LOWER BOUNDS FOR SCHRÖDINGER EVOLUTIONS

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HADE erc

(1) Uncertainty principles (with **L. Escauriaza**, **C. Kenig** and **G. Ponce**)

(2) A space time lower bound (**EKPV**)

(3) New results (with **M. Agirre**)

- A lower bound for a fixed time
- Uniqueness

(4) About the proof.

Hardy's uncertainty principle (Cowling, Price)

Define for $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx$$

Assume that

$$\begin{aligned} f(x) e^{\frac{|x|^2}{\alpha^2}} &\in L^2 \\ \widehat{f}(\xi) e^{\frac{|\xi|^2}{\beta^2}} &\in L^2 \\ \alpha^2 \beta^2 &\le 4. \end{aligned}$$

Then

$$f \equiv 0$$

Morgan and Paley-Wiener

If
$$\int |f|^2 e^{-|x|^p} dx < +\infty$$

$$\int |\widehat{f}|^2 e^{-|\xi|^{p'}} d\xi < +\infty$$

$$1 < p < \infty \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

then $u \equiv 0$.

Paley–Wiener

If $\displaystyle\int |f(x)| e^{-\epsilon|x|} dx < +\infty, \;\; \epsilon > 0,$

then $\widehat{f}$ can not have compact support.

Free Schrödinger Equation

S.E. $$\begin{cases} \partial_t u = i\Delta u & x \in \mathbb{R}^n \quad t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

$$\begin{aligned} u(x,t) &= \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy \\ &= \frac{1}{(it)^{n/2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} e^{-i\frac{x}{2t}\cdot y} e^{i\frac{|y|^2}{4t}} u_0(y)dy \end{aligned}$$

$$\begin{aligned} f(y) &= e^{i\frac{|y|^2}{4t}} u_0(y), \\ \widehat{f}(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} f(y)dy. \end{aligned}$$

Hence

$$u(0)e^{\frac{|y|^2}{\alpha^2}} \in L^2 \quad \Longleftrightarrow \quad f e^{\frac{|y|^2}{\alpha^2}} \in L^2$$

$$u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2 \quad \Longleftrightarrow \quad \widehat{f}\left(\frac{x}{2T}\right) e^{\frac{|x|^2}{\beta^2}} \in L^2.$$

Hardy's uncertainty principle:

$$\alpha\beta \le 4T \quad \text{then} \;\; u \equiv 0$$

Time/Energy uncertainty principle (Wave Packets)

$$u(x,t) = \frac{1}{(t+i)^{n/2}} e^{-\frac{|x|^2}{4(t+i)}}$$

$$|u|^2 = \frac{1}{(t^2+1)^{n/2}} e^{-\frac{|x|^2}{2(t^2+1)}}$$

- $u_R(x,t) = u\big(Rx, R^2 t\big)$ is also a solution.
- $u^{t_0}(x,t) = u\big(x, t - t_0\big)$ is also a solution.

Schrödinger Equation with a Potential

$$u = u(x,t)$$

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n$$

$$\left.\begin{array}{l} \bullet \quad e^{\frac{|x|^2}{\alpha^2}} u(0) \in L^2 \\ \bullet \quad e^{\frac{|x|^2}{\beta^2}} u(T) \in L^2 \\ \bullet \quad \alpha\beta \ \text{ small enough} \end{array}\right\} \Longrightarrow \quad u \equiv 0$$

With C. E. Kenig, L. Escauriaza and G. Ponce

Theorem 2.– $u \in \mathcal{C}\left([0,T] : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$u(0)e^{\frac{|x|^2}{\alpha^2}} \in L^2 \qquad ; \qquad u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2,$$

and $\alpha\beta < 4T$, then $u \equiv 0$.

Hypothesis on the potential:

H1 $V = V_0(x) +$ perturbation with a gaussian decay
V_0 real and bounded

H1* $V = V(x,t)$

$$\lim_{R\to\infty} \int_0^T \sup_{|x|\geq R} |V| dt = 0$$

Theorem 3.–There exists a non-trivial $u \in \mathcal{C}\left(\mathbb{R} : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in \mathbb{R},$$

with $V = V(x,t) \in \mathbb{C}$ and $|V(x,t)| \leq \frac{C}{1+|x|^2}$, such that

$$u(0)e^{\frac{|x|^2}{\alpha^2}} \in L^2 \qquad ; \qquad u(1)e^{\frac{|x|^2}{\beta^2}} \in L^2,$$

and $\alpha^2\beta^2 = 4$.

About the proofs in the constant coefficient case (key words):

- Uncertainty principle:
  - Positive Commutators.
- Hardy's theorem:
  - Analyticity, Cauchy–Riemann equations, Log convexity.
  - Liouville's theorem.

Log-Convexity (a dynamic uncertainty principle)

S symmetric; $\mathcal{A}$ antisymmetric:

$$\partial_t v = (S + \mathcal{A})v$$

$$H(t) = \langle v, v \rangle$$

$$\begin{aligned}\dot{H}(t) &= \langle v_t, v \rangle + \langle v, v_t \rangle \\ &= \langle (S + \mathcal{A})v, v \rangle + \langle v, (S + \mathcal{A})v \rangle \\ &= 2\langle Sv, v \rangle.\end{aligned}$$

$$\begin{aligned}\ddot{H}(t) &= 2\langle Sv_t, v \rangle + 2\langle Sv, v_t \rangle \\ &= 2\langle (S + \mathcal{A})v, Sv \rangle + 2\langle Sv, (\mathcal{A} + S)v \rangle \\ &= 4\langle Sv, Sv \rangle + 2\langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle.\end{aligned}$$

$$
\begin{aligned}
(\lg H(t))^{\cdot\cdot} &= \left(\frac{\dot H}{H}\right)^{\cdot} = \frac{\ddot H H - \dot H^2}{H^2} \\
&= \frac{1}{\langle v,v\rangle}\left\{4\langle Sv,Sv\rangle\langle v,v\rangle - 4\langle Sv,v\rangle^2 + 2\big\langle (S\mathcal{A}-\mathcal{A}S)v,v\big\rangle\right\} \\
&\geq 2\big\langle (S\mathcal{A}-\mathcal{A}S)v,v\big\rangle
\end{aligned}
$$

Hence if $2\big\langle (S\mathcal{A}-\mathcal{A}S)v,v\big\rangle \geq 0$ then

$$H(t) \leq H(0)^{1-t}H(1)^t$$

More generally if

$$2\big\langle (S\mathcal{A}-\mathcal{A}S)v,v\big\rangle \geq \psi(t)\langle v,v\rangle,$$

then

$$H(t)e^{-B(t)} \leq H(0)^{1-t}H(1)^t,$$

with

$$\ddot B = \Psi(t), \quad B(0) = B(1) = 0.$$

A Particular Example

- $\partial_t u = i\Delta u$
- $e^{\frac{|x|^2}{2}} u = v$
- $H(t) = \|v(t)\|_{L^2}^2 = \langle v(t), v(t)\rangle$
- $$\begin{aligned} \partial_t v &= \left(e^{\frac{|x|^2}{2}} i\Delta e^{-\frac{|x|^2}{2}}\right) v \\ &= i\sum_j e^{\frac{|x|^2}{2}} \partial_j^2 e^{-\frac{|x|^2}{2}} v \\ &= i\sum_j e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} v \\ &= i\sum_j (x_j - \partial_j)(x_j - \partial_j) v \\ &= i\left(|x|^2 - 2x\cdot\nabla - 1 + \Delta\right) v \qquad \textbf{(C–R)} \end{aligned}$$

$$\partial_t v = (S+\mathcal{A})v \quad ; \quad Sv = -i(2x\cdot\nabla + 1)v \quad ; \quad \mathcal{A}v = i(\Delta + |x|^2)$$

$$S\mathcal{A} - \mathcal{A}S = -4\Delta + 4|x|^2$$

$$\big\langle S\mathcal{A} - \mathcal{A}Sv, v\big\rangle = 4(\|\nabla v\|^2 + \|xv\|^2) \geq 4\langle v, v\rangle.$$

Hence

$$(\lg H(t))^{\cdot\cdot} \geq 8$$

and

$$H(t) \leq H(0)^{1-t} H(1)^t$$

- Therefore u has a gaussian decay for $0 < t < 1$!!!

Remark.– If u_0 has gaussian decay then $e^{it\Delta}u_0$ does not necessarily have it for $t > 0$ $\left(u_0 = (\operatorname{sig} x)e^{-|x|^2}\right)$.

A space time lower bound

Theorem (with L. Escauriaza, C. Kenig and G. Ponce).– Let $u \in \mathcal{C}\left([0,1] : H^1(\mathbb{R}^n)\right)$ be a strong solution of

$$i\partial_t u + \Delta u + Vu = 0, \quad t \in [0,1], \quad x \in \mathbb{R}^n$$

If

$$\int_0^1 \int_{\mathbb{R}^n} \left(|u|^2 + |\nabla_x u|^2\right)(x,t)dxdt \leq A^2,$$

$$\int_{1/2-1/8}^{1/2+1/8} \int_{|x|<1} |u|^2(x,t)dxdt \geq 1,$$

and

$$\|V\|_{L^\infty(\mathbb{R}^n \times [0,1])} \leq L,$$

then there exists $R_0 = R_0(n,A,L) > 0$ and a constant $c = c(n)$ such that for $R \geq R_0$ it follows that

$$\delta(R) \equiv \left(\int_0^1 \int_{R-1<|x|<R} \left(|u|^2 + |\nabla_x u|^2\right)(x,t)dxdt\right)^{1/2} (x,t)dxdt \geq ce^{-cR^2}.$$

Lemma (Carleman estimate).– Assume that $R > 0$ and $\varphi : [0,1] \longrightarrow \mathbb{R}$ is a smooth function. Then, there exists $c(n) = c(n, \|\varphi'\|_\infty + \|\varphi''\|_\infty) > 0$ such that the inequality

$$\frac{\sigma^{3/2}}{c(n)R^2} \|e^{\sigma \left|\frac{x}{R}+\varphi(t)e_1\right|^2} g\|_2 \leq \|e^{\sigma \left|\frac{x}{R}+\varphi(t)e_1\right|^2} (i\partial_t + \Delta) g\|_2$$

holds when $\sigma \geq c(n)R^2$ and $g \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$ has its support contained in the set

$$\left\{(x,t) \ : \ \left|\frac{x}{R} + \varphi(t)e_1\right| \geq 1\right\}$$

Theorem (with M. Aguirre).– Let $u \in \mathcal{C}([0,1] : H^1_{\rm loc}(\mathbb{R}^n))$ be a solution of

$$\begin{cases} \partial_t u = i(\Delta u + V(x,t)u) \\ u(x,0) = u_0(x) \end{cases}$$

where $V \in L^\infty(\mathbb{R}^n \times [0,1])$ is a complex potential and

$$\|V\|_{L^\infty(\mathbb{R}^n \times [0,1])} \le L$$

Let $R_1 \ge R_0 > 0$ be such that for some $c_0 > 0$,

$$\int_{B_{R_0}} |u_0|^2 dx = c_0^2,$$

and

$$\sup_{0 \le t \le 1} \int_{B_{R_1}} (|u(x,t)|^2 + |\nabla u(x,t)|^2) dx = A^2 < +\infty.$$

Then there exists $t^* = \min\left(R_0^2,\ L^{-2},\ \frac{A}{8c_0 L}, (\frac{2c_0^2}{16A^2})^2\right)$ such that if $0 < t < t^*$ and $\rho \ge R_0$

$$\frac{e^{c(n)\frac{\rho^2}{t}}}{t} \int_{t/2}^t \int_{\rho(1+s/t) \le |y| \le \rho(1+s/t)+\sqrt{t}} (|u(y,s)|^2 + s|\nabla_y u(y,s)|^2) dy ds \ge \frac{c_0^2}{4}$$

Theorem.– Assume that for any $u_0 \in H^1(\mathbb{R}^n)$ there exists a unique solution $u \in \mathcal{C}\big([0,1] : H^1(\mathbb{R}^n)\big)$ of

$$\begin{cases} \partial_t u = i\,(\Delta + V(x,t))\,u \quad x \in \mathbb{R}^n\,, \quad t \in (0,1) \\ \\ u(x,0) = u_0, \end{cases}$$

with $V \in L^\infty\big(\mathbb{R}^n \times [0,1]\big)$.

If there exist $R_j \to \infty$, $j \in \mathbb{N}$ such that for some u

$$\lim_{t\downarrow 0} \frac{1}{t} e^{c(n)\frac{R_j^2}{t}} \int_{t/2}^{t} \int_{R_j(1+s/t)\le|y|\le R_j(1+s/t)+\sqrt{t}} |u(x,s)|^2 + s\,|\nabla u(x,s)|^2\,dxds = 0,$$

then $u \equiv 0$.

About the proof

Lemma.– If $u(y,s)$ verifies

$$\partial_s u = i(\Delta u + V(y,s)u + F(y,s)), \qquad (y,s) \in \mathbb{R}^n \times [0,1]$$

and α and β are positive, then

$$\tilde{u}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2} u\left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}$$

verifies

$$\partial_t \tilde{u} = i(\Delta \tilde{u} + \tilde{V}(x,t)\tilde{u} + \tilde{F}(x,t)), \qquad (x,t) \in \mathbb{R}^n \times [0,1]$$

with

$$\tilde{V}(x,t) = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V\left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right)$$

and

$$\tilde{F}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2+2} F\left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}$$

The goal is to use Carleman's estimate (referencia) in a suitable way so that we can control both u and ∇u by the initial data. For this purpose we want to build a function g with specific functions. First, let $\gamma > 1$ and define $R = R_0\sqrt{\gamma}$. Define also the following cut-off functions, $\theta_R, \eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\varphi \in \mathcal{C}^\infty([0,1])$

$$\theta_R(x) = \begin{cases} 1, & |x| \leq R \\ 0, & |x| \geq R+1 \end{cases} \qquad \eta(x) = \begin{cases} 1, & |x| \geq 2 \\ 0, & |x| \leq 3/2 \end{cases}$$

$$\varphi(t) = \begin{cases} 4, & t \in [3/8, 5/8] \\ 0, & t \in [0, 1/4] \cup [3/4, 1] \end{cases}$$

We define the new unknown v,

$$v(x,t) = \alpha(t)^{n/2} u(\alpha(t)x, s(t)) e^{-\frac{i}{2}\beta(t)|x|^2} \quad , \quad (x,t) \in \mathbb{R}^n \times [0,1]$$

Given γ, define α, β and s

$$\alpha(t) = \frac{1}{\gamma^{1/2}(1-t) + \gamma^{-1/2}t} \leq \frac{1}{\gamma^{1/2}(1-t)} \leq \frac{4}{\gamma^{1/2}}$$

$$\beta(t) = \frac{1}{1-t+\gamma^{-1}t} - \frac{1}{\gamma(1-t)+t} \leq \frac{1}{1-t+\gamma^{-1}t} \leq 4$$

$$\frac{s(t)}{t} = \frac{1}{\gamma(1-t)+t} \leq \frac{1}{\gamma(1-t)} \leq \frac{4}{\gamma}$$

We use all the information gathered above to define the function g as follows:

$$g(x,t) = \theta_R(x)\eta\left(\frac{x}{R} + \varphi(t)e_1\right) v(x,t) \quad , \quad (x,t) \in \mathbb{R}^n \times [0,1]$$

$$\mathbf{R = R_0 \gamma^{1/2}}$$

Observe that due to the nature of the test functions, g is compactly supported and,

- $g = \theta_R v$ on $(x,t) \in \{|x| \leq R+1\} \times [3,8,5/8]$
- $\nabla_x v(x,t) = \alpha(t)^{n/2} e^{-\frac{i}{2}\beta(t)|x|^2} (\alpha(t)\nabla u - i\beta(t)x \cdot u)$
- $\operatorname{supp} g \subseteq \{\left|\frac{x}{R} + \varphi e_1\right| \geq 1\}$.

THANK YOU FOR YOUR ATTENTION