One-well metastability for an inelastic linear Boltzmann operator

Thomas Normand
joint work with Frédéric Hérau and Dorian Le Peutrec

Kinetic equation, Mathematical Physics and Probability

June 18th, 2024

Main equation

We are interested in the long-time behavior of the solutions of the linear Boltzmann equation

$$\begin{cases} h\partial_t f + v \cdot h\partial_x f - \partial_x \phi \cdot h\partial_v f + Q_h(f) = 0 \\ f_{|t=0} = f_0 \in L^2(\mathbb{R}^{2d}) \end{cases} \tag{1}$$

$\rightsquigarrow$ **semiclassical** study, i.e in the limit $h \to 0$ ("*low temperature*" regime) of the spectrum of the operator

$$P_h = v \cdot h\partial_x - \partial_x \phi \cdot h\partial_v + Q_h$$

associated to equation (1).

Notations and assumptions

- $\phi \in \mathcal{C}^\infty(\mathbb{R}^d_x, \mathbb{R})$ is a **Morse** coercive function, at most quadratic at infinity with only 1 local minimum (at $x = 0$).

  We denote $$\Phi_0 := \mathrm{Hess}_0\,\phi = \begin{pmatrix} p_1^2 & & \\ & \ddots & \\ & & p_d^2 \end{pmatrix}$$

- Π_1 is the **orthogonal projector** on $\mathrm{e}^{-\frac{v^2}{2h}} L^2(\mathbb{R}^d_x)$,
- Π_{v_k} is the **orthogonal projector** on $v_k \mathrm{e}^{-\frac{v^2}{2h}} L^2(\mathbb{R}^d_x)$,
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We consider the **"inelastic linear"** collision operator

$$Q_h = h\Big(\mathrm{Id} - \Pi_1 - \Pi_v\Big)$$

⇝ Local conservation of both mass and momentum

▷ Carrapatoso Dolbeault Hérau Mischler Mouhot Schmeiser 22 ($h = 1$)

First question :

What is $\mathrm{Spec}(P_h) \cap i\mathbb{R}$?

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▷ First, $\mathcal{M}_h(x,v) := \exp\left(-\frac{\phi(x) + v^2/2}{h}\right) \in \operatorname{Ker} P_h$.

▷ Since $\operatorname{Re}\langle P_h u, u\rangle = h\|(1 - \Pi_1 - \Pi_v)u\|^2$,

$$(\lambda, u) \text{ eigenpair with } \lambda \in i\mathbb{R} \iff \begin{cases} (1 - \Pi_1 - \Pi_v)u = 0 \\ v \cdot h\partial_x u - \partial_x \phi \cdot h\partial_v u = \lambda u \end{cases}$$

⇝ We look for solutions of the form

$$u(x,v) = \Big(r(x) + m(x) \cdot h^{-1/2} v\Big)\mathcal{M}_h$$

which gives

$$hv \cdot \partial_x r + \sqrt{h} D_x m v \cdot v - \sqrt{h}\partial_x \phi \cdot m = \lambda r + \lambda m \cdot h^{-1/2} v$$

$$\begin{cases} D_x^{\mathrm{sym}} m = 0 \\ \lambda m = h^{3/2} \partial_x r \\ \lambda r = -\sqrt{h} \partial_x \phi \cdot m \end{cases}$$

By the Schwarz Lemma,

$$(\mathrm{Hess}\, m_k)_{i,j} = \partial_{x_i} (D_x^{\mathrm{sym}} m)_{j,k} + \partial_{x_j} (D_x^{\mathrm{sym}} m)_{i,k} - \partial_{x_i k} (D_x^{\mathrm{sym}} m)_{i,j} = 0$$

and thus $\underline{\exists\, A \in \mathcal{M}_d^{\mathrm{skew}}(\mathbb{C})}$ s.t

$$m(x) = A h^{-1/2} x + b$$

$$\rightsquigarrow \boxed{\lambda A = 0}$$

2 cases :

▷ $\lambda = 0$: $\rightsquigarrow r \in \mathbb{C}$ and $\partial_x\phi \cdot Ax = 0$

$$\implies u \in \mathbb{C}\mathcal{M}_h \oplus \boxed{\mathcal{R}_\phi\, x \cdot v\mathcal{M}_h}$$ "rotational modes"

with $$\mathcal{R}_\phi = \{A \in \mathcal{M}_d^{\mathrm{skew}}(\mathbb{C})\,;\, \partial_x\phi \cdot Ax = 0\}$$
$$\subseteq \mathcal{M}_d^{\mathrm{skew}}(\mathbb{C}) \cap \Phi_0^{-1}\mathcal{M}_d^{\mathrm{skew}}(\mathbb{C}) =: \check{\mathcal{R}}_\phi$$

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▷ $\lambda \neq 0$: $\rightsquigarrow m = b \in \mathbb{C}^d$, $r(x) = h^{-3/2}\lambda b \cdot x$

$$\rightsquigarrow \left(\partial_x \phi + h^{-2}\lambda^2 x\right) \cdot b = 0 \qquad \Longrightarrow \qquad \boxed{\lambda \in \{\pm i p_k h\}}$$

In that case, denoting $I_\phi = \left\{k \in [\![1, d]\!]\,;\, \partial_{x_k}\phi(x) - p_k^2 x_k = 0\right\}$

$$u \in \boxed{\bigoplus_{k \in I_\phi} \mathbb{C}\,(\pm i p_k x_k + v_k)\,\mathcal{M}_h}$$ "harmonic directions"

To sum up :

Proposition [CDHMMS 2022, HLPN 2024]

We have

$$\operatorname{Spec} P_h \cap i\mathbb{R} = \{-ip_k h\,;\ k \in I_\phi\} \cup \{0\} \cup \{ip_k h\,;\ k \in I_\phi\}$$

and the associated eigenspaces are orthogonal to one another and given by

$$\operatorname{Ker} P_h = \mathbb{C}\mathcal{M}_h \overset{\perp}{\oplus} \mathcal{R}_\phi x \cdot v\mathcal{M}_h$$

and for $k \in I_\phi$

$$\operatorname{Ker}(P_h \mp ip_k h) = \mathbb{C}\,(\pm ip_k x_k + v_k)\,\mathcal{M}_h$$

Hypocoercivity

▷ Hérau, Villani (∼2006); Dolbeault-Mouhot-Schmeiser (2010); or recently Carrapatoso, Mischler, Robbe, Bernou, Tristani; Stoltz...

▷ [CDHMMS] : First result with multiple conservation laws

Hypocoercivity

Theorem

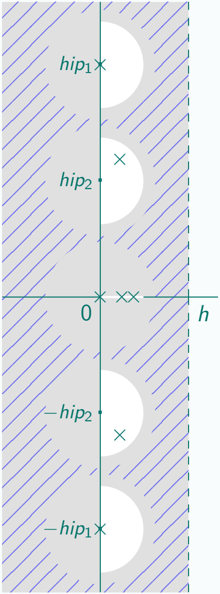
The spectrum of P_h in $\{0 < \operatorname{Re} z \leq h\}$ consists of exactly :

- $\left(\dim \check{\mathcal{R}}_\phi - \dim \mathcal{R}_\phi\right)$ *eigenvalues in* $]0, h^{3/2}]$
- $\left|\{j \notin I_\phi \,;\, p_j = p_k\}\right|$ *eigenvalues in* $B(ip_k h, h^{3/2}) \cap \{\operatorname{Re} z > 0\}$ *for each* $k \notin I_\phi$
- $\left|\{j \notin I_\phi \,;\, p_j = p_k\}\right|$ *eigenvalues in* $B(-ip_k h, h^{3/2}) \cap \{\operatorname{Re} z > 0\}$ *for each* $k \notin I_\phi$

Corresponding resolvent estimates of order $O(h^{-1})$ hold true.

$\rightsquigarrow$ Total of :

$1 + \dim \check{\mathcal{R}}_\phi + 2d$ eigenvalues in $\{0 \leq \operatorname{Re} z \leq h\}$



Ideas of proof

▷ 1st step : Introduce a family of *quasimodes* for P_h

$$\mathrm{Quasim} := \underbrace{\mathbb{C}\mathcal{M}_h \oplus \check{\mathcal{R}}_\phi x \cdot v\mathcal{M}_h}_{P_h = O(h^{3/2})} \oplus \underbrace{\left(\bigoplus_{k=1}^{d} \mathbb{C}\left(\pm i p_k x_k + v_k\right)\mathcal{M}_h\right)}_{P_h \mp i p_k h = O(h^{3/2})}$$

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▷ 2nd step : Find $\widetilde{P}_h$ such that $\mathrm{Re}\,\langle \widetilde{P}_h f, f\rangle \geq h\|f\|^2 \quad \forall f \in \mathrm{Quasim}^\perp$

⇝ We follow and adapt [CDHMMS] :

Let $f \in \mathrm{Quasim}^\perp$ decomposed as

$$f = \Big(r(x) + m(x) \cdot h^{-1/2} v\Big)\mathcal{M}_h + f^\perp \qquad \text{with} \qquad f^\perp \in (\mathrm{Ker}\, Q_h)^\perp$$

We already have $\boxed{\mathrm{Re}\,\langle P_h f, f\rangle = h\|f^\perp\|^2}$

Lemma (gains in m and r)

There exist L_1 and L_2 two self adjoint and $O(1)$ operators such that for all $f \in \mathrm{Quasim}^{\perp}$,

$$\boxed{\mathrm{Re}\,\langle L_1 P_h f, f\rangle \geq h\|m\mathcal{M}_h\|^2 - O(h\|f\|\|f^{\perp}\|)}$$

$$\boxed{\mathrm{Re}\,\langle L_2 P_h f, f\rangle \geq h\|r\mathcal{M}_h\|^2 - O(h\|f\|\|m\mathcal{M}_h\|)}$$

$\rightsquigarrow$ Taking $\widetilde{P}_h = \mathrm{Id} + \varepsilon_1 L_1 + \varepsilon_2 L_2$ gives indeed $\mathrm{Re}\,\langle \widetilde{P}_h f, f\rangle \geq h\|f\|^2$

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Sketch of proof of the Lemma :

- Find L_1 such that $\mathrm{Re}\,\langle L_1 P_h f, f\rangle \geq h\|D_x^{\mathrm{sym}} m\mathcal{M}_h\|^2$
  $+$ use a Korn-Poincaré inequality from [CDHMM]
- Find L_2 such that $\mathrm{Re}\,\langle L_2 P_h f, f\rangle \geq h\|\partial_x r\mathcal{M}_h\|^2$
  $+$ use a Poincaré inequality from [CDHMM] $\square$

▷ Conclusion :

- Deduce the resolvent estimate $(P_h - z)^{-1} = O(h^{-1})$
- Introduce the spectral projectors

$$\mathbb{P}_0 = \frac{1}{2i\pi} \int_{|z|=h} (z - P_h)^{-1} \mathrm{d}z$$

and

$$\mathbb{P}_{\pm i p_k} = \frac{1}{2i\pi} \int_{|z - i p_k h|=h} (z - P_h)^{-1} \mathrm{d}z$$

and compute their ranks.

Metastability

▷ Reversible processes : Bovier-Eckhoff-Gayrard-Klein 04; Helffer-Klein-Nier 04; DiGesu-Lelievre-Le Peutrec-Nectoux 10's, Michel 19

▷ Non reversible processes : Hérau-Nier 04; Hérau-Hitrik-Sjöstrand 10-15; Bouchet-Reygner 16; Landim Seo 18-22; Guillin Nectoux 20; Bony-Le Peutrec-Michel 22; Delande 24

▷ Boltzmann (1 conservation law) : Robbe 16; N. 23

⇝ First result with multiple conservation laws/one well

Metastability

Assumptions :

$p_k \neq p_j$ for $k \neq j$ (in particular $\check{\mathcal{R}}_\phi = \{0\}$) and $I_\phi \neq [\![1, d]\!]$

Consider the decomposition $\mathrm{Id} = \mathbb{P}_{\mathrm{Im}} + \mathbb{P}_{\mathrm{Re}<} + \mathbb{P}_{\mathrm{Re}>}$ where

$$\mathbb{P}_{\mathrm{Im}} := \mathbb{P}_0 + \sum_{k \in I_\phi} (\mathbb{P}_{ip_k} + \mathbb{P}_{-ip_k}) \qquad \mathbb{P}_{\mathrm{Re}<} := \sum_{k \notin I_\phi} (\mathbb{P}_{ip_k} + \mathbb{P}_{-ip_k})$$

Corollary

- $\mathrm{e}^{-tP_h}\mathbb{P}_{\mathrm{Im}} = \mathbb{P}_0 + \sum_{k \in I_\phi} \left(\mathrm{e}^{-ip_k ht}\mathbb{P}_{ip_k} + \mathrm{e}^{ip_k ht}\mathbb{P}_{-ip_k} \right)$ $\rightsquigarrow$ *no decay*
- $\mathrm{e}^{-tP_h}\mathbb{P}_{\mathrm{Re}>} = O(\mathrm{e}^{-ht})$ *(Gearhart-Prüss)* $\rightsquigarrow$ *quick decay*
- $\mathrm{e}^{-tP_h}\mathbb{P}_{\mathrm{Re}<} = \sum_{k \notin I_\phi} \left(\mathrm{e}^{-\lambda_{ip_k} t}\mathbb{P}_{ip_k} + \mathrm{e}^{\lambda_{ip_k} t}\mathbb{P}_{-ip_k} \right)$ $\rightsquigarrow$ *metastable part*

Perspectives

- Study the multi-well case
- Consider some potentials of the form $\phi(x) = \frac{x^2}{2}$ on $B(0, r)$
- Study on domains bounded in space [Nier, Lelièvre et al., Bernou et al.] : $\mathbb{R}^{2d} \rightsquigarrow \Omega \times \mathbb{R}^d_v$.

Thank you !