ASYMPTOTIC LOWER BOUNDS FOR SOLUTIONS TO DISPERSIVE EQUATIONS

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Asymptotics for Schrödinger Equations

$$\partial_t u = i\,(\Delta u + V u) \qquad x \in \mathbb{R}^d \quad t \in \mathbb{R}$$

- $V = V(x,t)$
- $V = \pm |u|^p$
- $\Delta \rightsquigarrow \mathcal{L} = \Delta_y - \Delta_z \qquad ; \qquad x = (y,z)$

(a) Defocusing

(b) Focusing

Defocusing Case $\qquad d \geq 3$

- $V = -|u|^p \qquad \frac{4}{d} < p \leq \frac{4}{d-2}$
- $V = V(x) \qquad$ short range and repulsive

Theorem 1 (with N. Visciglia).–

$$\lim_{R\to\infty} \frac{1}{R} \int_{B_R} \int_{-\infty}^{\infty} |\nabla u(x,t)|^2 = \|u(\cdot,0)\|^2_{\dot{H}_V^{1/2}(\mathbb{R}^\alpha)}$$

$$\left(= \pi \left(\|\varphi_+\|^2_{\dot{H}_x^{1/2}} + \|\varphi_-\|^2_{\dot{H}_x^{1/2}} \right) \right)$$

Remarks.– **(i)** $\nabla \hookrightarrow \partial_r$

(ii) $d = 3$ slightly different

(iii) Uniqueness result

Proofs: Morawetz/Virial Identity

$$H(t) = \int \psi(|x|)\, |u(x,t)|^2 \, dx$$

$$\dot{H}(t) = -2\mathrm{Im} \int_{\mathbb{R}^d} \nabla\psi \nabla u \bar{u} dx$$

$$\ddot{H} = ?$$

$$\lim_{T\to\infty} \int_{-T}^{T} \int_{\mathbb{R}^d} \left(\nabla_x \bar{u} D^2 \psi \nabla_x u - (\Delta^2 \psi + 4\partial_r V \partial_r \psi) \frac{|u|^2}{4} \right) dx dt$$
$$= \psi'(\infty) \|f\|^2_{\dot{H}_V^{1/2}(\mathbb{R}^d)}$$

- $\lim_{t\to\infty} \dot{H}(t) = \frac{1}{2} \psi'(\infty) \|f\|^2_{\dot{H}_V^{1/2}(\mathbb{R}^d)}$
- $-\Delta^2 \psi \geq 0 \quad \Longrightarrow \quad d \geq 3.$

Remark.– Similar result for the wave equation: equipartition of energy.

Focusing case (L. Escauriaza, C.E. Kenig, G. Ponce)

Two scenarios:

(i) Blowing up profiles.

$$\begin{aligned} \partial_t u &= i\left(\Delta u + |u|^{4/d} u\right) \qquad x \in \mathbb{R}^d \\ u(x,0) &= u_0(x) \in L^2(\mathbb{R}^d) \end{aligned}$$

$$u(x,t) = \frac{1}{(1-t)^{d/2}} e^{i|x|^2/4(1-t)} e^{iw/(1-t)} Q_w\left(\frac{x}{1-t}\right)$$

$$wQ_w = \Delta Q_w + Q_w^{\frac{4}{d}+1} \quad ; \quad w > 0 \quad Q_w = Q_w(r) \quad Q \geq 0$$

$$Q_w = w^{d/4} Q\left(w^{1/2} x\right) \quad ; \quad Q \text{ with linear exponential decay}$$

Q1: Is this linear exponential decay optimal?

Q2: What happens if $\Delta \hookrightarrow \mathcal{L} = \Delta_y - \Delta_z \quad x = (y,z)$?

(ii) Eigenfunctions/Eigenvalues:

$$\begin{aligned} \partial_t u &= i\,(-\Delta + V)\,u \\ (-\Delta + V)\,u &= wu \\ V &= V(x) \quad \text{real} \quad w < 0 \end{aligned}$$

Meshkov:

$(\Delta + V)\,u = 0 \quad x \in \mathbb{R}^n \qquad V = V(x) \in L^\infty(\mathbb{R}^n)$ maybe complex

If $\int e^{2a|x|^{4/3}} |u|^2 dx > 0 \quad \forall\, a > 0$ then $u \equiv 0.$

The result is sharp.

Cruz–Sampedro:

$$|V(x)| \leq \frac{C}{(1 + |x|)^{1/2^+}}$$

$$\int e^{2a|x|} |u|^2 dx < 0 \quad \forall\, a > 0 \quad \text{then} \quad u \equiv 0.$$

Q3: What happens if $V = V(x, t)$?

Theorem 2.– Assume u is a smooth solution of

$$\begin{cases} \partial_t u &= i\left(\mathcal{L}u \pm |u|^p u\right) \qquad x \in \mathbb{R}^d \quad t \in \mathbb{R} \quad p = \dfrac{4}{d} \quad \left(p \geq \dfrac{4}{d}\right) \\ u(x,0) &= u_0 \end{cases}$$

such that there exists $Q = Q(x)$ with

$$|u(x,t)| \leq \frac{1}{(1-t)^{d/2}} Q\left(\frac{x}{1-t}\right) \quad t \in (-1,1)$$

Then there exists λ_0 big enough such that if $\lambda \geq \lambda_0$ and

$$\int e^{\lambda |x|} Q^2(x) dx < +\infty,$$

then $u \equiv 0$.

Theorem 3.– If $u \in H^1(\mathbb{R}^d)$ is a solution of

$$\begin{cases} \bullet \quad \partial_t u = i\,(\mathcal{L} + V(x,t))\,u & ; \qquad V(x,t) \in \mathbb{R} \\ \bullet \quad |V(x,t)| \le \dfrac{C}{(1+|x|)^{1/2^{+}\epsilon}} & \epsilon > 0 \end{cases}$$

Then there exists $\lambda = \lambda_0\,(\|V\|_{L^\infty}, \epsilon)$ such that if

$$(*) \qquad \sup_t \int e^{2\lambda|x|}\,|u(x,t)|^2\,dx < +\infty$$

for $\lambda \ge \lambda_0$ then $u \equiv 0$.

Remarks.–

- Galilean invariance gives a similar result for traveling wave solutions.

- The result can be extended to Laplace equation. The question is then about the existence of wave guides (j.w. with L. Escauriaza and L. Fanelli).

- $(*)$ can be relaxed.

About the Proofs

Blow up:

Pseudo conformal transformation:

$$u(x,t) = \frac{1}{(1-t)^{d/2}} w\left(\frac{x}{1-t}, \frac{1}{1-t}\right) e^{i\frac{|x|^2}{4(1-t)}}$$

Then u satisfies a similar equation but $T = 1$ becomes $T = +\infty$.

The proof follows a similar argument to the one of **Theorem 3**.

Notice that the L^2 norm is a preserved quantity. Hence if it is zero at infinity the solution is zero.

Proof of Theorem 3

$$\begin{aligned} H(t) &= \int e^{2\lambda\varphi}\,|u(x,t)|^2\,dx \\ &= \int |w(x,t)|^2\,dx \end{aligned}$$

$\ddot{H}$?

$$\begin{aligned} w &= e^{\lambda\varphi}u. \\ \partial_t w &= (S+\mathcal{A})w + iV(x,t)w \end{aligned}$$

$$\mathcal{L} = \Delta : \quad S = -i\lambda\,(2\nabla\varphi\cdot\nabla + \Delta\varphi)$$

$$\mathcal{A} = i\left(\Delta + \lambda^2\,|\nabla\varphi|^2\right).$$

(If $V \equiv 0$)

$$\begin{aligned} \dot{H}(t) &= 2\langle Sw, w\rangle \\ \ddot{H}(t) &= 2\langle (S\mathcal{A} - \mathcal{A}S)\,w, w\rangle + 4\langle Sw, Sw\rangle \end{aligned}$$

$$\begin{aligned}\langle S\mathcal{A} - \mathcal{A}S, w\rangle &= 4\lambda \int \nabla w D^2\varphi \overline{\nabla w} - \lambda \int \Delta^2 \varphi w \overline{w} \\ &\quad + 4\lambda^3 \int \nabla\varphi D^2\varphi \nabla\varphi |w|^2\end{aligned}$$

- No obstruction in the dimension.
- $\varphi(x) = |x|^{4/3}$ is critical for “bounded” perturbations.
- $\langle Sw, w\rangle$ uniformly bounded in t?
- For general $\mathcal{L}$: $\nabla \hookrightarrow \tilde{\nabla} = (\nabla_y, -\nabla_z)$.

Two Identities

$$\partial_t w = (S + \mathcal{A})w + F$$

$$\begin{aligned}
\frac{d}{dt}\langle w, w\rangle &= 2\langle Sw, w\rangle + 2\mathrm{Re}\,\langle F, w\rangle \\
\frac{d}{dt}\langle Sw, w\rangle &= \langle (S\mathcal{A} - \mathcal{A}S)\, w, w\rangle + 2\mathrm{Re}\,\langle Sw, Fw\rangle + 2\langle Sw, Sw\rangle \\
\eta &= \eta(t)
\end{aligned}$$

Hence

$$\begin{aligned}
\int_a^b \dot{\eta}\langle Sw, w\rangle &= \frac{1}{2}\dot{\eta}\, \langle w, w\rangle|_a^b - \int_a^b \ddot{\eta}\langle w, w\rangle - \int_a^b \dot{\eta}\mathrm{Re}\,\langle F, w\rangle \\
\int_a^b \dot{\eta}\langle Sw, w\rangle &= \eta\, \langle Sw, w\rangle|_a^b - \int \eta\,(\langle (S\mathcal{A} - \mathcal{A}S)\, w, w\rangle + 2\langle Sw, Sw\rangle) \\
&\quad - 2\int \eta\mathrm{Re}\,\langle Sw, Fw\rangle
\end{aligned}$$

STEP 1 Use the two identities with $\eta(t) = |T - t| - \dfrac{1}{2}$ to obtain a uniform bound for

$$\int_{T-1/2}^{T+1/2} \eta(t) \langle Sw, Sw \rangle$$

Hence $\exists T_n \to \infty$ such that $\langle Sw(T_n), Sw(T_n) \rangle$ is uniformly bounded.

STEP 2 Take $\eta(t) \equiv 0,\ 0 \le t \le 1 \quad \eta(t) \equiv 1 \quad 2 < t < T_n \quad \eta \uparrow .$ Use the two identities to obtain

$$\int_2^{T_n} \langle (S\mathcal{A} - \mathcal{A}S) w, w \rangle dt$$

is uniformly bounded.

STEP 3

$$\int_2^\infty \int |u(x,t)|^2 \, dx dt \leq C \int_2^\infty \langle (S\mathcal{A} - \mathcal{A}S)w, w \rangle dt$$

Hence

$$\|u(\cdot,0)\|_{L^2} = \|u(\cdot,t)\|_{L^2} \longrightarrow 0$$

and

$$u \equiv 0$$

We use that $V \in \mathbb{R}$ in this last step.

Step 1 and **Step 2** follow from the positivity of

$$\langle (S\mathcal{A} - \mathcal{A}S)w + V(x,t)w, w\rangle.$$

This is proved taking λ big enough and

$$\varphi'(r) = \begin{cases} r & r \le 1 \\ 2 - \dfrac{1}{\lg r} & r > 2 \end{cases}$$

$\varphi' \uparrow$ and smooth.

GRACIAS POR SU ATENCIÓN