A Fully Automatic Goal-Oriented hp-Adaptive Strategy with Applications to Electromagnetics.
Part I: A DC Resistivity Logging Problem.

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Baker-Atlas, a Division of Baker-Hughes

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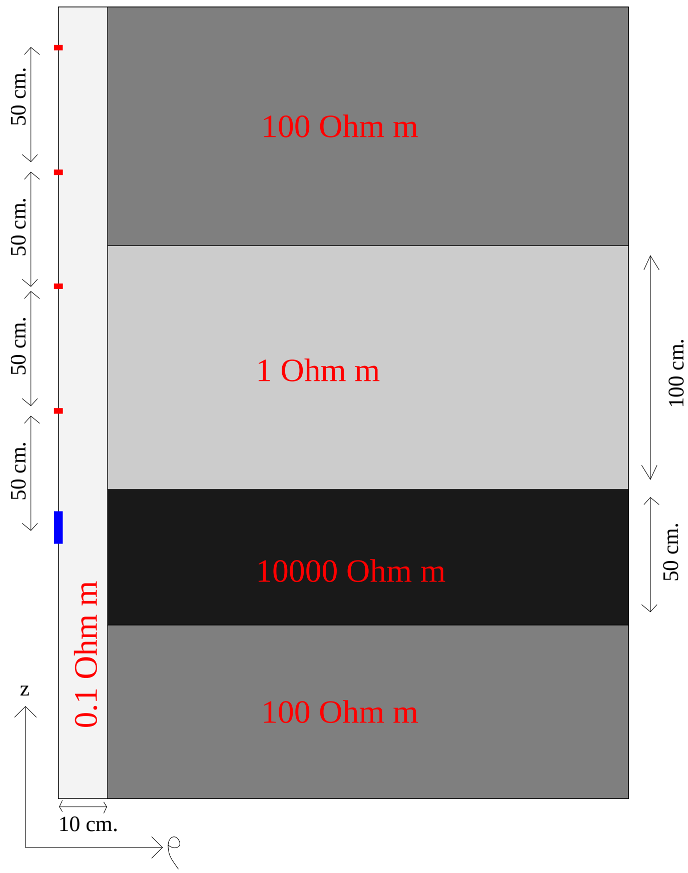
OVERVIEW

1. Motivation: A DC Resistivity Logging Problem.
2. Conductive Media Equation.
3. hp-Finite Elements.
4. Fully Automatic Energy Norm hp-Adaptive Strategy.
5. Fully Automatic Goal-Oriented hp-Adaptive Strategy.
6. Numerical Results.
7. Conclusions and Future Work.

MOTIVATION

A Direct Current (DC) Resistivity Logging Problem (Baker-Atlas)



Axisymmetric 3D problem.

Four different materials.

Material properties varying by up to FIVE orders of magnitude.

Objective:
Determine Electric Current on Receiving Electrodes.

CONDUCTIVE MEDIA EQUATION

Derivation of Conductive Media Equation:

Maxwell's Equations:

$$\begin{cases} \nabla \times \mathrm{H} = (\sigma - j\omega\epsilon)\mathrm{E} + \mathrm{J} \\ \nabla \times \mathrm{E} = (j\omega\mu\epsilon)\mathrm{H} \ , \\ \nabla \cdot \epsilon\mathrm{E} = \rho \ , \\ \nabla \cdot \mu\mathrm{H} = 0 \ , \end{cases}$$

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Steady state:

$$\left\{\begin{array}{l}\nabla \times \mathrm{H} = \sigma\mathrm{E} + \mathrm{J} \\ \nabla \times \mathrm{E} = 0\ , \\ \nabla \cdot \epsilon\mathrm{E} = \rho\ , \\ \nabla \cdot \mu\mathrm{H} = 0\ .\end{array}\right.$$

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Since $\nabla \times \mathrm{E} = 0$, then $\mathrm{E} = -\nabla\Psi$ for some Ψ:

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CONDUCTIVE MEDIA EQUATION

Derivation of Conductive Media Equation:

Maxwell's Equations: **Steady state:**

$$\begin{cases} \nabla \times H = (\sigma - j\omega\epsilon)E + J \\ \nabla \times E = (j\omega\mu\epsilon)H \, , \\ \nabla \cdot \epsilon E = \rho \, , \\ \nabla \cdot \mu H = 0 \, , \end{cases} \quad \overset{\omega=0}{\Longrightarrow} \quad \begin{cases} \nabla \times H = \sigma E + J \\ \nabla \times E = 0 \, , \\ \nabla \cdot \epsilon E = \rho \, , \\ \nabla \cdot \mu H = 0 \, . \end{cases}$$

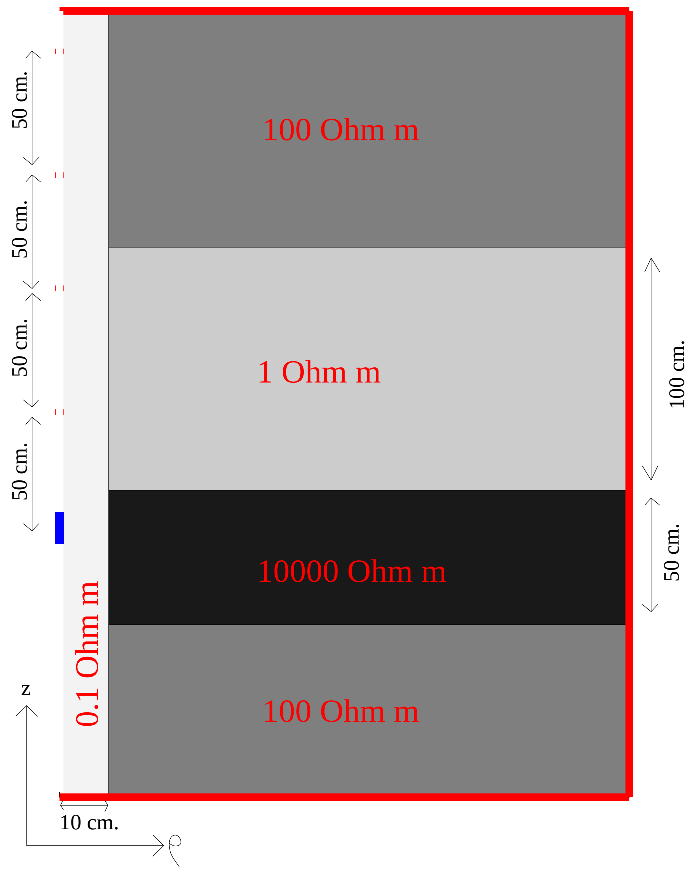
Since $\nabla \times E = 0$, then $E = -\nabla\Psi$ for some Ψ:

$$\begin{cases} \nabla \times H = -\sigma\nabla\Psi + J \\ -\nabla \cdot \epsilon\nabla\Psi = \rho \, , \\ \nabla \cdot \mu H = 0 \, . \end{cases} \quad \overset{\nabla\circ}{\Longrightarrow} \quad \begin{cases} -\nabla \cdot \sigma\nabla\Psi = \nabla \cdot J \, , \\ -\nabla \cdot \epsilon\nabla\Psi = \rho \, , \\ \nabla \cdot \mu H = 0 \, . \end{cases}$$

$$\boxed{-\nabla \cdot \sigma\nabla\Psi = \nabla \cdot J}$$

CONDUCTIVE MEDIA EQUATION

Boundary Conditions



Essential (Dirichlet BC) to make the computational domain finite.

No BC for the center of axisymmetry.

An extra boundary term to model the source electrode.

CONDUCTIVE MEDIA EQUATION

Variational Formulation

Multiplying the conductive media equation by a test function, integrating by parts, and incorporating the natural and essential boundary conditions:

$$\begin{cases} \text{Find } \Psi \in \Psi_D + V \text{ such that:} \\ \int_\Omega \sigma \nabla \Psi \nabla \xi \, dV = \int_\Omega \nabla \cdot \mathrm{J} \, \xi \, dV + \int_{\Gamma_N} g \, \xi \, dS \quad \forall \xi \in V \, . \end{cases}$$

Using Cylindrical Coordinates:

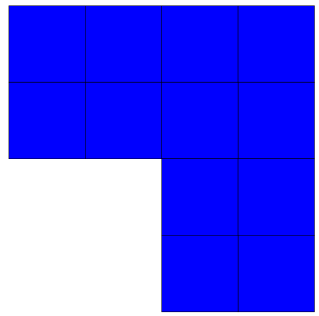
$$\begin{cases} \text{Find } \Psi \in \Psi_D + V \text{ such that:} \\ \int_\Omega \sigma \nabla \Psi \nabla \xi \, \rho \, d\rho d\psi dz = \int_\Omega \nabla \cdot \mathrm{J} \, \xi \, \rho \, d\rho d\psi dz + \int_{\Gamma_N} g \, \xi \, dS \quad \forall \xi \in V \, . \end{cases}$$

Using a different notation:

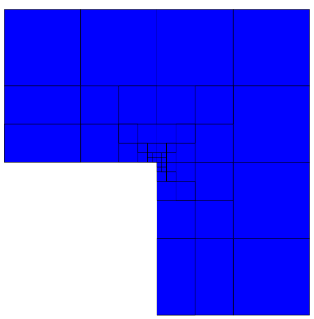
$$\begin{cases} \text{Find } \Psi \in \Psi_D + V \text{ such that:} \\ b(\Psi, \xi) = f(\xi) \quad \forall \xi \in V \, . \end{cases}$$

HP-FINITE ELEMENTS

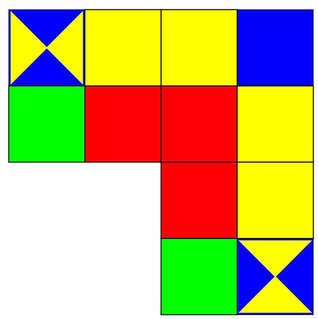
Different refinement strategies for finite elements:



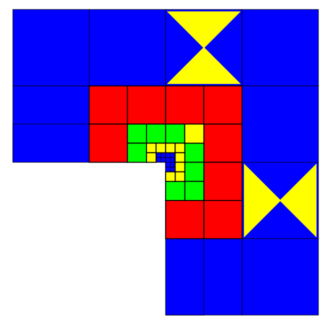
Given initial grid



***h*-refined grid**



***p*-refined grid**



***hp*-refined grid**

HP-FINITE ELEMENTS

Exponential convergence rates

for a number of regular and SINGULAR problems

if we orquestrate an optimal distribution of h and p within the same grid

Smaller dispersion (pollution) error

as *p* increases.

More geometrical details captured

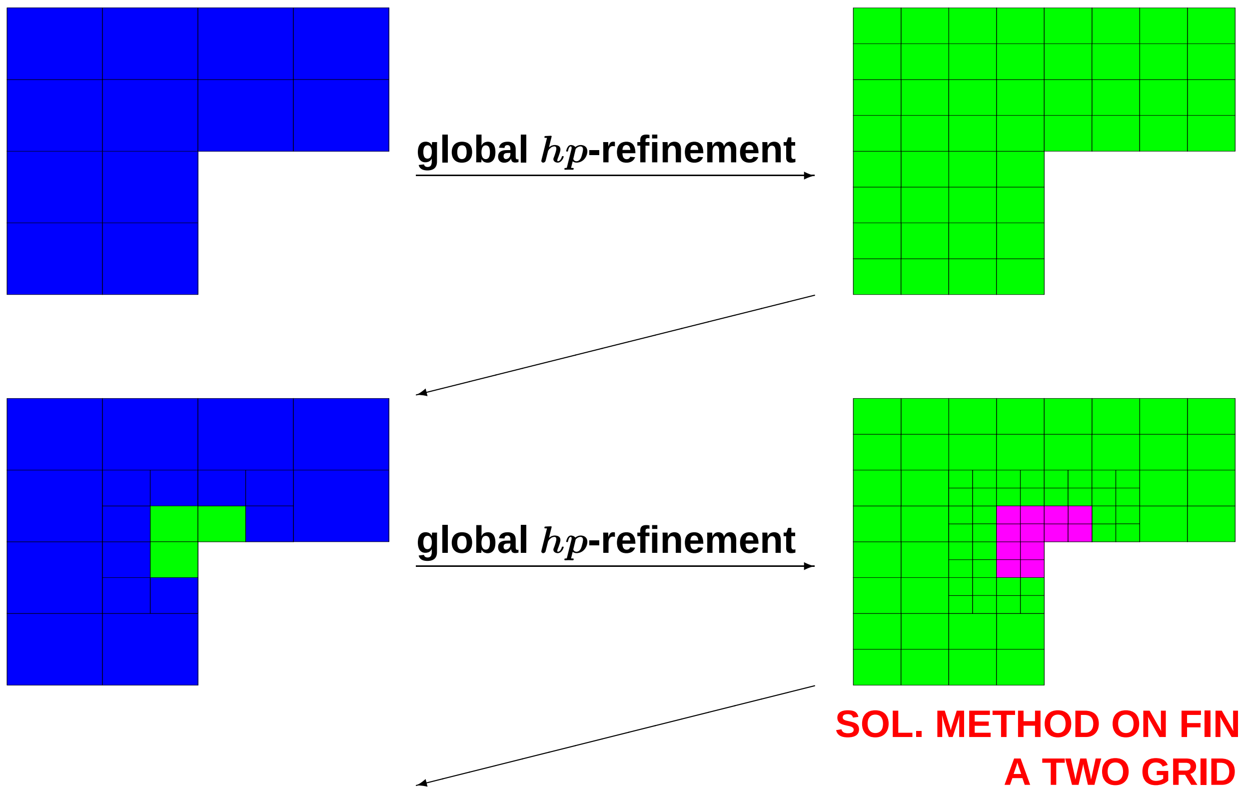
as *h* decreases.

FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Fully automatic *hp*-adaptive strategy

Coarse grids **(hp)**

Fine grids **($h/2, p+1$)**

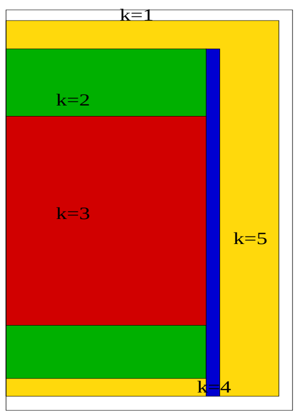


SOL. METHOD ON FINE GRIDS:
A TWO GRID SOLVER

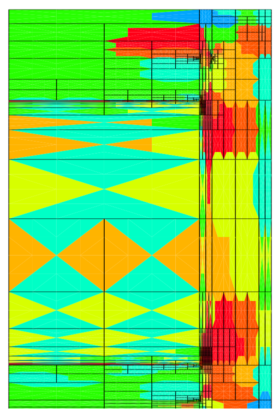
FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Convergence comparison: orthotropic heat conduction problem

Geometry



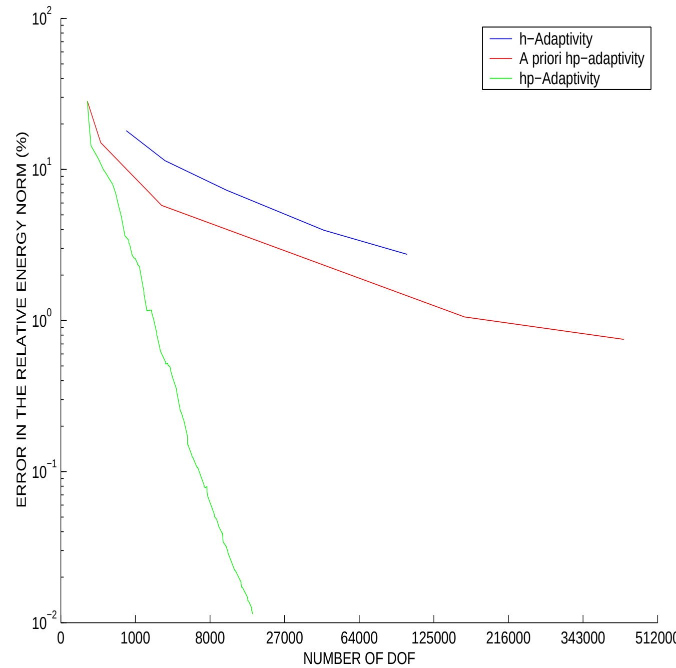
Final hp-grid



Equation: $\nabla(\mathrm{K}\nabla u) = f^{(k)}$

$$\mathrm{K} = \mathrm{K}^{(k)} = \begin{bmatrix} K_x^{(k)} & 0 \\ 0 & K_y^{(k)} \end{bmatrix}$$

$$K_x^{(k)} = (25,\ 7,\ 5,\ 0.2,\ 0.05)$$
$$K_y^{(k)} = (25,\ 0.8,\ 0.0001,\ 0.2,\ 0.05)$$

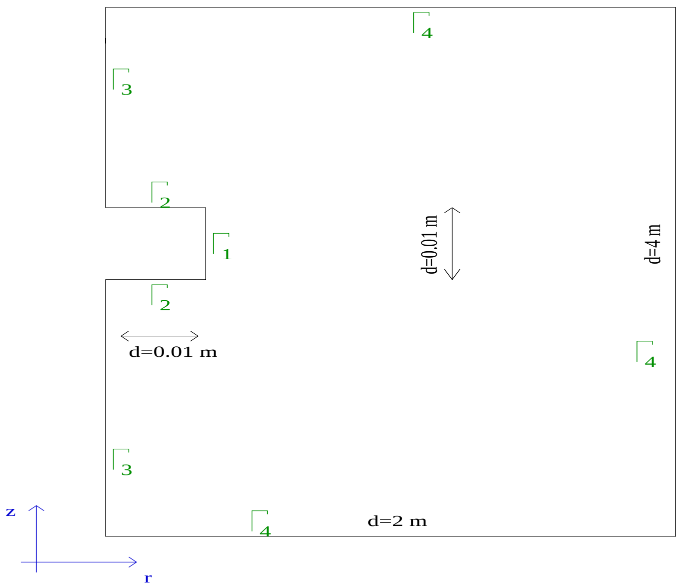


Automatic hp-adaptivity: 2K d.o.f.
***A priori* hp-adaptivity: 500K d.o.f.**
Automatic h-adaptivity: $>$5000K d.o.f.

FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Time Harmonic Maxwell's Equations

$$\nabla \times \mathrm{E} = -j\mu\omega \mathrm{H}$$
$$\nabla \times \mathrm{H} = j\omega\epsilon \mathrm{E} + \sigma \mathrm{E}$$



Reduced Wave Equation:

$$\nabla \times \left(\frac{1}{\mu}\nabla \times E\right) - (\omega^2\epsilon - j\omega\sigma)E = -j\omega J^{imp}$$

Boundary Conditions (BC):

Dirichlet BC at a PEC surface:

$$\mathrm{n} \times E = 0 \;\; on \;\; \Gamma_2 \cup \Gamma_4$$

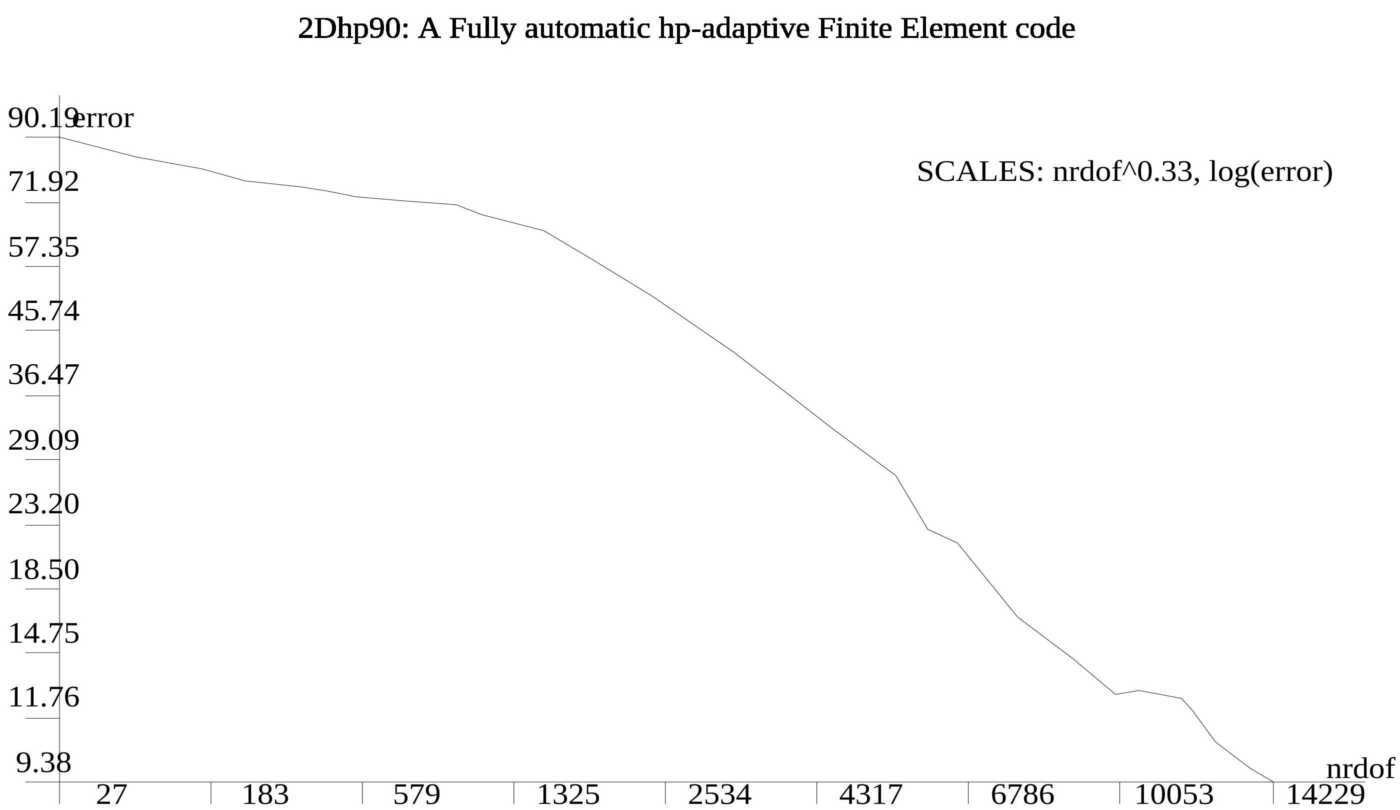
Neumann BC's:

$$\mathrm{n} \times \frac{1}{\mu}\nabla \times E = -j\omega \;\; on \;\; \Gamma_1$$

$$\mathrm{n} \times \frac{1}{\mu}\nabla \times E = 0 \;\; on \;\; \Gamma_3$$

FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Battery example: Convergence history



FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Why results were so bad if we had such a small error?

Optimization is based on minimization of the ENERGY NORM of the error, given by:

$$\| error \|^2 = \int | error |^2 + \int | \nabla \times error |^2$$

Interpretation of results:

- **The grid is optimal for the selected refinement criteria,**
- **but our refinement criteria is inadequate for our pourposes.**

AUTOMATIC GOAL-ORIENTED *HP*-ADAPTIVITY

What does it mean *Goal-Oriented* Adaptivity?

We consider the following problem (in variational form):

$$\begin{cases} \text{Find } \Psi \in V \text{ such that:} \\ b(\Psi, \xi) = f(\xi) \quad \forall \xi \in V \ . \end{cases}$$

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$MISLEADING$!!!!!

The problem we *really* want to solve is:

$$\begin{cases} \text{Find } L(\Psi)\text{, where } \Psi \in V \text{ such that:} \\ b(\Psi, \xi) = f(\xi) \quad \forall \xi \in V \ , \end{cases}$$

where $L(\Psi)$ is our goal (for example, $L(\Psi) = \Psi(b) - \Psi(a)$).

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HP goal-oriented adaptivity consists of constructing an optimal grid:

$$\arg \min_{hp:|L(e_{hp})| \leq TOL} N_{hp}$$

AUTOMATIC GOAL-ORIENTED *HP*-ADAPTIVITY

Mathematical Formulation (Goal-Oriented Adaptivity)

We consider the following problem (in variational form):

$$\begin{cases} \text{Find } L(\Psi)\text{, where } \Psi \in V \text{ such that:} \\ b(\Psi, \xi) = f(\xi) \quad \forall \xi \in V \ . \end{cases}$$

We define residual $r_{hp}(\xi) = b(e_{hp}, \xi)$. We seek for solution G of:

$$\begin{cases} \text{Find } G \in V \text{ such that:} \\ r(G) = L(e_{hp}) \ . \end{cases}$$

This is necessarily solved if we find the solution of the *dual* problem:

$$\begin{cases} \text{Find } G \in V \text{ such that:} \\ b(\Psi, G) = L(\Psi) \quad \forall \Psi \in V \ . \end{cases}$$

Notice that $L(e) = b(e, G)$.

AUTOMATIC GOAL-ORIENTED *HP*-ADAPTIVITY

Solution of Dual Problem

Dual problem:

$$\begin{cases} \text{Find } G \in V \text{ such that:} \\ b(\Psi, G) = L(\Psi) \quad \forall \Psi \in V \ . \end{cases}$$

where $L(\Psi) = \sigma(\Psi(b) - \Psi(a))$. But L is NOT a continuous functional !!!!.

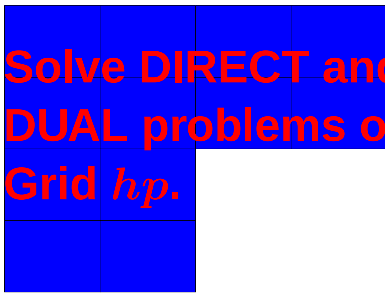
Thus, G CANNOT be computed by solving this dual problem using FEM.

We need a postprocessing formula to obtain a functional $\tilde{L}$ asymptotically equivalent to L.

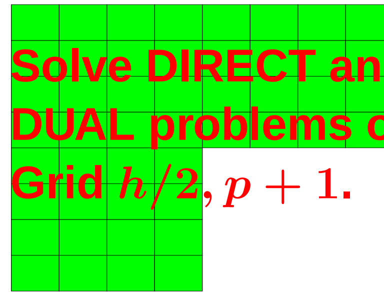
I. Babuska, A. Miller, *The Post-Processing Approach in the FEM*, 1984.

AUTOMATIC GOAL-ORIENTED *HP*-ADAPTIVITY

Algorithm for Goal-Oriented Adaptivity



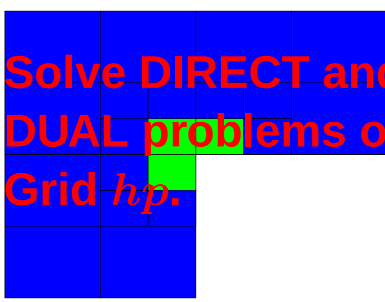
$\longrightarrow$



Compute $e = e_{h/2,p+1} - e_{hp}$**, and** $\epsilon = G_{h/2,p+1} - G_{hp}$**.**

Use estimate $|b(e,\epsilon)| \leq \sum_K |b_K(e,\epsilon)|$**.**

Apply ENERGY NORM BASED hp**-Adaptivity using** $\sum_K |b_K(e,\epsilon)|$ **instead of** $b(e,e)$**.**



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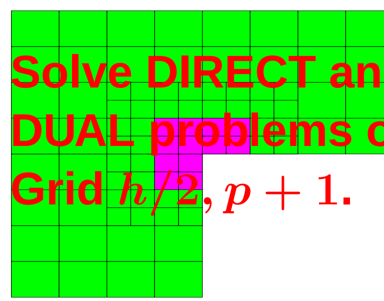
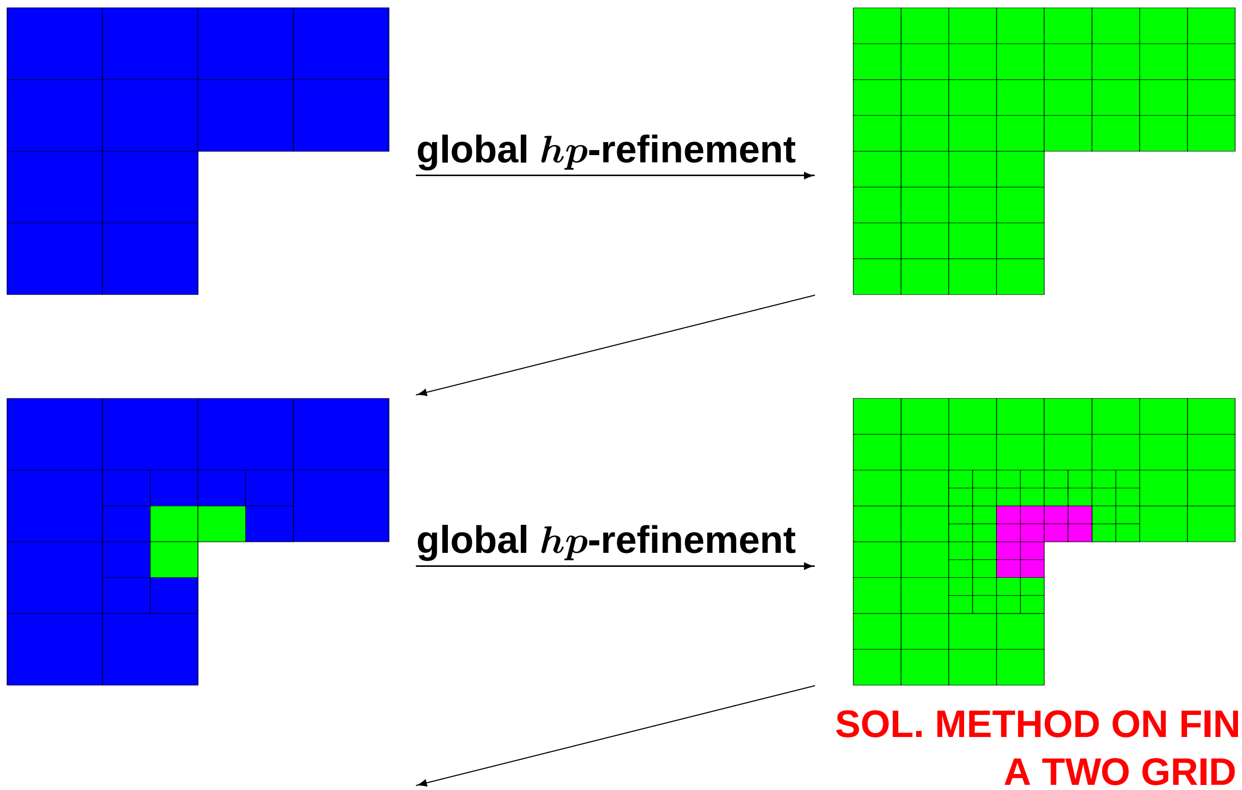


AUTOMATIC GOAL-ORIENTED *HP*-ADAPTIVITY

Fully automatic *hp*-adaptive strategy

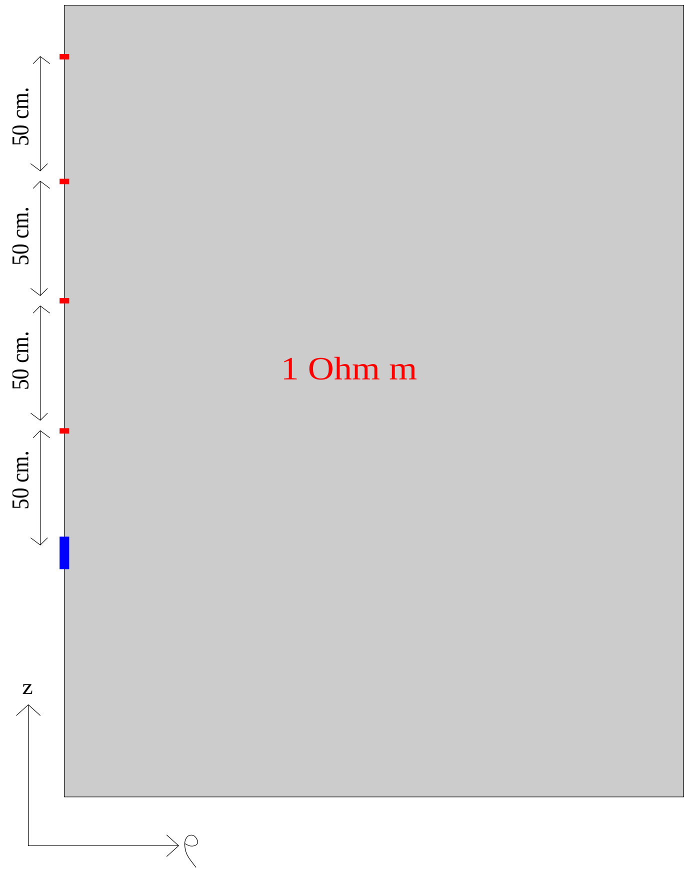
Coarse grids (hp) **Fine grids** ($h/2, p+1$)



SOL. METHOD ON FINE GRIDS:
A TWO GRID SOLVER

NUMERICAL RESULTS

A Simpler Problem with a Homogeneous Material



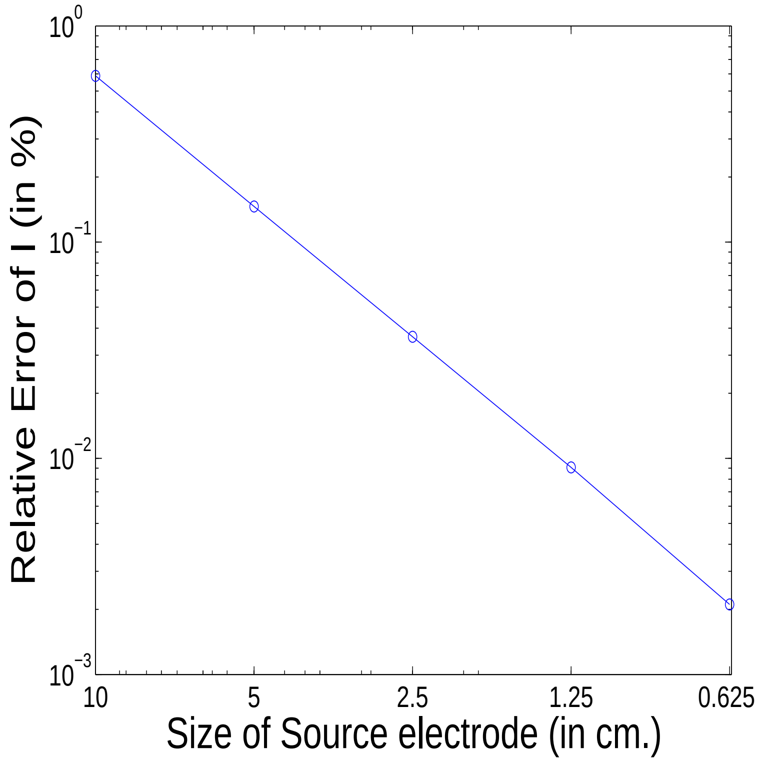
Point Source.

Exact Solution: $\Psi = \frac{1}{4\pi r}$.

A boundary integral term to model the point source.

NUMERICAL RESULTS

Study of Modeling Error due to Finite Size of the Source

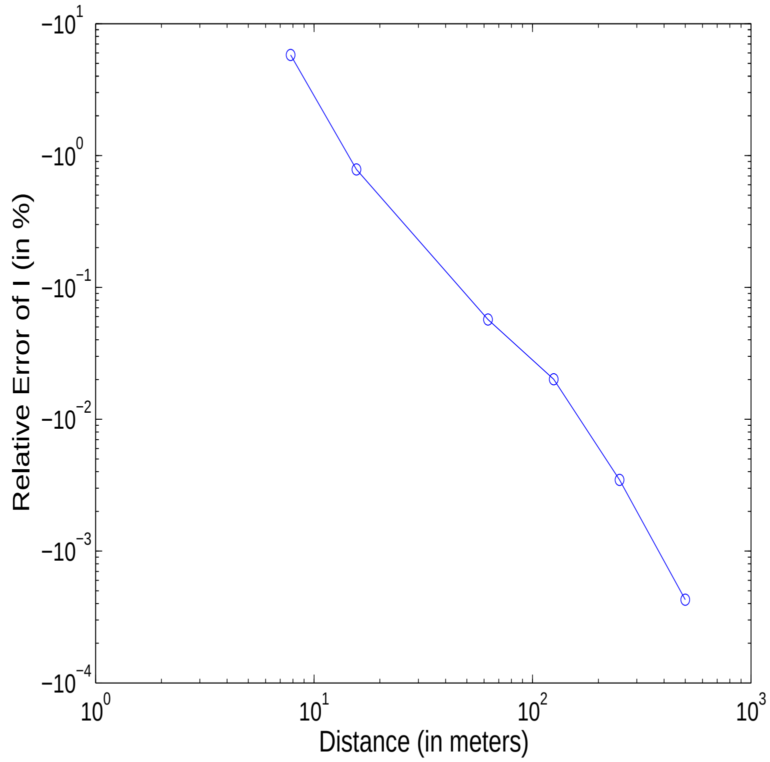


Simple Problem with a Homogeneous Material.

Distance Between Source and Receiving Electrodes: 50-100 cm.

NUMERICAL RESULTS

Study of Modeling Error due to Finite Size of Computational Domain

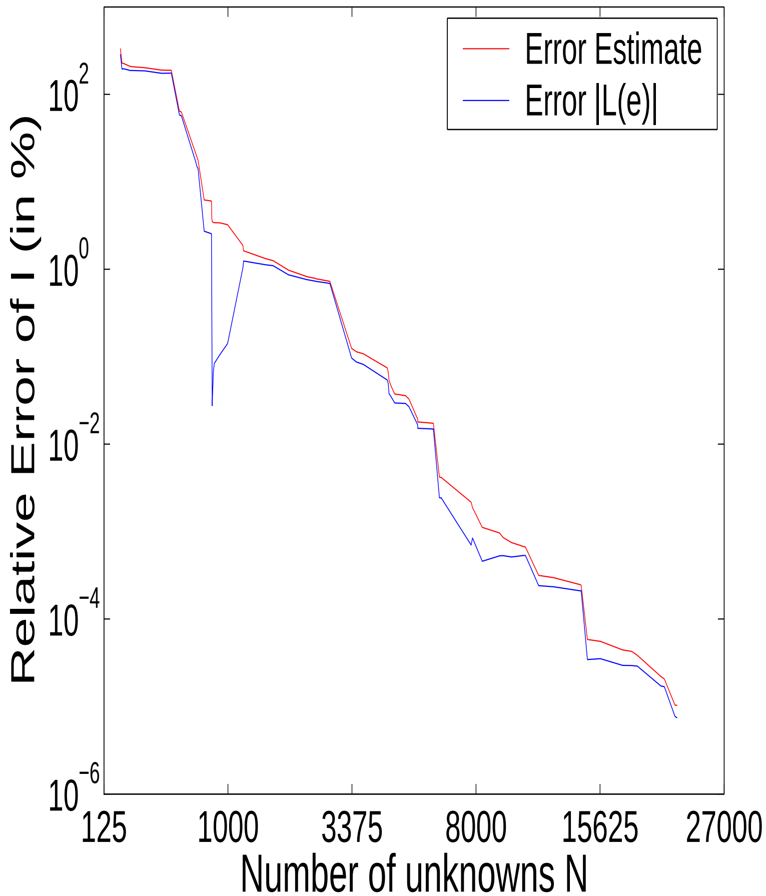


DC Resistivity Logging Problem with Different Materials.

Distance Between Source and Receiving Electrodes: 150-200 cm.

NUMERICAL RESULTS

Convergence History



DC Resistivity Logging Problem with Different Materials.

Distance Between Source and Receiving Electrode: $150cm$.

$|L(e)| \leq \sum_K |b(e,\epsilon)| =$ **Error Estimate.**

Relative Error (in $\%$**) vs dB**

10^{-6} **% =**10^{-7} **dB**

10^{-4} **% =**10^{-5} **dB**

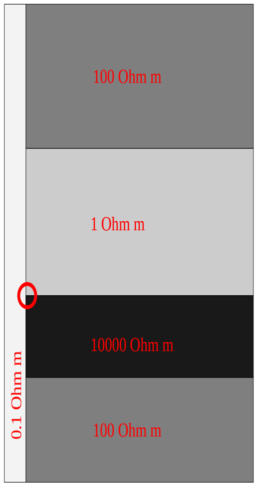
10^{-2} **% =**10^{-3} **dB**

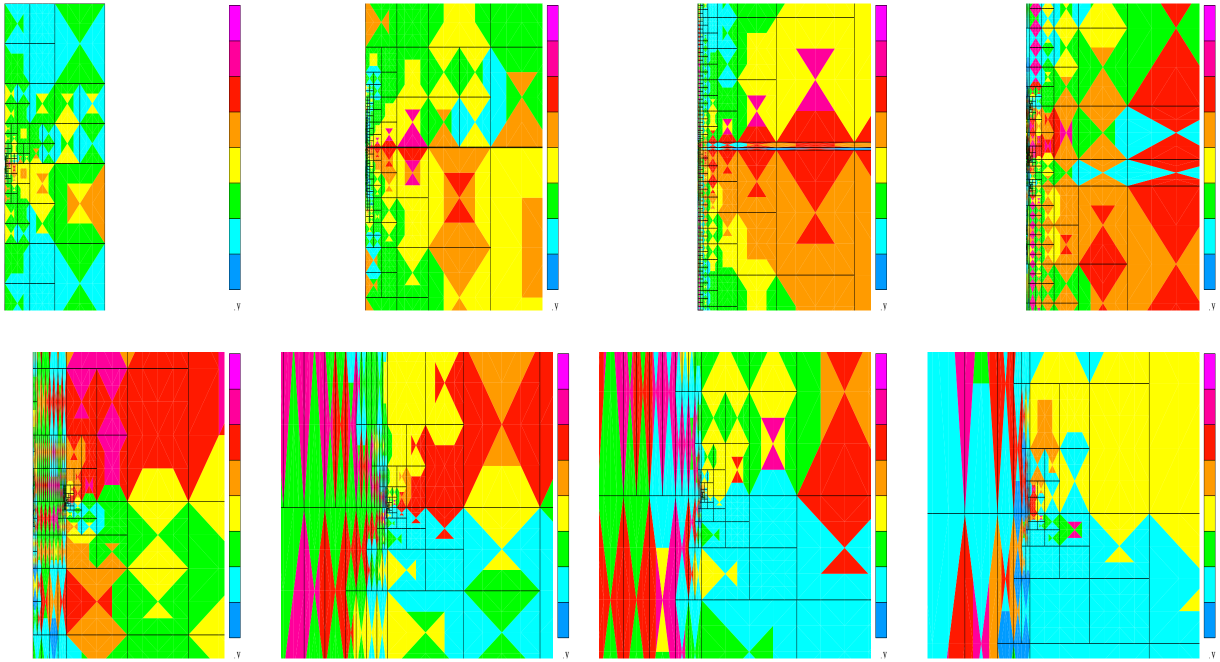
10^{0} **% =** 10^{-1} **dB**

10^{2} **% =** 10^{-1} **dB**

NUMERICAL RESULTS

Final hp-grid (Zooms by factor of 10)





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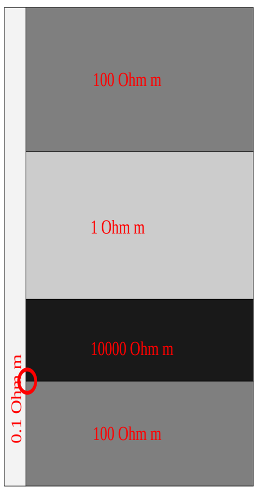

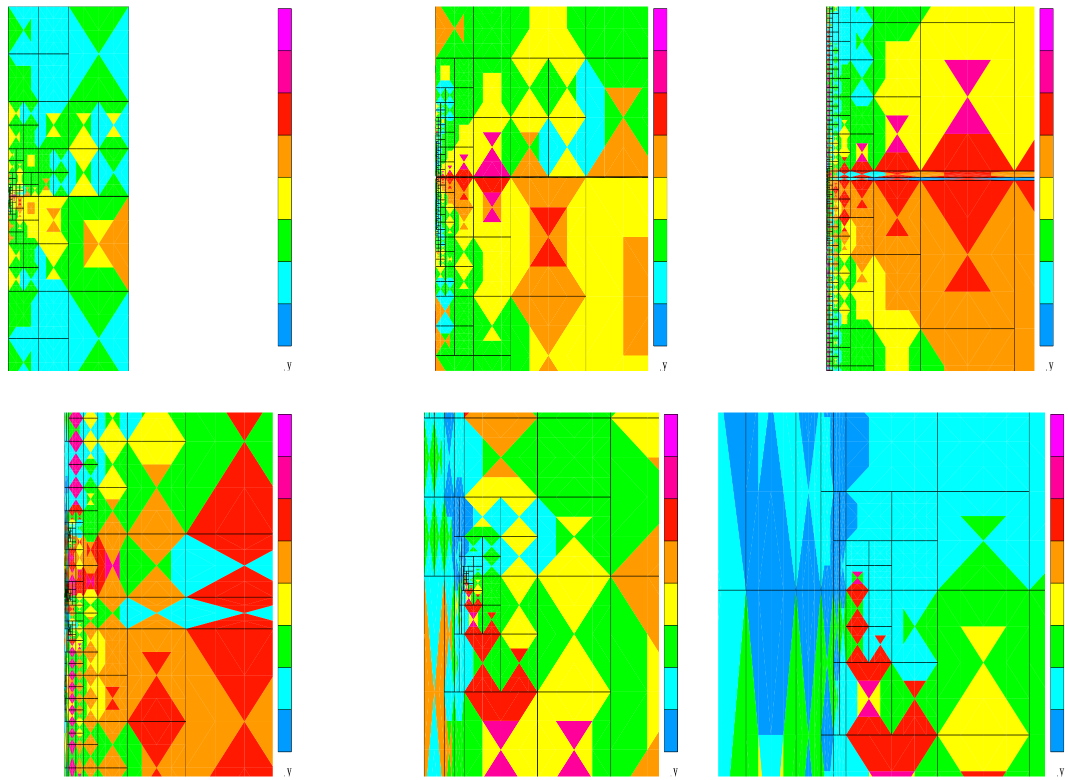

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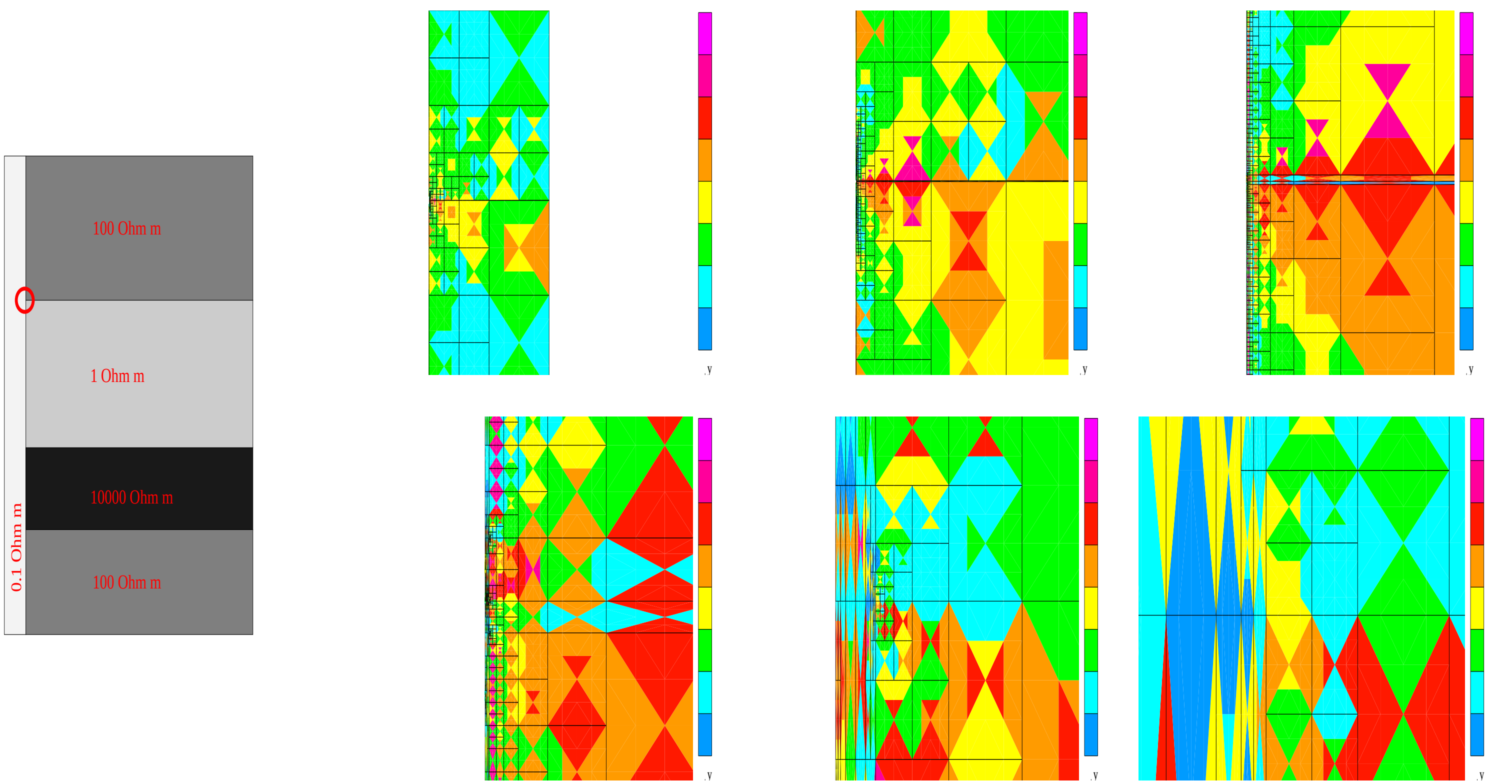
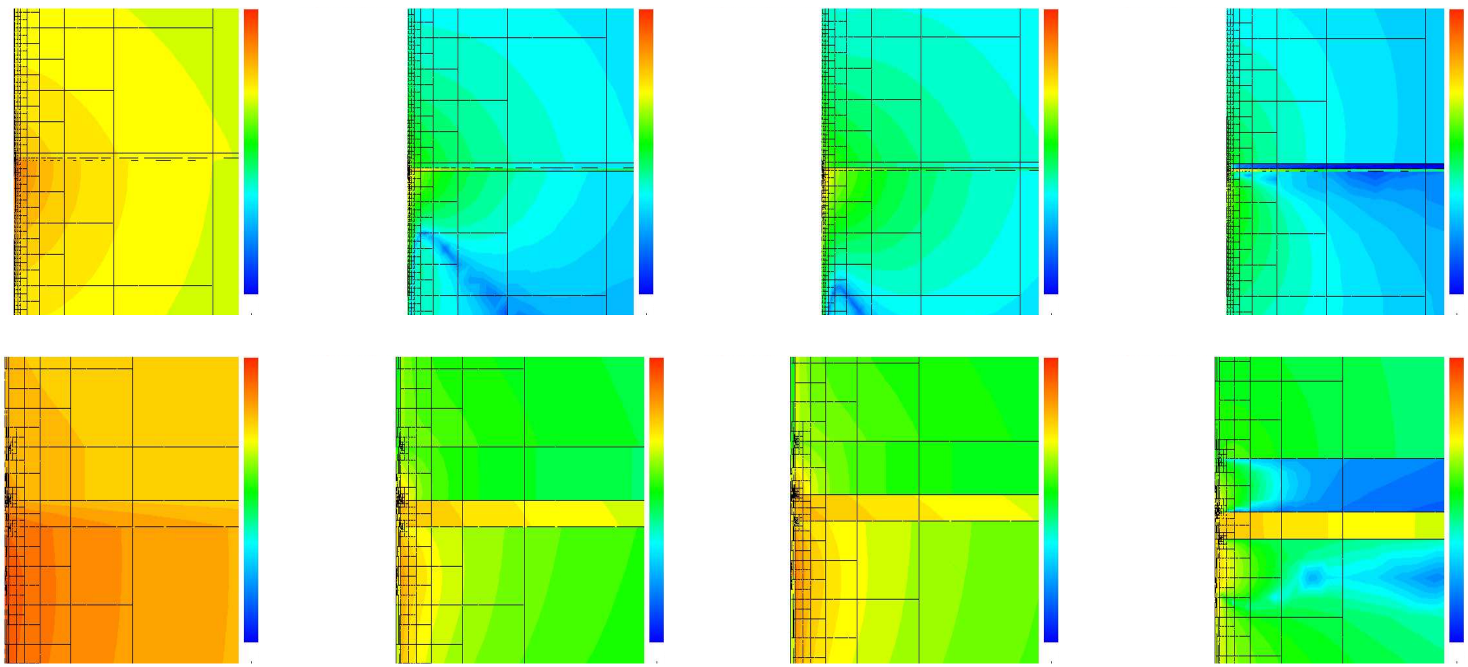


NUMERICAL RESULTS

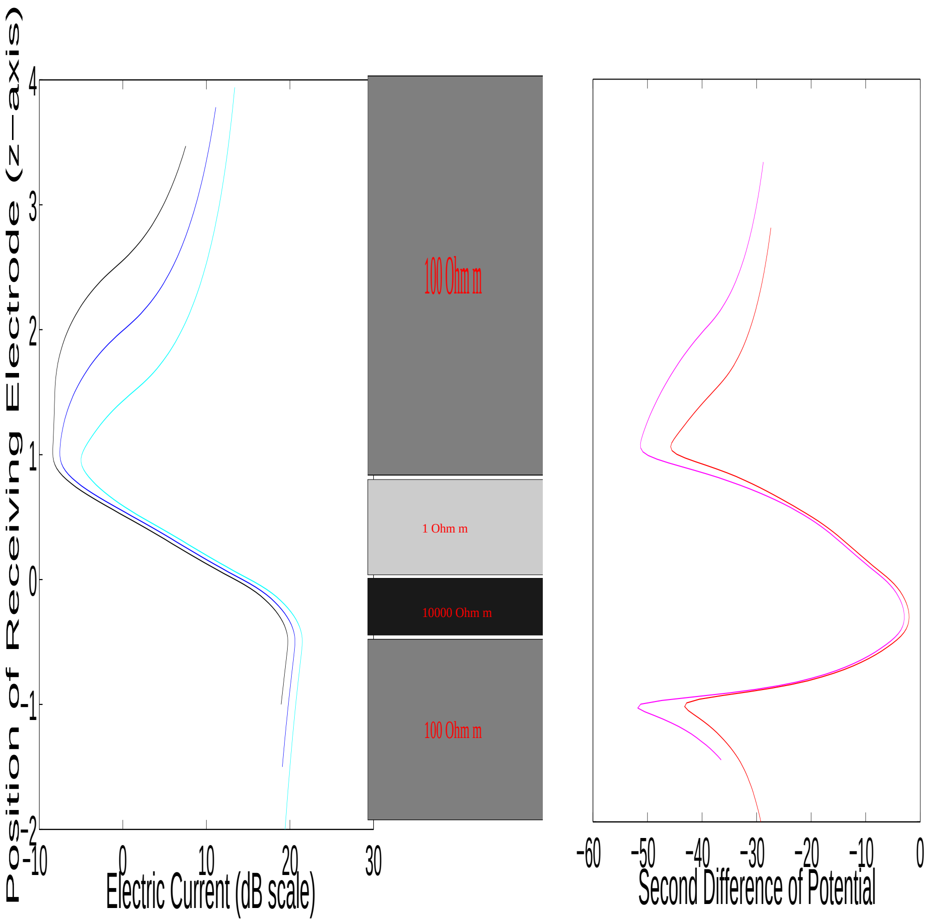
Solution (Zooms by factors of 10^3 and 10^4)

Ψ $\qquad$ $\nabla\Psi = -\mathbf{E}$ $\qquad$ $\dfrac{\partial\Psi}{\partial\rho}$ $\qquad$ $\dfrac{\partial\Psi}{\partial z}$



NUMERICAL RESULTS

The Main Result



DC Resistivity Logging Problem with Different Materials.

Electric Current (in the decibel scale) vs Position of the receiving electrode (with respect z).

Distance between source and first receiving electrode:

$0.5m$ **-light blue-**
$0.5m$ **-red-**
$1.0m$ **-dark blue-**
$1.0m$ **-magenta-**
$1.5m$ **-black-**

CONCLUSIONS AND FUTURE WORK

Conclusions

- **It is possible to construct a self-adaptive Goal-Oriented algorithm based on a self-adaptive energy norm based algorithm, for h-, p-, and hp-Finite Elements.**

- **The Fully Automatic hp-Adaptive Algorithm produces a sequence of grids that converges exponentially in terms of the quantity of interest vs the CPU time.**

- **We obtained high-accuracy approximations of DC Resistivity Logging Problems by using only a small number of unknowns.**

CONCLUSIONS AND FUTURE WORK

Future Work

- To extend the self-adaptive Goal-Oriented algorithm to AC resistivity logging problems.
- To extend the self-adaptive Goal-Oriented algorithm to 3D problems.
- To improve performance of the self-adaptive Goal-Oriented algorithm.
- To solve inverse problems by using hp Goal-Oriented adaptivity.