SHELL INTERACTIONS FOR DIRAC OPERATORS: AN ISOPERIMETRIC-TYPE INEQUALITY

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Zaragoza, 27–01–2017

The Operator

- $\partial_t \psi = iH\psi \quad ; \quad H = H_0 + \mathbb{V} \quad , \quad \psi = \psi(x,t)$

- $H_0 = \frac{1}{i}\alpha\nabla + m\beta$

- $H_0^2 = -\Delta + m^2$

$$\begin{array}{l} \alpha \cdot \alpha = \mathbb{1} \qquad \alpha = (\alpha_j) \\ \alpha\beta + \alpha\beta = 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad j \neq k \quad ; \quad \alpha_j^2 = 1 \qquad j = 1,2,3 \end{array}$$

- If $x \in \mathbb{R}^3$ then $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, $\phi, \chi \in \mathbb{C}^2$ (spinors).

- $\mathbb{V}$: "critical" $\qquad \frac{1}{\lambda}\mathbb{V}\left(\frac{x}{\lambda}\right) \sim \mathbb{V}(x)$

Example: Coulomb $\qquad \mathbb{V} = \frac{-\lambda}{|x|}\mathbb{1}$

General Questions

(a) Self–adjointness.

(b) Spectrum: Characterization of the ground state by the "right inequality".

Similar questions for a non linear $\mathbb{V}$ always assume some smallness condition on $\mathbb{V}$.

(c) What is a small/big perturbation of H_0?

The Laplacian

$$\mathcal{H} = -\Delta + V\,, \quad V(x) \in \mathbb{R}$$

$$\langle \mathcal{H} f, f \rangle = \int |\nabla f|^2 + \int V |f|^2$$

(a) $V \geq 0$ or $\int |V||f|^2 < 1^- \int |\nabla f|^2$

(b) $\min\limits_{\|f\|_{L^2}=1} \left\{ \int |\nabla f|^2 + V|f|^2 \right\} \quad (V \leq 0)$

Example: $V(x) = \dfrac{\lambda}{|x|^2}$

$$\int_{\mathbb{R}^3} \frac{|f|^2}{|x|^2} \leq 4 \int_{\mathbb{R}^3} |\nabla f|^2 \qquad \textbf{(Hardy's inequality)}$$

$$f = f(|x|) \qquad \int_0^\infty |f(r)|^2 dr \leq 4 \int_0^\infty r^2 |f'(r)|^2 dr$$

$$\lambda > -1/4$$

Dirac: $\alpha_j = \begin{pmatrix} 0 & \widehat{\sigma_j} \\ \widehat{\sigma_j} & 0 \end{pmatrix} \qquad j = 1, 2, 3$

$$\widehat{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \widehat{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \widehat{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Remark: If $(H_0 + V)\psi_0 = a\psi_0$, $\psi_0 = \begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix}$ and $\widetilde{V}(x) = -V(-x)$, $\widetilde{\psi}_0 = \begin{pmatrix} \chi_0(-x) \\ \phi_0(-x) \end{pmatrix}$, then $(H_0 + \widetilde{V})\widetilde{\psi}_0 = -a\widetilde{\psi}_0$.
In particular if $V = 0$ then the spectrum is $(-\infty, -m] \cup [m, \infty)$.

Coulomb Potential

- $H_0 - \dfrac{\lambda}{|x|}$

(a) Self–adjointness: **Rellich '53**, **Schminke '72**, **Wust '75**, **Nencim '76**, **Kato '80– '83** (Kato–Nencin inequality)

Final answer: $|\lambda| < 1$.

(b) "Ground state" ($\lambda \geq 0$) Minimization process (**Dolbeault**, **Esteban**, **Séré '00**):

– Variational inequality for ϕ $\left(\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}\right)$.

– Hardy–Kato–Nencin type inequalities (**Dolbeault**, **Duoandikoetxea**, **Esteban**, **Loss**, **Vega '04 '07**).

- $\widehat{\sigma} \cdot A \quad \widehat{\sigma} \cdot B = A \cdot B + i\widehat{\sigma} A \wedge B$

- $\widehat{\sigma} \frac{x}{|x|} \quad \widehat{\sigma} \cdot \nabla = \frac{x}{|x|} \cdot \nabla + i\widehat{\sigma} \frac{x}{|x|} \wedge \nabla = \partial_r - \frac{1}{r} \widehat{\sigma} \cdot L$

- $(1 + \widehat{\sigma} \cdot L)^2 \geq 1$

Electrostatic Shell Interactions:

$\Omega \subset \mathbb{R}^3$ bounded smooth domain

$\sigma =$ surface measure on $\partial\Omega$

$N =$ outward unit normal vector field on $\partial\Omega$

Electrostatic shell potential $V_\lambda = \lambda\delta_{\partial\Omega}$:

$$\lambda \in \mathbb{R}, \qquad V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$$

$\varphi_\pm =$ non–tangential boundary values of $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ when approaching from Ω or $\mathbb{R}^3 \setminus \overline{\Omega}$

Electrostatic shell interaction for H: $H + V_\lambda$

(a) Self-Adjointess

If $\lambda \neq \pm 2 \quad \Longrightarrow \quad H + V_\lambda$ is self-adjoint on $\mathcal{D}(H + V_\lambda)$.

(**[Arrizabalaga, Mas, Vega, 2014]**,
more general **[Posilicano, 2008]**
Ω ball $\longrightarrow$ **[Dittrich, Exner, Seba, 1989]**)

$a \in (-m, m)$

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}\,|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2 - a^2}\,|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right]$$

$$= \text{ fundamental solution of } H - a$$

$$\mathcal{D}(H + V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (G dx + g d\sigma),\ G \in L^2((R)^3)^4 \ \ g \in L^2(\partial\Omega)^4,$$

$$\lambda\left(\phi^0 * (G dx)\right)\big|_{\partial\Omega} = -\left(1 + \lambda C^0_{\partial\Omega}\right) g)\Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \phi^a(x-y) g(y) d\sigma(y)\ ,\ x \in \partial\Omega.$

$\lambda^2 = 4$ **[Durmieres, Bonafos, Vega '17]**: Essentially self-adjoint.

$$g \in H^{1/2}(\Sigma) \text{ if } \lambda^2 \neq 4. \text{ Boundary identity is in } H^{1/2}(\Sigma).$$

$$g \notin H^{1/2}(\Sigma) \text{ if } \lambda^2 = 4. \text{ Boundary identity is in } H^{-1/2}(\Sigma).$$

- $-4\left(C^0_{\partial\Omega}(\alpha \cdot N)\right)^2 = \mathbb{1}$ $\quad N$: outer normal

- $$\left\{\alpha \cdot N, C^0_{\partial\Omega}\right\} = \alpha \cdot N C^0_{\partial\Omega} + C^0_{\partial\Omega}\alpha \cdot N$$
  $$= i\left\{\mathcal{C}^*_\pm - \mathcal{C}_\pm\right\}$$

  $\mathcal{C}_\pm$: Calderón projector operator.

  $\mathcal{C}^*_\pm - \mathcal{C}_\pm$ is a compact operator.

(b) Point Spectrum on $(-m,m)$ for $H+V_\lambda$

Birman–Schwinger principle: $a\in(-m,m), \quad \lambda\in\mathbb{R}\setminus\{0\},$

$$\ker(H+V_\lambda-a)\neq 0 \quad \Longleftrightarrow \quad \ker\left(\frac{1}{\lambda}+C^a_{\partial\Omega}\right)\neq 0$$

(problem in $\mathbb{R}^3$) **(problem in $\partial\Omega$)**

Properties of $C^a_{\partial\Omega}$, $a\in[-m,m]$**:**

(a) $C^a_{\partial\Omega}$ bounded self–adjoint operator in $L^2(\partial\Omega)^4$.

(b) $\left[C^a_{\partial\Omega}(\alpha\cdot N)\right]^2=-\frac{1}{4}I_d.$ $\left(\alpha\cdot N=\sum_{j=1}^{3}\alpha_j N_j \text{ multiplication operator}\right)$

$$\ker\left(\frac{1}{\lambda}+C^a_{\partial\Omega}\right)\neq 0 \quad \begin{cases} \overset{(a)}{\Longrightarrow} & |\lambda|\geq\lambda_l(\partial\Omega)>0 \quad \text{and} \quad \lambda_l(\partial\Omega)\leq 2 \\ \overset{(b)}{\Longrightarrow} & |\lambda|\leq\lambda_u(\partial\Omega)<+\infty \quad \text{and} \quad \lambda_u(\partial\Omega)\geq 2 \end{cases}$$

Therefore, $\ker(H+V_\lambda-a)\neq 0 \quad \Longrightarrow \quad |\lambda|\in[\lambda_l(\partial\Omega),\lambda_u(\partial\Omega)]$

Main result:

Question: How small can $[\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$ be?

(Isoperimetric–type statement w.r.t. Ω)

(Find optimizers)

Examples: $\Omega \subset \mathbb{R}^3$ bounded smooth domain

- Isoperimetric inequality: $\mathrm{Vol}\,(\Omega)^2 \leq \frac{1}{36}\,\mathrm{Area}\,(\partial\Omega)^3.$
- Pólya–Szegö inequality:

$$\mathrm{Cap}\,(\overline{\Omega}) = \left(\inf_{\nu} \iint \frac{d\nu(x)d\nu(y)}{4\pi|x-y|}\right)^{-1}$$

ν probability
Borel measure
$\mathrm{supp}\,\nu \subset \overline{\Omega}$

$$\mathrm{Cap}\,(\overline{\Omega}) \geq 2\big(6\pi^2\mathrm{Vol}\,(\Omega)\big)^{1/3}. \qquad \longleftarrow \qquad \text{[Pólya, Szegö, 1951]}$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Theorem [AMV2016].– $\Omega \subset \mathbb{R}^3$ bounded smooth domain. If

$$m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} > \frac{1}{4\sqrt{2}},$$

then

$$\sup\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \text{ for some } a \in (-m, m)\} \geq 4\left(m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}(\partial\Omega)^2}{\text{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

and

$$\inf\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \text{ for some } a \in (-m, m)\} \leq 4\left(-m\frac{\text{Area}(\partial\Omega)}{\text{Cap}(\overline{\Omega})} + \sqrt{m^2\frac{\text{Area}(\partial\Omega)^2}{\text{Cap}(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

In both cases, $=$ holds $\iff$ Ω is a ball.

Proof:

(1) $$\ker\left(\frac{1}{\lambda(a)} + C^a_{\partial\Omega}\right) \neq 0 \quad \Longrightarrow \quad C^a_{\partial\Omega} g_a = \frac{1}{\lambda(a)} g_a \ , \ \|g_a\| = 1$$

$$\Longrightarrow \quad \frac{1}{\lambda(a)} = \frac{1}{\lambda(a)} \langle g_a, g_a \rangle = \langle C^a_{\partial\Omega} g_a, g_a \rangle$$

$$C^a_{\partial\Omega} \hookrightarrow (H-a)^{-1} \quad \Longrightarrow \quad \frac{d}{da} C^a_{\partial\Omega} \hookrightarrow (H-a)^{-2}$$

$$\Longrightarrow \quad \frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a \rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

(2)

$$\left.\begin{array}{ll} Kf(x) = \dfrac{1}{4\pi}\displaystyle\int \frac{f(y)}{|x-y|}d\sigma y & \begin{pmatrix}\text{compact}\\ \text{positive}\end{pmatrix} \\ Wf(x) = \dfrac{1}{4\pi}\displaystyle\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} i\cdot\widehat{\sigma}\cdot\frac{x-y}{|x-y|^3}f(y)\,d\sigma(y) & \\ & \text{(SIO)} \end{array}\right\} C^a_{\partial\Omega} = \begin{pmatrix} 2mK & W \\ W & 0 \end{pmatrix}$$

$$\left(\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right)$$

Then,

$$[C^m_{\partial\Omega}(\alpha\cdot N)]^2 = -\frac{1}{4} \quad\Longrightarrow\quad \begin{cases} \{(\widehat{\sigma}\cdot N)K, (\widehat{\sigma}\cdot N)W\} = 0 \\ [(\widehat{\sigma}\cdot N)W]^2 = -\dfrac{1}{4} \end{cases} \qquad (**)$$

$$\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0 \quad\Longrightarrow\quad C^m_{\partial\Omega}g=\frac{1}{\lambda}g \qquad g=\begin{pmatrix}\mu\\ h\end{pmatrix}$$

$$\Longrightarrow\quad \left\{\begin{array}{rcl} 2mK\mu+Wh & = & -\frac{1}{\lambda}\mu \\ \\ W\mu & = & -\frac{1}{\lambda}h \end{array}\right.$$

$$\overset{(**)}{\Longrightarrow}\quad \exists f\in L^2(\partial\Omega)^2,\ f\neq 0 \text{ such that } \left(-\frac{8m}{\lambda}K+1-\frac{16}{\lambda^2}W^2\right)f=0$$

Multiply by $\overline{f}$ and integrate on $\partial\Omega$:

$$\left(\begin{array}{c}\text{decreasing}\\ \text{on } \lambda>0\end{array}\right)\qquad \left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\underbrace{\int_{\partial\Omega}Kf\cdot\overline{f}}_{\geq 0}=\int_{\partial\Omega}|f|^2$$

Quadratic form inequality:

$$\lambda_\Omega=\inf\left\{\lambda>0:\left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\int_{\partial\Omega}Kf\cdot\overline{f}\leq\int_{\partial\Omega}|f|^2\qquad\forall f\in L^2(\partial\Omega)^2\right\}$$

[Esteban,Séré, 1997]

Theorem [AMV2015].–

(a) $$4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\frac{1}{4}}\right)$$

$$\leq \lambda_\Omega \leq 4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\|W\|^2_{\partial\Omega}}\right).$$

(b) If $\lambda>0$ and $\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0 \quad\Longrightarrow\quad \lambda\leq\lambda_\Omega.$

(c) If $\lambda_\Omega>2\sqrt{2}$ $\left(\hookrightarrow (\frac{1}{4\sqrt{2}}\right)$, equality is attained and minimizers $\hookrightarrow$ $\ker\left(\frac{1}{\lambda_\Omega}+C^m_{\partial\Omega}\right)\neq 0.$

Recall Birman–Schwinger principle:

$$\frac{d}{da}\left(\frac{1}{\lambda(a)}\right) \sim \langle (H-a)^{-2} g_a, g_a \rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$$

(assume g_a independent of a)

This suggests another way of obtaining the ground state for the Coulomb potential $V(x) = -\frac{\lambda}{|x|}$:

$$\frac{m^2 - a^2}{m^2} \int \frac{|\psi^2}{|x|} \leq \int \left| \left(\frac{1}{i} \alpha \cdot \nabla + m\beta + a \right) \psi \right|^2 |x|$$

(**Arrizabalaga**, **Duoandikoetxea**, **Vega ’13**; **Cassano**, **Pizzichilo**, **Vega ’17**)

The inequality is optimal and it is achieved for $A > 0$ by the ground state of $V_a(x) = -\frac{m^2 - a^2}{m^2} \frac{1}{|x|}$.

The proof is a consequence of the ‘uncertainty principle”.

- $2\mathrm{Re}\,\langle S\psi, A\psi\rangle = \langle (SA - AS)\psi, \psi\rangle$ if $S^* = S$ and $A^* = -A$.

- $2\mathrm{Re}\,\langle A_1\psi, A_2\psi\rangle = -\langle (A_1A_2 + A_2A_1)\psi, \psi\rangle$ if $A_1^* = -A_1$ and $A_2^* = -A_2$.

In our case the right choice is:

$$2\mathrm{Re}\,\langle \underbrace{(\alpha\cdot\nabla + i(m\beta + a))}_{A_1}\,\psi, (1+\sigma\cdot L)\mathbb{1}\,\alpha\cdot\frac{x}{|x|}(\underbrace{m\beta}_{S} + \underbrace{a}_{A_2})\rangle.$$

THANK YOU FOR YOUR ATTENTION

Ingredients of the proof:

(1) The monotonicity of $\lambda(a)$ in $\ker\left(\frac{1}{\lambda(a)} + C^a_{\partial\Omega}\right)$ reduces the study of (∗) to $a = \pm m$.

(2) The quadratic form inequality relates $\sup\{|\lambda| : \ker(H + V_\lambda - a)) \neq 0 \text{ for some } a \in (-m, m)\}$ in (∗) with the optimal constant of an inequality involving the single layer potential K and a SIO. (Here appears the $1/4\sqrt{2}$)

(3) Isoperimetric type statement for K in terms of $\text{Area}\,(\partial\Omega) \setminus \text{Cap}\,(\overline{\Omega})$.

Review :

(3) $\longrightarrow$ Isoperimetric–type result for λ_Ω.

(2) $\longrightarrow$ Theorem **(b)** and **(c)** ensure

$$\lambda_\Omega = \sup\{|\lambda| : \ker(1/\lambda + C^m_{\delta\Omega}) \neq 0\}.$$

(1) $\longrightarrow$ Use monotonicity to replace "for some $a \in (-m, m)$" by $a = m$.

(3) Ω ball $\Longrightarrow$ $\|W\|^2_{\partial\Omega} = \dfrac{1}{4}$

(" $\Longleftarrow$ " [**Hofmann, Marmdejo–Olea, Mitrea, Pérez–Esteva, Taylor, 2009**])

$$\Longrightarrow \quad \lambda_\Omega = 4\left(m\|K\|_{\partial\Omega} + \sqrt{m^2\|K\|^2_{\partial\Omega} + \|W\|^2_{\partial\Omega}}\right)$$

Ω general,

$$\|K\|_{\partial\Omega} = \sup_{f\neq 0} \frac{1}{\|f\|^2_{\partial\Omega}} \int_{\partial\Omega} Kf \cdot \overline{f} \geq \iint \overbrace{\frac{d\sigma(y)}{4\pi|x-y|}}^{\inf \mathbf{D}} \frac{d\sigma(x)}{\sigma(\partial\Omega)} \geq \frac{\text{Area}\,(\partial\Omega)}{\text{Cap}\,(\overline{\Omega})}$$

$(\mathbf{f} = \mathbf{1})$

(" = " $\Longleftrightarrow$ Ω is a ball: **Gruber's conjecture**)

- Neumann eigenvalue problem $(\Omega \subset \mathbb{R}^2)$

$$\begin{cases} -\Delta u = \mu u \ \text{ in } \ \Omega \\ \dfrac{\partial u}{\partial N} = 0 \ \text{ on } \ \partial\Omega \end{cases} \implies \mu_1(\Omega) \leq \frac{C}{\text{Area}\,(\Omega)}$$

[Szegö, 1954]

(disks give "=")

- Steklov eigenvalue problem $(\Omega \subset \mathbb{R}^2)$

$$\begin{cases} \Delta u = 0 \ \text{ in } \ \Omega \\ \dfrac{\partial u}{\partial N} = \omega u \ \text{ on } \ \partial\Omega \end{cases} \implies \omega_1(\Omega) \leq \frac{2\pi}{\text{Length}\,(\partial\Omega)}$$

[Weirstock, 1954]

Free Dirac operator in $\mathbf{R}^3$: $H = -i\alpha \cdot \nabla + m\beta$

$(m = \text{mass} > 0)$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \left(\begin{pmatrix} 0 & \widehat{\sigma}_1 \\ \widehat{\sigma}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_2 \\ \widehat{\sigma}_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_3 \\ \widehat{\sigma}_3 & 0 \end{pmatrix} \right)$$

$$\widehat{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \alpha_i^2 = \beta^2 = I_d & i = 1, 2, 3 \\ \{\alpha_i, \beta\} = \{\alpha_i, \alpha_j\} = 0 & i \neq j \end{cases}$$

(Clifford Algebra Structure)

$$H^2 = (-\Delta + m^2) I_d \quad \Longrightarrow \quad \begin{cases} H \text{ local version of } \sqrt{-\Delta + m^2} \\ \text{1st order symmetric differential operator} \\ \text{Introduced by Dirac (1928)} \longrightarrow \textbf{electron} \end{cases}$$

$a \in (-m, m)$

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}\,|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2-a^2}\,|x|\right) i\alpha \cdot \frac{x}{|x|^2}\right]$$

$$= \text{ fundamental solution of } H - a$$

$$\mathcal{D}(H + V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (Gdx + gd\sigma),\ G \in L^2((R)^3)^4 \ \ g \in L^2(\partial\Omega)^4,$$

$$\lambda\left(\phi^0 * (Gdx)\right)\Big|_{\partial\Omega} = -\left(1 + \lambda C^0_{\partial\Omega}\right) g)\Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \phi^a(x-y)g(y)d\sigma(y)\ ,\ x \in \partial\Omega.$

If $\lambda \neq \pm 2 \quad \Longrightarrow \quad H + V_\lambda$ is self–adjoint on $\mathcal{D}(H + V_\lambda)$.

(**[AMV, 2014]**, more general **[Posilicano,2008]**
Ω ball $\longrightarrow$ **[Dittrich, Exner, Seba,1989]**)