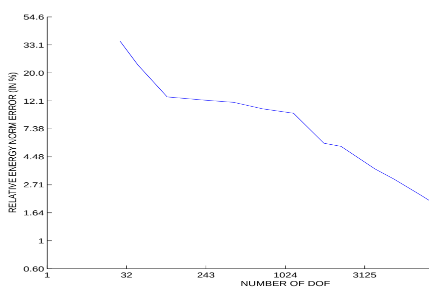
Fickera problem



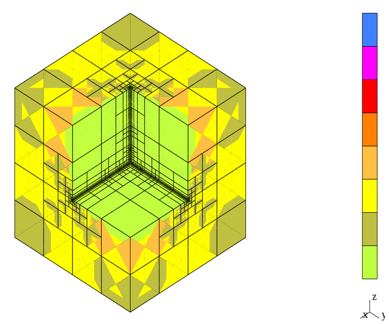
Equation: $-\Delta u = 0$
Boundary Conditions: Neumann, Dirichlet



Solution: unknown



Convergence history
(tolerance error= 1 %)

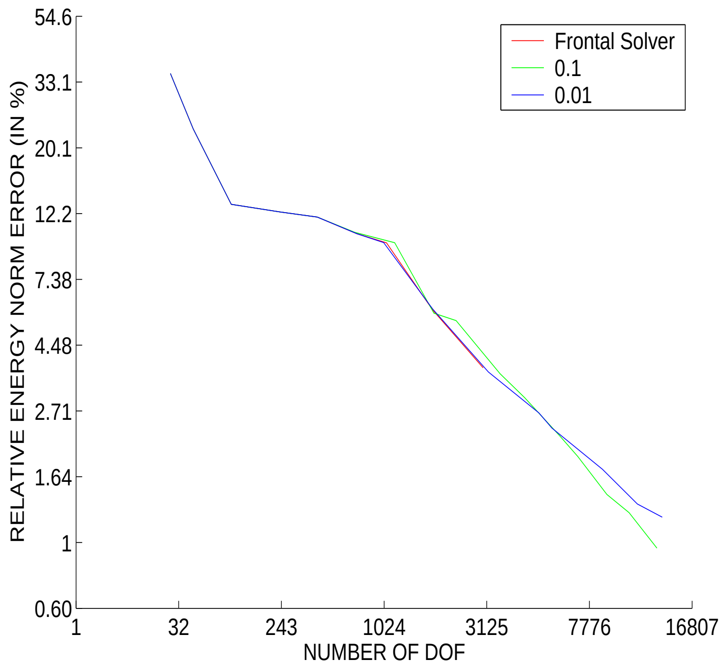


Final hp grid

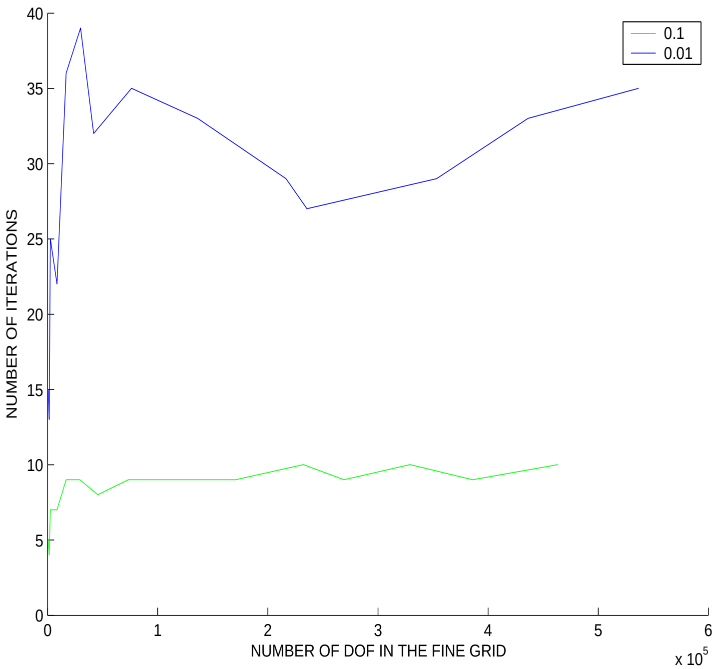
A TWO GRID SOLVER FOR ELECTROSTATICS

Guiding automatic hp-refinements

Fickera problem. Guiding hp-refinements with a partially converged solution.



Discretization error estimate



Number of iterations

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We seek x such that $Ax = b$. Consider the following iterative scheme:

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$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg\ \min \| x^{(n+1)} - x \|_B = \frac{(A^{-1} r^{(n)}, S r^{(n)})_B}{(S r^{(n)}, S r^{(n)})_B}$$ (NOT COMPUTABLE)

Then, we define our two grid solver for Electromagnetics as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$

1 iteration with $S = S_\nabla = \sum G_i^{-1}$ $+$

1 iteration with $S = S_C = P A_C^{-1} R$

A TWO GRID SOLVER FOR ELECTRODYNAMICS

A two grid solver for discretization of Maxwell's equations using hp-FE

Consider the following two problems:

Problem I: $\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = \mathbf{J}$

Matrix form: $Au = v$

Two grid solver V-cycle:

$TG = (I - \alpha_1 S_F A)(I - \alpha_2 S_\nabla A)(I - S_C A_C)$

Problem II: $\nabla \times \nabla \times \mathbf{E} + \mathbf{E} = \mathbf{J}$

Matrix form: $\hat{A}u = v$

Two grid solver V-cycle:

$\widehat{TG} = (I - \alpha_1 \hat{S}_F \hat{A})(I - \alpha_2 \hat{S}_\nabla \hat{A})(I - \hat{S}_C \hat{A}_C)$

Theorem: If h is small enough, then:

$$\| TGe^{(n)} \| \leq \| \widehat{TG}e^{(n)} \| + Ch$$

Notice that C is independent of h and p.

A TWO GRID SOLVER FOR ELECTRODYNAMICS

A two grid solver for discretization of Maxwell's equations using hp-FE

Helmholtz decomposition:

$$H_D(\text{curl};\Omega) = (Ker(\text{curl})) \oplus (Ker(\text{curl}))^{\perp}$$

We define the following subspaces (T =grid, K =element, v =vertex, e =edge):

$\Omega^v_{k,i} = \text{int}(\bigcup\{\bar{K} \in T_k : v_{k,i} \in \partial K\})$; $\Omega^e_{k,i} = \text{int}(\bigcup\{\bar{K} \in T_k : e_{k,i} \in \partial K\})$ *Domain decomposition*

$M^v_{k,i} = \{u \in M_k : \text{supp}(u) \subset \Omega^v_{k,i}\}$; $M^e_{k,i} = \{u \in M_k : \text{supp(u)} \subset \Omega^e_{k,i}\}$ *Nedelec's elements decomposition*

$W^v_{k,i} = \{u \in W_k : \text{supp}(u) \subset \Omega^v_{k,i}\}$; $W^e_{k,i} = \{u \in W_k : \text{supp}(u) \subset \Omega^e_{k,i}\} = \emptyset$ *Polynomial spaces decomposition*

Hiptmair proposed the following decomposition of M_k:

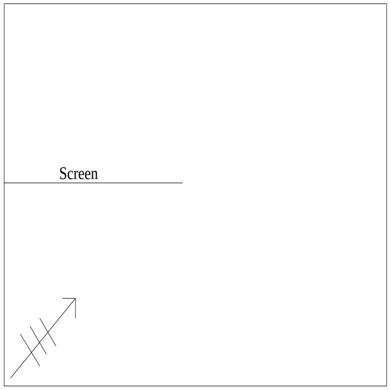
$$M_k = \sum_e M^e_{k,i} + \sum_v \nabla W^v_{k,i}$$

Arnold *et. al* proposed the following decomposition of M_k:

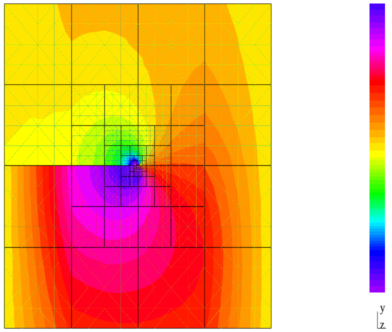
$$M_k = \sum_v M^v_{k,i}$$

A TWO GRID SOLVER FOR ELECTRODYNAMICS

Plane Wave incident into a screen (diffraction problem)

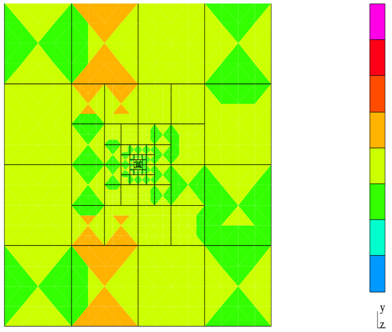


Geometry



Second component of electric field

Convergence history
(tolerance error= 0.1 %)

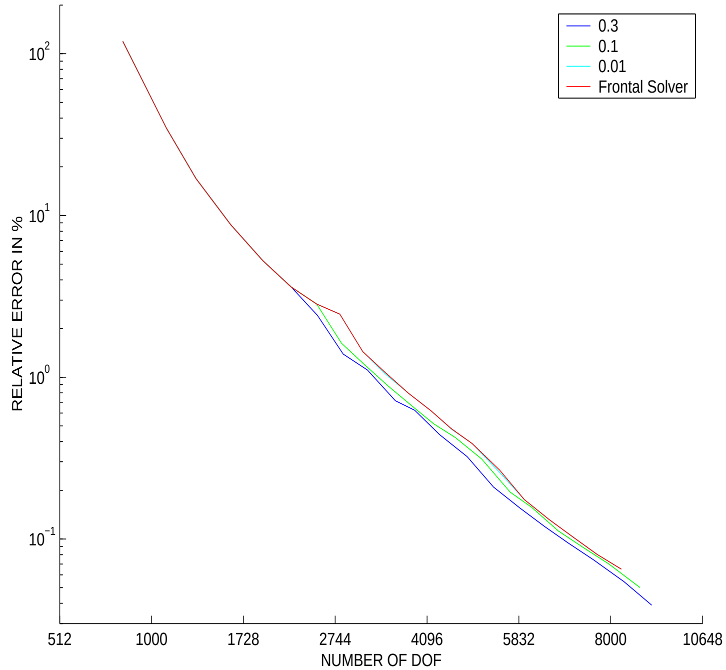


Final hp grid

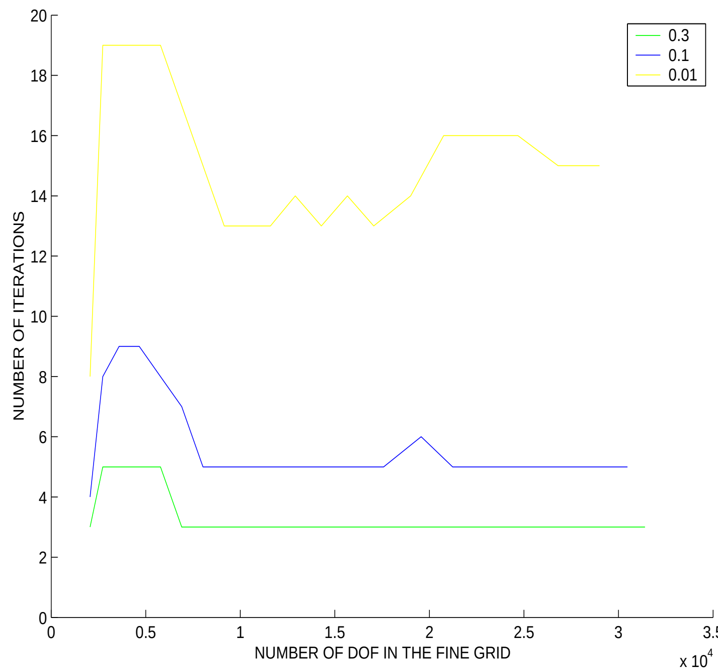
A TWO GRID SOLVER FOR ELECTRODYNAMICS

Guiding automatic hp-refinements

Diffraction problem. Guiding hp-refinements with a partially converged solution.



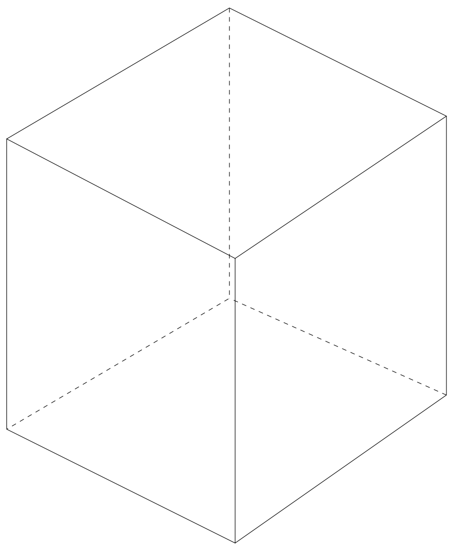
Discretization error



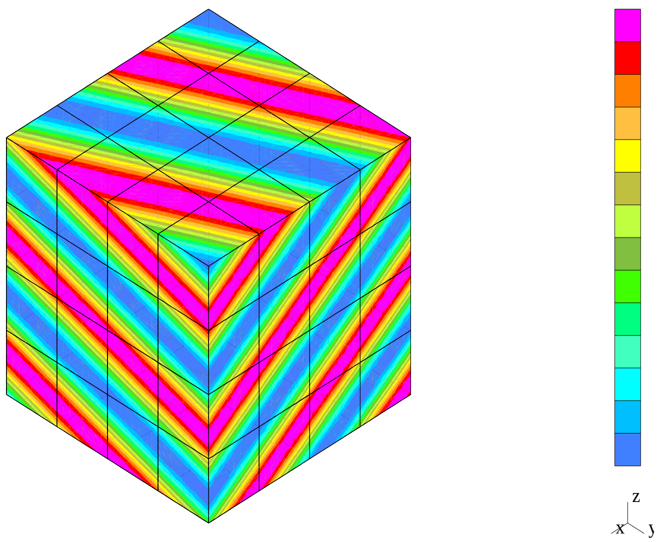
Number of iterations

A TWO GRID SOLVER FOR ELECTRODYNAMICS

3D EM Model Problem



Maxwell's equations
Boundary Conditions: Dirichlet, Cauchy

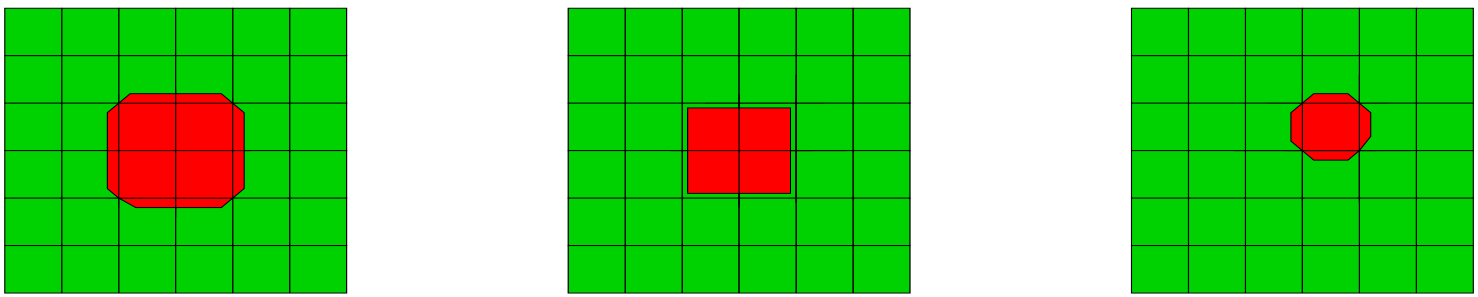


Solution: Plane wave

A TWO GRID SOLVER FOR ELECTRODYNAMICS

Selection of patches (for block Jacobi smoother)

Three examples of patches (blocks) for the Block Jacobi smoother:

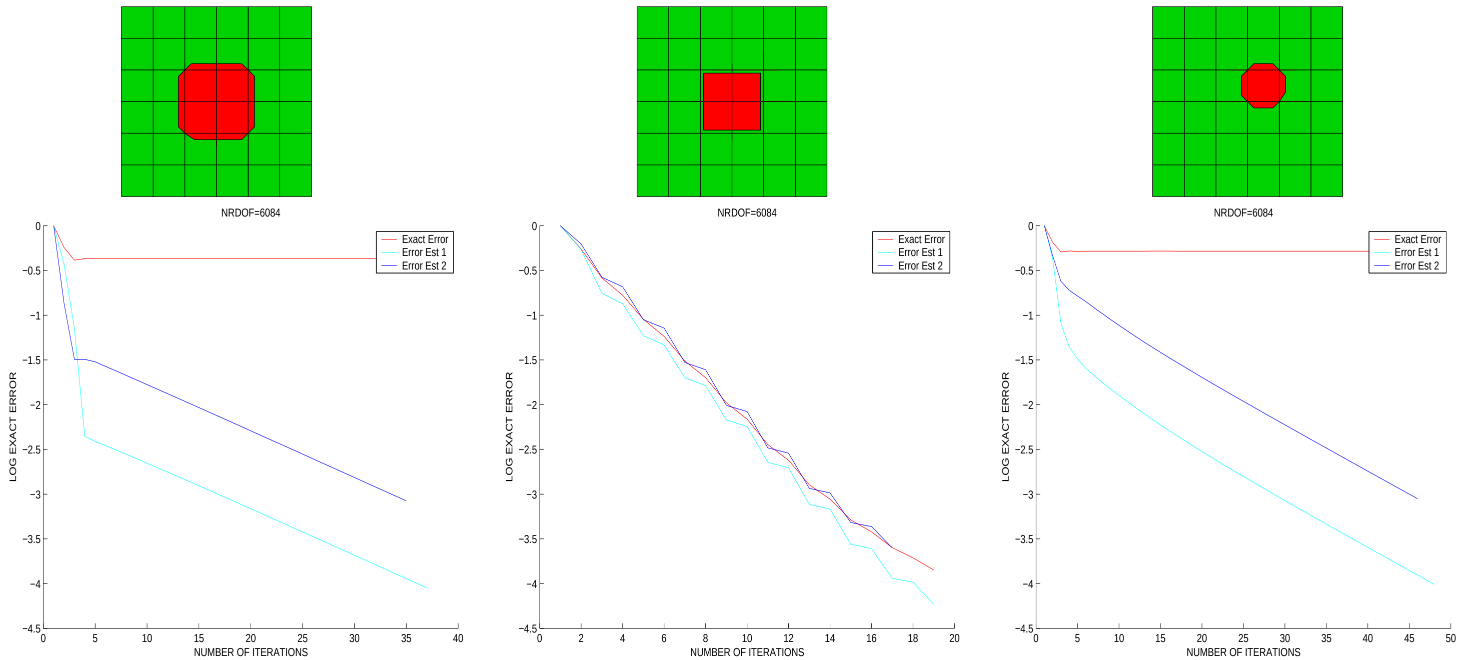


Example 1: span of basis functions corresponding to element stiffness matrices for all elements adjacent to a vertex.
Example 2: span of basis functions with support contained in the support of a fine grid vertex node basis function.
Example 3: span of basis functions corresponding to an element stiffness matrix.

A TWO GRID SOLVER FOR ELECTRODYNAMICS

Performance of different smoothers

3D EM Model Problem



ONLY SMOOTHER 2 CONVERGES

ELECTROMAGNETIC APPLICATIONS

Design of an initial uniform hp-grid.

3D EM Model Problem

(length of main diagonal of the cube varying from 1 to 50 wavelengths)

Nr. of λ vs p		p=1	p=2	p=3	p=4	p=5
1	ERROR	5.0 %	4.2 %	1.2 %	1.8 %	0.3 %
	D.O.F.	40K	946	1033	308	548
2	ERROR	5.0 %	4.2 %	2.9 %	1.9 %	0.3 %
	D.O.F.	(>300K)	6427	2764	2226	4109
4	ERROR	5.0 %	5.0 %	5.0 %	1.9 %	1.2 %
	D.O.F.	(>2300K)	(>82K)	12K	14K	12K
8	ERROR	5.0 %	5.0 %	5.0 %	5.0 %	2.8 %
	D.O.F.	(>20M)	(>650K)	(>167K)	(>71K)	51K
50	ERROR	5.0 %	5.0 %	5.0 %	5.0 %	5.0 %
	D.O.F.	(>5000M)	(>122M)	(>25M)	(>14M)	(>9.5M)

Large p controls dispersion error